ON NOWHERE DENSE CLOSED $P$-SETS

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ABSTRACT. We show that no compact space of weight $\omega_1$ can be covered by nowhere dense closed $P$-sets. In addition, we construct a compact space of weight $\omega_2$ which can be covered by nowhere dense closed $P$-sets. As an application, we show that CH is equivalent to the statement that each small nonpseudocompact space has a remote point.

0. Definitions and notation. All spaces considered are completely regular.

As usual we identify a cardinal with an initial ordinal, and an ordinal with the set of smaller ordinals. Ordinals carry the order topology. A cardinal $\kappa$ is regular if $\kappa$ is not the sum of fewer, smaller cardinals.

Let $\kappa$ be any uncountable cardinal. A subset $B$ of a space $X$ is called a $P_\kappa$-set provided that each intersection of fewer than $\kappa$ neighborhoods of $B$ is again a neighborhood of $B$. As usual, a $P_{\omega_1}$-set is simply called a $P$-set. A space $X$ is a $P$-space if each singleton is a $P$-set.

$\beta X$ denotes the Čech-Stone compactification of $X$ and $X^*$ is $\beta X - X$. A point $x$ of $X^*$ is called a remote point of $X$ if $x \notin \text{cl}_{\beta X} A$ for each nowhere dense subset $A$ of $X$.

A $\pi$-base $\mathcal{B}$ for a space $X$ is a family of nonempty open subsets of $X$ such that each nonempty open set in $X$ contains some $B \in \mathcal{B}$. The $\pi$-weight, $\pi(X)$, of $X$ is the least cardinal $\kappa$ for which there is a $\pi$-base for $X$ of cardinality $\kappa$.

$(X_\alpha, f_{\alpha\beta}, \kappa)$ means that $\kappa$ is an ordinal, that for each $\alpha < \kappa$, $X_\alpha$ is a space and that, for each $\beta < \alpha$, $f_{\alpha\beta}$ is a map from $X_\alpha$ into $X_\beta$ such that if $\beta < \alpha < \gamma$ then $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$. The triple $(X_\alpha, f_{\alpha\beta}, \kappa)$ is called an inverse system. The inverse limit $\lim_{\leftarrow} (X_\alpha, f_{\alpha\beta}, \kappa)$ of the inverse system $(X_\alpha, f_{\alpha\beta}, \kappa)$ is the subspace

$$\left\{ x \in \prod_{\alpha < \kappa} X_\alpha \mid \forall \beta < \alpha < \kappa x_\beta = f_{\alpha\beta}(x_\alpha) \right\}$$

of $\prod_{\alpha < \kappa} X_\alpha$. The projection from $\lim_{\leftarrow} (X_\alpha, f_{\alpha\beta}, \kappa)$ into $X_\alpha$ is denoted by $f_{\alpha\alpha}$. An inverse system $(X_\alpha, f_{\alpha\beta}, \kappa)$ is called continuous provided that $X_\beta = \lim_{\leftarrow} (X_\alpha, f_{\alpha\beta}, \kappa)$ for each limit ordinal $\beta < \kappa$.

A space $X$ is called small provided that $|C^*(X)| < 2^\omega$.

1. Introduction. It is well known that a pseudocompact $P$-space is finite [GH]; hence a compact infinite space cannot have too many singletons which are $P$-sets. This leaves open the question whether a compact infinite space

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can have “many” “small” $P$-sets. An appropriate topological translation of “smallness” is nowhere denseness, hence we are interested in nowhere dense closed $P$-sets. We were somewhat surprised to find the following partial answer to the above question.

1.1. **Theorem.** Let $X$ be a compact space of $\pi$-weight $< \kappa$ ($\kappa > \omega$). Then there is an $x \in X$ such that $x \notin K$ for all closed nowhere dense $P_\kappa$-sets $K \subset X$.

Notice that in case $\kappa = \omega_1$, this theorem states that no compact of $\pi$-weight $\omega_1$ can be covered by nowhere dense closed $P$-sets.

This result suggests a host of questions: among others, whether every compact space of weight $\kappa^+$ contains a point which is not in any nowhere dense closed $P_\kappa$-set. We answer this question in the negative.

1.2. **Example.** For each uncountable $\kappa$ there is a compact space $X_\kappa$ of weight $\kappa^+$ such that each point of $X_\kappa$ is contained in some nowhere dense closed $P_\kappa$-set of $X_\kappa$.

As an immediate consequence, CH is equivalent to the statement that no compact space of weight $2^\omega$ can be covered by nowhere dense closed $P$-sets.

We find an application of our results in the construction of remote points.

1.3. **Theorem.** CH is equivalent to the statement that each small nonpseudocompact space has a remote point.

2. **Proof of Theorem 1.1.** We start with a simple lemma.

2.1. **Lemma.** If $X = \lim_{\leftarrow} (X_\alpha, f_{\alpha\beta}, \kappa)$, where

(a) $\kappa$ is regular,

(b) $\pi(X_\alpha) < \kappa$ for each $\alpha < \kappa$,

(c) $(X_\alpha, f_{\alpha\beta}, \kappa)$ is continuous;

then for each closed subset $A$ of $X$ with empty interior there is some $\alpha < \kappa$ such that $f_{\alpha\alpha}[A]$ has empty interior.

