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ON NOWHERE DENSE CLOSED P-SETS

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ABSTRACT. We show that no compact space of weight ω_1 can be covered by nowhere dense closed P-sets. In addition, we construct a compact space of weight ω_2 which can be covered by nowhere dense closed P-sets. As an application, we show that CH is equivalent to the statement that each small nonpseudocompact space has a remote point.

0. Definitions and notation. All spaces considered are completely regular.

As usual we identify a cardinal with an initial ordinal, and an ordinal with the set of smaller ordinals. Ordinals carry the order topology. A cardinal κ is *regular* if κ is not the sum of fewer, smaller cardinals.

Let κ be any uncountable cardinal. A subset B of a space X is called a P_κ*-set* provided that each intersection of fewer than κ neighborhoods of B is again a neighborhood of B. As usual, a P_{ω_1}-set is simply called a *P-set.* A space X is a *P-space* if each singleton is a P-set.

βX denotes the Čech-Stone compactification of X and X^* is $\beta X - X$. A point x of X^* is called a *remote point* of X if $x \notin \mathrm{cl}_{\beta X} A$ for each nowhere dense subset A of X.

A *π-base* $\mathfrak{B}$ for a space X is a family of nonempty open subsets of X such that each nonempty open set in X contains some $B \in \mathfrak{B}$. The *π-weight*, $\pi(X)$, of X is the least cardinal κ for which there is a π-base for X of cardinality κ.

$(X_\alpha, f_{\alpha\beta}, \kappa)$ means that κ is an ordinal, that for each $\alpha < \kappa$, X_α is a space and that, for each $\beta < \alpha, f_{\alpha\beta}$ is a map from X_α into X_β such that if $\beta < \alpha < \gamma$ then $f_{\gamma\beta} = f_{\alpha\beta} \circ f_{\gamma\alpha}$. The triple $(X_\alpha, f_{\alpha\beta}, \kappa)$ is called an *inverse system.* The *inverse limit* $\varprojlim (X_\alpha, f_{\alpha\beta}, \kappa)$ of the inverse system $(X_\alpha, f_{\alpha\beta}, \kappa)$ is the subspace

$$\left\{ x \in \prod_{\alpha<\kappa} X_\alpha \middle| \, \forall_{\beta<\alpha<\kappa} x_\beta = f_{\alpha\beta}(x_\alpha) \right\}$$

of $\prod_{\alpha<\kappa} X_\alpha$. The projection from $\varprojlim (X_\alpha, f_{\alpha\beta}, \kappa)$ into X_α is denoted by $f_{\kappa\alpha}$. An inverse system $(X_\alpha, f_{\alpha\beta}, \kappa)$ is called *continuous* provided that $X_\beta = \varprojlim (X_\alpha, f_{\alpha\gamma}, \beta)$ for each limit ordinal $\beta < \kappa$.

A space X is called *small* provided that $|C^*(X)| \leqslant 2^\omega$.

1. Introduction. It is well known that a pseudocompact P-space is finite [GH]; hence a compact infinite space cannot have too many singletons which are P-sets. This leaves open the question whether a compact infinite space

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can have "many" "small" P-sets. An appropriate topological translation of "smallness" is nowhere denseness, hence we are interested in nowhere dense closed P-sets. We were somewhat surprised to find the following partial answer to the above question.

1.1. THEOREM. *Let X be a compact space of π-weight $\leqslant \kappa$ ($\kappa > \omega$). Then there is an $x \in X$ such that $x \notin K$ for all closed nowhere dense P_κ-sets $K \subset X$.*

Notice that in case $\kappa = \omega_1$, this theorem states that no compact of π-weight ω_1 can be covered by nowhere dense closed P-sets.

This result suggests a host of questions: among others, whether every compact space of weight κ^+ contains a point which is not in any nowhere dense closed P_κ-set. We answer this question in the negative.

1.2. EXAMPLE. For each uncountable κ there is a compact space X_κ of weight κ^+ such that each point of X_κ is contained in some nowhere dense closed P_κ-set of X_κ.

As an immediate consequence, CH is equivalent to the statement that no compact space of weight 2^ω can be covered by nowhere dense closed P-sets.

We find an application of our results in the construction of remote points.

1.3. THEOREM. CH *is equivalent to the statement that each small nonpseudocompact space has a remote point.*

2. Proof of Theorem 1.1. We start with a simple lemma.

2.1. LEMMA. *If $X = \varprojlim (X_\alpha, f_{\alpha\beta}, \kappa)$, where*

(a) *κ is regular,*

(b) *$\pi(X_\alpha) < \kappa$ for each $\alpha < \kappa$,*

(c) *$(X_\alpha, f_{\alpha\beta}, \kappa)$ is continuous;*

then for each closed subset A of X with empty interior there is some $\alpha < \kappa$ such that $f_{\kappa\alpha}[A]$ has empty interior.

