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AR-MAPS OBTAINED FROM CELL-LIKE MAPS

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ABSTRACT. The recent solution by J. van Mill of a problem of Borsuk involves using a convexification procedure in order to produce a map f from the Hilbert cube Q to a non-AR X so that each point-inverse $f^{-1}(x)$ is a Hilbert cube. A different method of obtaining AR-maps from cell-like maps is described and is used to show that if there is a dimension raising cell-like map, then there is an integer n and a map f from Q to a non-AR X so that each point-inverse $f^{-1}(x)$ is an n-cell or a point.

Introduction. As is well known, Taylor's Example [5] can be used to construct a cell-like map $f: Q \to X$ which is not a shape equivalence (Keesling [2]). This map has the remarkable property that $\sup\{\dim f^{-1}(x): x \in X\} < \infty$. Taylor's Example was recently used by van Mill [4] to construct a map $g: Q \to Y$ onto a non-AR Y so that each point-inverse $g^{-1}(y)$ is an AR. A map which has AR's for point-inverses is called, for convenience, an AR-map.

There is a convexification procedure in [4] which is used to replace a cell-like map with an AR-map but this procedure does not yield control on the dimension of the point-inverses of the AR-map. We present a method for obtaining an AR-map from a cell-like map which, for maps with finite dimensional domains, produces an AR-map with finite dimensional point-inverses. As an application, we prove that if there is a cell-like dimension raising map, then there is an AR-map f from Q onto a non-AR X so that $\sup\{\dim f^{-1}(x): x \in X\} < \infty$. Since it is unknown whether there is a cell-like dimension raising map we have not constructed an AR-map as above. However, we identify a type of "convexification" procedure which would suffice to construct such an example from the Taylor Example (see §3).

1. Preliminaries. Our terminology is standard. A *cell-like map* is a proper map with each point-inverse having trivial shape. The Hilbert cube is denoted by both Q and I^∞. If $f: X \to Y$ is onto, then the *mapping cylinder* $M(f)$ of f is the space which is obtained from $X \times [-1, 1]$ by identifying the set $f^{-1}(y) \times \{-1\}$ to a point for $y \in Y$ and the *double mapping cylinder* $DM(f)$ of f is the space which is obtained from $X \times [-1, 1]$ by identifying each of the sets $f^{-1}(y) \times \{-1\}$ and $f^{-1}(y) \times \{1\}$ to points for $y \in Y$. The natural collapse to the base from $M(f)$ to Y (resp., $DM(f)$ to Y) is denoted $r(f)$ (resp., $Dr(f)$). A proper map $f: X \to Y$ is called a *hereditary shape equivalence* (Kozlowski [3]) if $f|_{f^{-1}(A)}: f^{-1}(A) \to A$ is a shape equivalence for

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each closed subset $A \subset Y$. Absolute retracts and absolute neighborhood retracts are compact.

Our construction heavily relies on the following results due to Kozlowski [3]. Parts (b) and (c) are proved in [3] and part (a) follows from techniques of proof therein; we have included a proof of (a) in the Appendix.

1.1. THEOREM. *Let X be compact and let $f\colon X \to Y$ be a cell-like map. Then*

(a) *f is a hereditary shape equivalence if and only if $Dr(f)\colon DM(f) \to Y$ is a hereditary shape equivalence;*

(b) *if X is an* AR (ANR), *then Y is an* AR (ANR) *if and only if f is a hereditary shape equivalence;*

(c) *if $Z \subset X$ is a closed subset and contains all the nondegenerate point-inverses of f, then f is a hereditary shape equivalence if and only if $f|_Z$ is a hereditary shape equivalence.*

2. The construction. We now present our main result.

2.1. THEOREM. *Let X be compact and let $f\colon X \to Y$ be cell-like. Then there is a compact space Z containing Y and an open retraction $s\colon Z \to Y$ so that*

(1) *s is an* AR-*map;*

(2) *s is a hereditary shape equivalence if and only if f is a hereditary shape equivalence;*

(3) $\sup\{\dim s^{-1}(y)\colon y \in Y\} \leqslant 2 \cdot \dim X + 1$.

