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A COUNTEREXAMPLE IN ANR THEORY

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There is an AR X and a non-AR Y and a continuous surjection $f: X \to Y$ so that each point-inverse $f^{-1}(y)$ is an AR. This solves a problem of Borsuk.

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AR AR-map CE-map

0. Introduction

Borsuk [2, 12.16] has raised the following well-known problem: If X is a compact ANR and the continuous surjection $f: X \to Y$ has AR's for point-inverses, is Y an ANR? Smale's theorem, [10, p. 604], gives a positive answer to this question under the additional hypothesis that Y is finite dimensional, [2, 12.14]. Interesting partial answer's to Borsuk's question were also obtained by Kozlowski [7]. In this paper we will give a counterexample to Borsuk's problem. Our construction can be adapted to obtain a continuous surjection $f: Q \to Y$ from the Hilbert cube Q onto a non-AR Y so that each point-inverse is a copy of Q embedded in Q as a Z-set (Anderson [1]).

The corresponding problem for cell-like (or CE) maps was solved by Taylor [11]. He produces a CE map $f: X \to Q$ from a compactum X onto Q which is not a shape equivalence and considering X as a subset of Q and taking the adjunction $F: Q \to Q \cup_f Q = Y$ gives a CE map of Q onto a non-AR. Keesling [5] observes that Y is not movable and therefore does not have the shape of any ANR.

In our construction we use Taylor's example and the map F and the space Y described above. We embed Q in a copy Q_0 of Q in a very special way allowing us to find a closed subspace Z of Q_0 containing Q and an extension $f: Z \to Y$ of F so that each point-inverse $f^{-1}(x)$ is the union of two Hilbert cubes intersecting precisely in $F^{-1}(x)$. This is achieved by a general process which we call *convexification*. Then

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we define two decomposition spaces Y_0 and Y_2 of Q_0 and a decomposition space Y_1 of Y_0 with decomposition maps which all have Hilbert cubes or points for point-inverses. Using results of Chapman [4] and Kozlowski [7] we then prove that Y_0 is an AR and that, assuming Y_1 is an AR, the space Y described above has trivial shape, thus contradicting Keesling's result.

Throughout this paper all space are separable metric. Let X be a compact space, let $A \subset X$ be closed and let $f: A \to Y$ be a continuous surjection. The *adjunction space* $X \cup_f Y$ is the space we get from X by identifying each set of the form $f^{-1}(y)$ to a single point. We usually regard Y to be a subspace of $X \cup_f Y$.

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1. Convexification

Let X be a compact space. A closed subset $Y \subset X$ is called a *Z-set* (Anderson [1]) provided that for each $\varepsilon > 0$ there is a continuous function $f_\varepsilon : X \to X - Y$ which moves the points less than ε, i.e. $d(f_\varepsilon, \mathrm{id}) < \varepsilon$.

Q denotes the Hilbert cube $\prod_1^\infty [-1, 1]$ and $\pi_n : Q \to [-1, 1]$ is the projection onto the nth coordinate. A Hilbert cube is a space homeomorphic to Q. If $A \subset Q$, then $C(A)$ denotes the closed convex hull of A, i.e. the intersection of all closed linearly convex subspaces of Q containing A. In addition, a closed subspace $X \subset Q$ is said to be *anti-convex* in Q provided that for any two disjoint closed subsets $A, B \subset X$ the convex hulls $C(A)$ and $C(B)$ of A and B are also disjoint.

The following Lemma is the key to our construction.