**Proof.** Since $(X_\alpha, f_{\alpha\beta}, \kappa)$ is continuous, for each limit ordinal $\alpha < \kappa$ the collection

$$\bigcup_{\beta < \alpha} \{ f_{\alpha\beta}^{-1}[U] \mid U \text{ is open in } X_\beta \}$$

is a base for $X_\alpha$. This implies that for each $\alpha < \kappa$ we may choose a $\pi$-base $\mathfrak{B}_\alpha$ for $X_\alpha$ such that:

(i) $\alpha < \beta \Rightarrow f_{\beta\alpha}^{-1}[\mathfrak{B}_\alpha] \subset \mathfrak{B}_\beta$;

(ii) if $\beta < \kappa$ is a limit ordinal then $\mathfrak{B}_\beta = \bigcup_{\alpha < \beta} f_{\beta\alpha}^{-1}[\mathfrak{B}_\alpha]$;

(iii) if $\beta < \kappa$ then $|\mathfrak{B}_\beta| < \kappa$.

Write $\mathfrak{B}_\alpha = \{ U_\alpha^\gamma \mid \gamma < \alpha' \}$ where $\alpha' < \kappa$. Fix $\alpha$ for awhile. For each $\gamma < \alpha'$ there is some $\gamma(\alpha) < \kappa$ such that $f_{\alpha\alpha}^{-1}[U_\alpha^\gamma] \subset f_{\alpha\alpha}[A]$. Write $\beta_0(\alpha) = \sup_{\gamma < \alpha'} \gamma(\alpha)$. Then $\beta_0(\alpha) < \kappa$ since $\kappa$ is regular. In addition, define $\beta_{n+1}(\alpha) = \beta_0(\beta_n(\alpha))$ for each $n < \omega$.

Write $\beta = \beta_\omega(0) = \sup_{n < \omega} \beta_n(0)$. Then $\beta < \kappa$ since $\kappa$ is regular. We claim that $f_{\alpha\beta}[A]$ has empty interior. For if $f_{\alpha\beta}[A]$ contains a member $V$ of $\mathfrak{B}_{\beta}$, then
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$V = f_{\beta, \kappa}(U)$ for some $U \in \beta, \kappa$ and some $n < \omega$. But then $f_{\beta, \kappa}(U) \subset f_{\kappa, \alpha}(A)$ whence $V \subset f_{\kappa, \alpha}(A)$, a contradiction. \[ \Box \]

2.2. Lemma. If $X$ is a compact space of $\pi$-weight $\kappa$ then there is an irreducible map $f: X \to Y$ where $Y$ has weight $\kappa$.

Proof. Assume $X \subset I^\lambda$, where $I$ is the closed unit interval, and let $\{ F_\alpha: \alpha < \kappa \}$ be a $\pi$-basis for $X$ such that

$$F_\alpha = \bigcap_{i < n_\alpha} \pi_{\alpha, i}^{-1}(U_\alpha^\alpha), \quad \text{where } U_\alpha^\alpha \text{ is open in } I.$$

Let $Y$ be the image of $X$ under the projection onto the coordinates $\{ \alpha_i: \alpha \in \kappa, i < n_\alpha \}$. One sees easily that this $Y$ and this map satisfy our conclusion. \[ \Box \]

2.3. Proof of Theorem 1.1. Assume first that $\kappa$ is regular. Fix an irreducible map $f: X \to Y$ where $Y \subset I^\kappa$. Let $\pi_{\beta, \alpha}: I^\beta \to I^\alpha$ be the projection ($\alpha < \beta < \kappa$) and let $X_\alpha = \pi_{\kappa, \alpha}(Y)$. Also, let $f_{\beta, \alpha} = \pi_{\beta, \alpha} \upharpoonright X_\beta$. Notice that $w(X_\alpha) < \kappa$ for each $\alpha < \kappa$. If $K \subset X$ is a closed $P_\kappa$-set and $\alpha < \kappa$ then

$$K \subset f^{-1} \circ f_{\kappa, \alpha}^{-1} \circ f_{\kappa, \alpha} \circ f[K],$$

and the latter is an intersection of less than $\kappa$ open sets, since $w(X_\alpha) < \kappa$. So

$$K \subset \text{int}_X f^{-1} \circ f_{\kappa, \alpha}^{-1} \circ f_{\kappa, \alpha} \circ f[K].$$

Also, by Lemma 2.1, if $K \subset X$ has empty interior then $f_{\kappa, \alpha} \circ f[K]$ has empty interior in $X_\alpha$ for some $\alpha < \kappa$ (since $f$ is irreducible).