PROOF. Since $(X_\alpha, f_{\alpha\beta}, \kappa)$ is continuous, for each limit ordinal $\alpha < \kappa$ the collection

$$\bigcup_{\beta<\alpha} \{ f_{\alpha\beta}^{-1}[U] \mid U \text{ is open in } X_\beta \}$$

is a base for X_α. This implies that for each $\alpha < \kappa$ we may choose a π-base $\mathfrak{B}_\alpha$ for X_α such that:

(i) $\alpha < \beta \Rightarrow f_{\beta\alpha}^{-1}[\mathfrak{B}_\alpha] \subset \mathfrak{B}_\beta$;

(ii) if $\beta < \kappa$ is a limit ordinal then $\mathfrak{B}_\beta = \bigcup_{\alpha<\beta} f_{\beta\alpha}^{-1}[\mathfrak{B}_\alpha]$;

(iii) if $\beta < \kappa$ then $|\mathfrak{B}_\beta| < \kappa$.

Write $\mathfrak{B}_\alpha = \{ U_\gamma^\alpha \mid \gamma < \alpha' \}$ where $\alpha' < \kappa$. Fix α for awhile. For each $\gamma < \alpha'$ there is some $\gamma(\alpha) < \kappa$ such that $f_{\gamma(\alpha)\gamma}^{-1}[U_\gamma^\alpha] \not\subset f_{\kappa\gamma(\alpha)}[A]$. Write $\beta_0(\alpha) = \sup_{\gamma<\alpha'} \gamma(\alpha)$. Then $\beta_0(\alpha) < \kappa$ since κ is regular. In addition, define $\beta_{n+1}(\alpha) = \beta_0(\beta_n(\alpha))$ for each $n < \omega$.

Write $\beta = \beta_\omega(0) = \sup_{n<\omega} \beta_n(0)$. Then $\beta < \kappa$ since κ is regular. We claim that $f_{\kappa\beta}[A]$ has empty interior. For if $f_{\kappa\beta}[A]$ contains a member V of $\mathfrak{B}_\beta$, then

$V = f_{\beta\beta_n(0)}^{-1}[U]$ for some $U \in \mathcal{B}_{\beta_n(0)}$ and some $n < \omega$. But then $f_{\beta_{n+1}(0)\beta_n(0)}^{-1}[U] \not\subset f_{\kappa\beta_{n+1}(0)}[A]$ whence $V \not\subset f_{\kappa\beta}[A]$, a contradiction. $\square$

2.2. LEMMA. *If X is a compact space of π-weight κ then there is an irreducible map f: $X \to Y$ where Y has weight κ.*

PROOF. Assume $X \subset I^\lambda$, where I is the closed unit interval, and let $\{F_\alpha: \alpha < \kappa\}$ be a π-basis for X such that

$$F_\alpha = \bigcap_{i<n_\alpha} \pi_{\alpha_i}^{-1}(U_i^\alpha), \quad \text{where } U_i^\alpha \text{ is open in } I.$$

Let Y be the image of X under the projection onto the coordinates $\{\alpha_i: \alpha \in \kappa, i < n_\alpha\}$. One sees easily that this Y and this map satisfy our conclusion. $\square$

2.3. PROOF OF THEOREM 1.1. Assume first that κ is regular. Fix an irreducible map f: $X \to Y$ where $Y \subset I^\kappa$. Let $\pi_{\beta\alpha}$: $I^\beta \to I^\alpha$ be the projection $(\alpha < \beta \leqslant \kappa)$ and let $X_\alpha = \pi_{\kappa\alpha}[Y]$. Also, let $f_{\beta\alpha} = \pi_{\beta\alpha} \restriction X_\beta$. Notice that $w(X_\alpha) < \kappa$ for each $\alpha < \kappa$. If $K \subset X$ is a closed P_κ-set and $\alpha < \kappa$ then

$$K \subset f^{-1} \circ f_{\kappa\alpha}^{-1} \circ f_{\kappa\alpha} \circ f[K],$$

and the latter is an intersection of less than κ open sets, since $w(X_\alpha) < \kappa$. So

$$K \subset \operatorname{int}_X f^{-1} \circ f_{\kappa\alpha}^{-1} \circ f_{\kappa\alpha} \circ f[K].$$

Also, by Lemma 2.1, if $K \subset X$ has empty interior then $f_{\kappa\alpha} \circ f[K]$ has empty interior in X_α for some $\alpha < \kappa$ (since f is irreducible).