PROOF. We choose an embedding $X \subset I^n$ where $n \geqslant 2 \cdot \dim X + 1$ (n can be infinite) and let $\Gamma(f) \subseteq I^n \times Y$ be the graph of f. The space Z is the quotient space obtained from $I^n \times Y$ identifying each set $f^{-1}(y) \times \{y\}$ to a point for $y \in Y$ and the retraction $s\colon Z \to Y$ is the mapping induced by the projection of $I^n \times Y$ to Y. Since each $s^{-1}(y)$ is an n-cell or Hilbert cube with a single cell-like set $(f^{-1}(y) \times \{y\})$ identified to a point, s is an AR-map. The map s satisfies condition (3) and is easily seen to be open.

In order to establish the theorem it suffices to consider the commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\pi_0} & Z \\ \pi_1\downarrow & & \downarrow s \\ DM(f) & \xrightarrow[Dr(f)]{} & Y \end{array}$$

where T is the quotient space obtained from the disjoint union $(I^n \times Y) \cup M(f)$ by identifying $(x, f(x))$ and $(x, 1)$ for each $(x, f(x)) \in \Gamma(f)$, where $\pi_0 | I^n \times Y - \Gamma(f)$ is the identity and $\pi_0 | M(f)$ is the collapse to the base, and where $\pi_1 | M(f) - X \times \{1\}$ is the identity and $\pi_1 | I^n \times Y$ is the projection to Y. Part (c) of Theorem 1.1 can be used to verify that both π_0 and π_1 are hereditary shape equivalences. An easy diagram "chase" establishes that s is a hereditary shape equivalence iff $Dr(f)$ is a hereditary shape equivalence and an appeal to part (a) of Theorem 1.1 completes the proof.

2.2. REMARK. It is an easy matter to choose the embedding $X \subset I^n$ ($n > 5$) so that each point-inverse $s^{-1}(y)$ is an n-cell (or a Hilbert cube for X infinite dimensional); specifically, for n finite choose a 1-LCC embedding of X into the interior of I^n and for n infinite choose a Z-embedding of X. In fact, with these choices the map s is completely regular in the sense of Dyer-Hamstrom [1]. A consequence of this is that the AR-map F in the next corollary can be chosen so that each point-inverse $F^{-1}(s)$ is a point or an n-cell.

2.3. COROLLARY. *If there is a cell-like dimension raising map, then there is an* AR-*map* F: $Q \to S$ *onto a non*-AR S *so that* $\sup\{\dim F^{-1}(s): s \in S\} < \infty$.

PROOF. It is known (cf. Kozlowski [3]) that if there is a cell-like dimension raising map, say f: $X \to Y$, then f is not a hereditary shape equivalence. By Theorem 2.1, there is a compact space Z containing Y and an AR retraction s: $Z \to Y$ so that s is not a hereditary shape equivalence and so that $\sup\{\dim s^{-1}(y): y \in Y\} < \infty$. Let $Z \subset Q$ and let $S = Q \cup_s Y$ be the adjunction space. The induced quotient map F: $Q \to S$ is not a hereditary shape equivalence since s is not. Consequently, by part (b) of Theorem 1.1, S is not an AR.

2.4. REMARK. Starting with a cell-like mapping f: $Q \to Y$ which is not a hereditary shape equivalence, the argument in the proof of Corollary 2.3 yields an AR-map F: $Q \to Y$ onto a non-AR Y. This map and the space Y are similar to those in van Mill [4].

3. A question. The Taylor Example yields a cell-like map f: $X \to Y$ between infinite dimensional compacta which is not a hereditary shape equivalence with $\sup\{\dim f^{-1}(y): y \in Y\} < \infty$. If the following question has an affirmative answer, then Corollary 2.3 holds without the assumption that there exists a cell-like dimension raising map. We have answered the question for finite dimensional X.

3.1. QUESTION. Let f: $X \to Y$ be a cell-like map between compacta. Does there exist a compact space W containing X and an extension F: $W \to Y$ of f such that

(1) F is an AR-map;

(2) F is a hereditary shape equivalence;

(3) if $\sup\{\dim f^{-1}(y): y \in Y\} < \infty$, then $\sup\{\dim F^{-1}(y): y \in Y\} < \infty$.