It is thus sufficient to choose $p \in X$ such that for each $\alpha < \kappa$ and each closed nowhere dense $H \subset X_\alpha$ we have that

$$p \notin \text{int}_X f^{-1} \circ f_{\kappa, \alpha}^{-1}[H].$$

If such a choice is impossible, then there are $\alpha_i < \kappa$ ($i < n$) and closed nowhere dense $H_i \subset X_\alpha$ such that

$$X = \bigcup_{i < n} \text{int}_X f^{-1} \circ f_{\kappa, \alpha}^{-1}[H_i].$$

Since a finite union of nowhere dense sets is nowhere dense, we may assume that $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \kappa$. Now, inductively define open sets $U_i \subset X_{\alpha_i}$ so that $U_0 = X_{\alpha_0} - H_0$ and $U_{i+1} = f_{\alpha_i, \alpha_{i+1}}^{-1}[U_i] - H_{i+1}$. Then $f^{-1} \circ f_{\alpha_i, \alpha_{i+1}}^{-1}[U_{n}]$ is nonempty and misses each $f^{-1} \circ f_{\kappa, \alpha}^{-1}[H_i]$, a contradiction.

Now observe that if $\kappa$ is singular, then any $P_\kappa$-set of $X$ is a $P_\kappa$-set; then the theorem for singular $\kappa$ follows from the theorem for regular $\kappa$. \[ \Box \]

3. The example.

3.1. Construction of Example 1.2. Let

$$X_\kappa = \{ f \in (\kappa + 1)^\delta | f \text{ is nondecreasing} \} = \{ f \in (\kappa + 1)^\delta | \forall \alpha < \beta < \kappa: f(\alpha) < f(\beta) \}.$$ 

It is trivial to verify that $X_\kappa$ is compact and that $w(X_\kappa) = \kappa^+$. If $f \in X_\kappa$, either $f(\alpha) = \kappa$ for some $\alpha < \kappa^+$, in which case $f$ is in the nowhere dense closed
$P_{\kappa^+}*\{ g \in X_\kappa \mid g(\alpha) = \kappa \}$, or there is some $\xi < \kappa$ for which $f(\alpha) < \xi$ for each $\alpha < \kappa^+$, in which case $f$ is in the nowhere dense closed $P_{\kappa^+}*\{ g \in X_\kappa \mid g(\alpha) < \xi \text{ for each } \alpha < \kappa^+ \} = \bigcap_{\alpha < \kappa^+} \{ g \in X_\kappa \mid g(\alpha) < \xi \}$ (observe that this intersection is decreasing). \hfill \Box

3.2. Corollary. CH is equivalent to the statement that no compact space of weight $2^\omega$ can be covered by nowhere dense closed $P$-sets.

3.3. Question. Is there, in ZFC, an $x \in \beta_\omega - \omega$ such that $x \not\in K$ for all closed nowhere dense $P$-sets $K$ of $\beta_\omega - \omega$?²

4. Remote points. Let us note that van Douwen [vD] has shown that each nonpseudocompact space of countable $\pi$-weight has a remote point. Not every nonpseudocompact space has a remote point [vDvM] and it is open whether or not every separable space has a remote point [vDvM] (the answer is yes under CH; this follows from a construction in [FG]).

4.1. Proof of Theorem 1.3. Assume CH and let $X$ be any nonpseudocompact small space. Let $Z$ be a nonempty closed $G_\delta$ of $\beta X$ which misses $X$ [GJ, 6.1] and let $Y = \beta X - Z$. Then $Y$ is locally compact and $\sigma$-compact, $X \subset Y$ and $\beta Y = \beta X$ [GJ, 6.7]. It is clear that it suffices to show that $Y$ has a remote point.

Since $X$ is small, $w(\beta X) = w(\beta Y) < 2^\omega$, hence $w(\beta Y - Y) < 2^\omega$. By [vMM, 4.1], for each locally compact $\sigma$-compact space $S$ and for each closed subspace $A \subset S$, it is true that $cl_{\beta S} A \cap S^*$ is a $P$-set of $S^*$. Hence, by [W, 2.11],

\[ \{ cl_{\beta Y} D \cap Y^* \mid D \text{ is nowhere dense in } Y \} \]

consists of nowhere dense closed $P$-sets of $Y^*$. By Theorem 1.1 we may find a point which is in none of them; clearly, it is a remote point.

Now assume that every small nonpseudocompact space has a remote point. Let $X = X_\omega$, (cf. Example 1.2) and let $Z = X \times \omega$. Then

\[ |C^*(Z)| < w(X) = \omega_2 < \omega_2 = 2^\omega, \]

hence $Z$ is small if CH fails. Since $X$ can be covered by nowhere dense $P$-sets, $Z = X \times \omega$ has no remote points by [vDvM]. \hfill \Box

4.2. Remark. With a similar proof the reader can easily verify the following fact: CH implies that, if $X$ is small, each nonempty closed $G_\delta$ of $\beta X$ which misses $X$ contains $2^{2^\omega}$ remote points of $X$. In particular, whenever $X$ is a small noncompact realcompact space, the set of remote points of $X$ is dense in $X^*$.

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²Frankiewicz and Mills, More on nowhere dense closed $P$-sets, have recently shown that Con(ZFC + $\omega^*$ is covered by nowhere dense closed $P$-sets).


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