1.1. Lemma. *Let X be a compact space. Then there is an embedding $\phi : X \to Q$ so that*
(i) *$\phi(X)$ is a Z-set;*
(ii) *for any two disjoint closed subsets $A, B \subset X$ there is an $n \in \mathbb{N}$ so that $\pi_n\phi(A) = -1$ and $\pi_n\phi(B) = 1$. In particular, $\phi(X)$ is anti-convex in Q.*

Proof. Let $\mathcal{U}$ be a countable open basis for X and let $\mathcal{B} = \{X - U : U \in \mathcal{U}\}$. In addition, let $\mathcal{E}$ be the collection of all finite intersections of elements of $\mathcal{B}$. Since each closed subset A of X is the intersection of all elements of $\mathcal{E}$ containing A and since X is compact, disjoint closed subsets of X are contained in disjoint elements of $\mathcal{E}$. For any pair (E, F) of disjoint elements of $\mathcal{E}$ pick a Urysohn function $f : X \to [-1, 1]$ so that $f(E) = -1$ and $f(F) = 1$. Since $\mathcal{E}$ is countable we have chosen only countably many functions. Let $\{f_n : n \geq 2\}$ denote the set of all those functions and let $f_1 : X \to [-1, 1]$ be the function with constant value 1. Define $\phi : X \to Q$ by $\phi(x)_n = f_n(x)$ $(n \in \mathbb{N})$. It is clear that ϕ is as required. □

The following lemma summarizes some relevant information on anti-convex subspaces of Q.

1.2. Lemma. *Let $X \subset Q$ be anti-convex and let $A \subset X$ be closed. Then*
(i) $C(A) \cap X = A$;
(ii) *X is a Z-set in $C(X)$* if *X is nondegenerate*;
(iii) *if X is infinite then $C(X)$ is infinite-dimensional, in particular, $C(X) \approx Q$.*

Proof. (i) is trivial. For (ii), let $\varepsilon > 0$. Let $F \subset X$ be finite so that the nearest point retraction $r: C(X) \to C(F)$ moves the points less than $\frac{1}{2}\varepsilon$. By an easy application of Radon's [8] Theorem that any subset $G \subset \mathbb{R}^n$ of $n+2$ points admits a Radon partition, i.e. a partition $\{A, B\}$ of G so that $C(A) \cap C(B) \neq \emptyset$, it follows that $C(F)$ is homeomorphic to a simplex σ under a homeomorphism mapping F onto the vertices of σ. Therefore, there is a map $f: C(F) \to C(F) - F$ so that $d(f, \mathrm{id}) < \frac{1}{2}\varepsilon$. Then $g = f \circ r$ moves the points less than ε while moreover $g(C(X)) \cap X = \emptyset$, since, by (i), $C(F) \cap X = F$. For (iii), notice that if $F \subset X$ has cardinality $n+1$, then $C(F)$ has dimension n. Hence $C(X)$ contains sets of dimension n for all $n \geqslant 1$. We conclude thet $C(X)$ is infinite dimensional and that $C(X) \approx Q$ now follows from Keller [6]. □

We include the following Lemma for completeness sake; it is well-known and the proof is straightforward.

1.3. Lemma. *For each $n \in \mathbb{N}$ let $A_n \subset Q$ be closed so that $A_{n+1} \subset A_n$. Then $C(\bigcap_1^\infty A_n) = \bigcap_1^\infty C(A_n)$.*

Proof. Clearly $C(\bigcap_1^\infty A_n) \subset \bigcap_1^\infty C(A_n)$. Suppose that there is a point

$$x \in \bigcap_1^\infty C(A_n) - C\left(\bigcap_1^\infty A_n\right).$$

By the Hahn–Banach Theorem (see Rudin [9]) there are closed and convex sets C, $D \subset Q$ so that $x \in C - D$, $C(\bigcap_1^\infty A_n) \subset D - C$ and $D \cup C = Q$. The compactness of Q implies that $A_n \subset D - C$ for some $n \in \mathbb{N}$. Consequently,

$$x \in \bigcap_1^\infty C(A_n) \subset C(A_n) \subset D,$$

which is a contradiction. □

The following Proposition shows why we think of our process as convexification.

1.4. Proposition. *Let $X \subset Q$ be anti-convex and let $f: X \to Y$ be a continuous surjection. If $Z = \bigcup\{C(f^{-1}(y)) : y \in Y\}$ then*
(i) *Z is closed in Q*;
(ii) *the function $F: Z \to Y$ defined by $F(z) = y$ if $z \in C(f^{-1}(y))$ is continuous.*

Proof. Both statements follow from the following Fact.