It is thus sufficient to choose $p \in X$ such that *for each $\alpha < \kappa$ and each closed nowhere dense $H \subset X_\alpha$ we have that*

$$p \notin \operatorname{int}_X f^{-1} \circ f_{\kappa\alpha}^{-1}[H].$$

If such a choice is impossible, then there are $\alpha_i < \kappa$ $(i < n)$ and closed nowhere dense $H_i \subset X_{\alpha_i}$ such that

$$X = \bigcup_{i<n} \operatorname{int}_X f^{-1} \circ f_{\kappa\alpha_i}^{-1}[H_i].$$

Since a finite union of nowhere dense sets is nowhere dense, we may assume that $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$. Now, inductively define open sets $U_i \subset X_{\alpha_i}$, so that $U_0 = X_{\alpha_0} - H_0$ and $U_{i+1} = f_{\alpha_{i+1}\alpha_i}^{-1}[U_i] - H_{i+1}$. Then $f^{-1} \circ f_{\kappa\alpha_{n-1}}^{-1}[U_{n-1}]$ is nonempty and misses each $f^{-1} \circ f_{\kappa\alpha_i}^{-1}[H_i]$, a contradiction.

Now observe that if κ is singular, then any P_κ-set of X is a P_{κ^+}-set; then the theorem for singular κ follows from the theorem for regular κ. $\square$

3. The example.

3.1. CONSTRUCTION OF EXAMPLE 1.2. Let

$$X_\kappa = \{f \in (\kappa+1)^{\kappa^+} \mid f \text{ is nondecreasing}\}$$
$$= \{f \in (\kappa+1)^{\kappa^+} \mid \forall_{\alpha<\beta<\kappa^+} f(\alpha) \leqslant f(\beta)\}.$$

It is trivial to verify that X_κ is compact and that $w(X_\kappa) = \kappa^+$. If $f \in X_\kappa$, either $f(\alpha) = \kappa$ for some $\alpha < \kappa^+$, in which case f is in the nowhere dense closed

P_κ-set$\{g \in X_\kappa | g(\alpha) = \kappa\}$, or there is some $\xi < \kappa$ for which $f(\alpha) \leqslant \xi$ for each $\alpha < \kappa^+$, in which case f is in the nowhere dense closed P_{κ^+}-set $\{g \in X_\kappa | g(\alpha) \leqslant \xi$ for each $\alpha < \kappa^+\} = \bigcap_{\alpha<\kappa^+}\{g \in X_\kappa | g(\alpha) \leqslant \xi\}$ (observe that this intersection is decreasing). □

3.2. COROLLARY. CH *is equivalent to the statement that no compact space of weight* 2^ω *can be covered by nowhere dense closed P-sets.*

3.3. *Question. Is there, in* ZFC, *an* $x \in \beta\omega - \omega$ *such that* $x \notin K$ *for all closed nowhere dense P-sets* K *of* $\beta\omega - \omega$?[2]

4. Remote points. Let us note that van Douwen **[vD]** has shown that each nonpseudocompact space of countable π-weight has a remote point. Not every nonpseudocompact space has a remote point **[vDvM]** and it is open whether or not every separable space has a remote point **[vDvM]** (the answer is yes under CH; this follows from a construction in **[FG]**).

4.1. PROOF OF THEOREM 1.3. Assume CH and let X be any nonpseudocompact small space. Let Z be a nonempty closed G_δ of βX which misses X **[GJ**, 6.1] and let $Y = \beta X - Z$. Then Y is locally compact and σ-compact, $X \subset Y$ and $\beta Y = \beta X$ **[GJ**, 6.7]. It is clear that it suffices to show that Y has a remote point.

Since X is small, $w(\beta X) = w(\beta Y) \leqslant 2^\omega$, hence $w(\beta Y - Y) \leqslant 2^\omega$. By **[vMM**, 4.1], for each locally compact σ-compact space S and for each closed subspace $A \subset S$, it is true that $\mathrm{cl}_{\beta S} A \cap S^*$ is a P-set of S^*. Hence, by **[W**, 2.11],

$$\{\mathrm{cl}_{\beta Y} D \cap Y^* | D \text{ is nowhere dense in } Y\}$$

consists of nowhere dense closed P-sets of Y^*. By Theorem 1.1 we may find a point which is in none of them; clearly, it is a remote point.

Now assume that every small nonpseudocompact space has a remote point. Let $X = X_{\omega_1}$ (cf. Example 1.2) and let $Z = X \times \omega$. Then

$$|C^*(Z)| \leqslant w(X)^\omega = \omega_2^\omega = \omega_2 \cdot 2^\omega,$$

hence Z is small if CH fails. Since X can be covered by nowhere dense P-sets, $Z = X \times \omega$ has no remote points by **[vDvM]**. □

4.2. REMARK. With a similar proof the reader can easily verify the following fact: CH implies that, if X is small, each nonempty closed G_δ of βX which misses X contains 2^{2^ω} remote points of X. In particular, whenever X is a small noncompact realcompact space, the set of remote points of X is dense in X^*.

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