Starting with such a map F: $W \to Y$, if Z is the quotient space obtained from W by identifying each set $f^{-1}(y)$ to a point for $y \in Y$ and if s: $Z \to Y$ is the map induced by F, then the proof of Theorem 2.1 applies with W taking the place of $I^n \times Y$ and shows that the AR-map s is a hereditary shape equivalence if and only if f is a hereditary shape equivalence.

Appendix. Let f: $X' \to X$ be a map. For a subset $A \subset X$, the inverse set $f^{-1}(A)$ is denoted A' and the mapping cylinder of the restriction of f to A' is considered to be a subset of the mapping cylinder $M(f)$ and is denoted $M(f, A)$. Let $Z \subset X$ be a closed subset and let α: $Z \to Y$ be a surjective map. The *adjunction space of* α, denoted $X \cup_\alpha Y$ is the quotient space obtained from X by identifying each point-inverse $\alpha^{-1}(y)$ to a point; the induced quotient map is denoted π_α: $X \to X \cup_\alpha Y$. The space Y is "naturally" identified with a subspace of $X \cup_\alpha Y$.

PROPOSITION. *Let $f: X' \to X$ be a cell-like map between compact spaces. Then f is a hereditary shape equivalence if and only if $Dr(f)$: $DM(f) \to X$ is a hereditary shape equivalence.*

PROOF. Suppose that f is a hereditary shape equivalence and consider X' to be a subset of an ANR W. Let α: $X' \times \{0, 1\} \to X \times \{0, 1\}$ be defined by $\alpha(x, i) = (f(x), i)$ for $i = 0, 1$. The map α is a hereditary shape equivalence and, therefore, parts (b) and (c) of Theorem 1.1 combine to imply that $(W \times I) \cup_\alpha (X \times \{0, 1\})$ is an ANR. For the same reasons, $W \cup_f X$ is an ANR. The composition of the projection of $W \times I$ to W and the quotient map π_f: $W \to W \cup_f X$ induces a map q: $(W \times I) \cup_\alpha (X \times \{0, 1\}) \to W \cup_f X$. Since q is a cell-like map between ANR's, part (b) of Theorem 1.1 implies that q is a hereditary shape equivalence. The restriction of q to $q^{-1}(X)$ is the map $Dr(f)$: $DM(f) \to X$ and, hence, the latter map is a hereditary shape equivalence.

Using basic results from [3], in order to show that f is a hereditary shape equivalence, it suffices to show that, for any pair of closed subsets $A \subset B$ of X, any map h: $M(f, A) \cup B' \to P$ into an ANR P extends to a map H: $M(f, B) \to P$. Since f is a cell-like map, there is a closed cover $A_0 \supset A, A_1, \dots, A_n$ of B and maps h_i: $M(f, A_i) \to P$ with $h_0 = h$ on $M(f, A)$ and, for $i \geq 1$, $h_i = h$ on A_i'.

Let $H_0 = h_0$ and inductively assume that H_i: $M(f, A_0 \cup \cdots \cup A_i) \to P$ agrees with h on $M(f, A) \cup (A_1' \cup \cdots \cup A_i')$. Let $D = (A_0 \cup \cdots \cup A_i) \cap A_{i+1}$ and let g be the restriction of f to D'. Define a map β: $DM(g) \to P$ by using the restriction of H_i on one copy of the mapping cylinder and the restriction of h_{i+1} on the other copy. By assumption the map $Dr(g)$: $DM(g) \to D$ is a shape equivalence and, therefore, β extends over the mapping cylinder of $Dr(g)$. This extension yields a homotopy rel. D' between the restrictions of H_i and h_{i+1} to $M(f, D)$. Since h_{i+1} extends over $M(f, A_{i+1})$, the map H_i also extends producing the map H_{i+1}. A more detailed argument can be found in [3].

REFERENCES

1. E. Dyer and M. E. Hamstrom, *Completely regular mappings*, Fund. Math. **45** (1957), 103–118.
2. J. Keesling, *A non-movable trivial-shape decomposition of the Hilbert cube*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **23** (1975), 997–998.
3. G. Kozlowski, *Images of* ANR'*s*, Trans. Amer. Math. Soc. (to appear).
4. J. van Mill, *A counterexample to a question of Borsuk on decomposition of absolute retracts* (to appear).
5. L. Taylor, *A counterexample in shape theory*, Bull. Amer. Math. Soc. **81** (1975), 629–632.

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