Fact. If $A \subset Y$ is closed, then $\bigcup\{C(f^{-1}(y)) : y \in A\}$ is closed.

For convenience put $Z_0 = \bigcup\{C(f^{-1}(y)) : y \in A\}$. Let $z_n \in Z_0$ $(n \in \mathbb{N})$ be a sequence converging to some point $z \in Q$. For each $n \in \mathbb{N}$ let $y_n \in A$ be the unique point for which $z_n \in C(f^{-1}(y_n))$. Some subsequence $\{y_{n_k} : k \in \mathbb{N}\}$ of $\{y_n : n \in \mathbb{N}\}$ converges to a point $y \in A$. We claim that $z \in C(f^{-1}(y))$. Suppose that this is not true. By Lemma 1.3 there is a closed neighborhood V of y so that $z \notin C(f^{-1}(V))$. There is an $m \in \mathbb{N}$ so that $\{y_{n_k} : k \geq m\} \subset V$. Consequently,

$$\bigcup\{C(f^{-1}(y_{n_k})) : k \geq m\} \subset C(f^{-1}(V)) \subset Q - \{z\}.$$

Since the sequence $\{z_{n_k} : k \geq m\}$ converges to z and since $C(f^{-1}(V))$ is closed, we have derived a contradiction. We conclude that $z \in C(f^{-1}(y)) \subset Z_0$. □

Notice that the extension F of f in the above Proposition is CE. In fact, each point-inverse $F^{-1}(y)$ is an AR, being closed and convex in Q.

It is interesting that any map can be "convexified". That any map can be made CE is known. If $f : X \to Y$ is continuous, then the mapping cylinder $M(f)$ of f can be regarded to be an extension of X (we identify X and $X \times \{0\}$) and the natural collapse to base is then an extension of f with contractible point-inverses.

We are now prepared to construct the Example.

2. The construction

We will only consider compact AR's and compact ANR's. A compactum X has *trivial shape* if X has the shape of a point (i.e. when X is embedded in an ANR, it can be contracted to a point within any of its neighborhoods). The map $f : X \to Y$ between compacta X and Y is *cell-like* (or CE) if f is onto and each point-inverse $f^{-1}(x)$ has trivial shape. For more information on shape theory see Borsuk [3].

We will make use of the following results.

2.1. Theorem (a) (Chapman [4]). *If $A \subset Q$ is a Z-set of trivial shape then the space we obtain from Q by identifying A to a point is homeomorphic to Q;*

(b) (Kozlowski [7]). *If X and Y are* ANR's *and if $f : X \to Y$ is* CE *then $f^{-1}(A)$ has trivial shape for any closed subset $A \subset Y$ of trivial shape;*

(c) (Kozlowski [7]). *If the continuous surjection $f : Q \to X$ has the property that each point-inverse $f^{-1}(x)$ is linearly convex, then X is an AR.*

(Both Chapman and Kozlowski prove far more general results than stated here, but since we only need the above special cases, we did not bother to state their results in full generality.)

Let Q_0 and Q_1 be copies of Q so that $Q_2 = Q_0 \cap Q_1$ is a Hilbert cube which is an anti-convex Z-set both in Q_0 and Q_1 (Lemma 1.1). As is well-known, the Homeomorphism Extension Theorem implies that $Q_0 \cup Q_1 \approx Q$ (Anderson [1]). Let C_i $(i = 0, 1)$ denote the convex closure operator in Q_i. In addition, let $f: Q_2 \to X$ be a CE map so that X does not have trivial shape (see the introduction). Without loss of generality we assume that each point-inverse $f^{-1}(x)$ of f is infinite (the map described in the introduction does not have this property, however by replacing Q by $Q \times [-1, 1]$ and defining $g: Q \times [-1, 1] \to X$ by $g = f \circ \pi$ (here π is the projection onto the first coordinate) we find a CE map from Q onto X so that each point-inverse is infinite). Define

$$Z_i = \bigcup\{C_i(f^{-1}(x)) : x \in X\}$$

and $F_i: Z_i \to X$ by $F_i(z) = x$ if $z \in C_i(f^{-1}(x))$ $(i = 0, 1)$. By Lemma 1.2 (iii) each point-inverse of F_i is a Hilbert cube (notice that a closed subset of an anti-convex subspace of Q is anti-convex). In addition, by Proposition 1.4, Z_i is closed and F_i is continuous.

Step 1. Let $Y_0 = (Q_0 \cup Q_1) \cup_{F_0} X$.
Notice that Y_0 is obtained from $Q_0 \cup Q_1$ by identifying each $C_0(f^{-1}(x))$ to a point. Let π_0: $Q_0 \cup Q_1 \to Y_0$ be the quotient map. Notice that each point-inverse $\pi_0^{-1}(y)$ is either a point or a Hilbert cube. Also, $\pi_0 | Q_2 = f$, in particular, $\pi_0(Q_2) = X$. By Theorem 2.1 (c), $\pi_0(Q_1)$ is an AR. By Whitehead's Theorem (see [2, 9.17]) this implies that Y_0 is an AR.

Let $x \in X$ be arbitrarily chosen and consider $C_1(f^{-1}(x))$. Since, by Lemma 1.2 (ii), $f^{-1}(x)$ is a Z-set in $C_1(f^{-1}(x))$ and since $f^{-1}(x)$ has trivial shape, we have that

$$\pi_0(C_1(f^{-1}(x))) = C_1(f^{-1}(x))/f^{-1}(x) \approx Q \qquad \text{(Theorem 2.1(a))}.$$

Define $G: \pi_0(Z_1) \to X$ by $G(\pi_0(z)) = x$ if $z \in C_1(f^{-1}(x))$. By the remark above, each point-inverse $G^{-1}(x)$ of G is a Hilbert cube. Clearly G is continuous.

Step 2. Let $Y_1 = Y_0 \cup_G X$.
Notice that Y_1 is obtained from Y_0 by identifying each $C_1(f^{-1}(x))/f^{-1}(x)$ to a point. Let $\pi_1: Y_0 \to Y_1$ be the quotient map. Notice that each point-inverse $\pi_1^{-1}(y)$ is either a point or a Hilbert cube. Also, $\pi_1 | X = \text{id}$, in particular, $\pi_1(X) = X$.

Step 3. Let $Y_2 = Q_1 \cup_{F_1} X$.
Notice that Y_2 is obtained from Q_1 by identifying each $C_1(f^{-1}(x))$ to a point. Let $\pi_2: Q_1 \to Y_2$ be the quotient map. It is clear that $Y_2 = \pi_1\pi_0(Q_1)$ and that $\pi_2 = \pi_1\pi_0 | Q_1$. By Theorem 2.1 (c), Y_2 is an AR.

Claim. Y_1 is not an ANR.
Suppose, to the contrary, that Y_1 is an ANR. Since $\pi_1: Y_0 \to Y_1$ is a CE map between ANR's, and since Y_2 is an AR, by Theorem 2.1 (b), $\pi_1^{-1}(Y_2)$ has trivial shape.

Put $Y_3 = Q_1 \cup_f X$. Since $Y_3 = \pi_1^{-1}(Y_2)$ the space Y_3 has trivial shape. Let $r: Q_1 \to Q_2$ be a retraction. Define $s: Y_3 \to X$ by

$$\begin{cases} s(x) = x & (x \in X), \\ s(y) = f(r(y)) & (y \notin X). \end{cases}$$

It is clear that s is a retraction. Since Y_3 has trivial shape and since X is a retract of Y_3 it follows that X has trivial shape (Borsuk [3]). We have derived a contradiction. This clearly solves Borsuk's problem.

2.2. Remark. It can be shown that Y_1 is not movable and therefore does not have the shape of any ANR.

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