SHORTER NOTES

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elegant and polished character, for which there is no other outlet.

\[ \beta \omega - \omega \text{ IS NOT FIRST ORDER HOMOGENEOUS} \]

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ABSTRACT. We find a first order property shared by some but not all point of
\( \beta \omega - \omega \).

Our result. Throughout, cardinals carry the discrete topology, and \( X^* \) denotes
\( \beta X - X \). The purpose of this note is to point out the following consequence of
known results.

THEOREM. Some but not all points \( x \) of \( \omega^* \) have the following property.
\( \mathcal{P} \): There is a closed subspace \( Y \) of \( \omega^* \) which is extremally disconnected and has \( x \)
as its cluster point.

Our property is simpler than previously known properties shared by some, but
not by all, points, the simplest of which is
\( \mathcal{Q} \): there is a countable \( A \subseteq \omega^* \) with \( x \in \overline{A} - A \),
see [K]; see also [F] and [R]. The reason that \( \mathcal{P} \) is simpler is that it can be
formulated in a much simpler language: in order to formulate \( \mathcal{Q} \) one needs the
notion “countable” (or an infinitely long expression), while \( \mathcal{P} \) can be formulated
with an expression of finite length which only uses the notion of a closed subset.
(Since \( \omega^* \) is a \( T_1 \)-space we can discuss \( x \in \omega^* \) by talking about \( \{x\} \).) Properties of
this sort are called first order; see [HJRT] for a more accurate description. To see
that our property \( \mathcal{P} \) is first order, note that for closed \( Y \subseteq \omega^* \) and for \( x \in \omega^* \).

\( Y \) is extremally disconnected iff \( \forall \) closed \( F, G \subseteq \omega^* \exists \) closed \( F', G' \subseteq \omega^* \) such that
\[ \left[ F \cup G = Y \Rightarrow ((F' \cup G' = Y) \land (F' \subseteq F) \land (G' \subseteq G) \land (F' \cap G' = \emptyset)) \right] \]
and \( x \) is a cluster point of \( Y \) iff \( \forall \) closed \( F \subseteq \omega^*[F \cup \{x\} = Y \Rightarrow F = Y] \).

The fact that not all points of \( \omega^* \) have the same first order properties answers a
question of Hensel, Jockusch, Rubel and Takeuti, [HJRT, §10, Q9]. Actually they
ask if every two points of \( \omega^* \) have the same first order properties in \( \beta \omega \) (this looks

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like a slip of the pen). Since for closed $Y \subseteq \beta\omega$ one has $Y \subseteq \omega^*$ iff $Y$ contains no isolated points of $\beta\omega$, which is a first order property of $Y$ in $\beta\omega$, our theorem implies that the answer is no.

**Remarks.** (a) If not all points of a space $X$ have the same first order properties then certainly $X$ is not homogeneous. We mention without proof that the converse is false, even for zero-dimensional compact spaces.

(b) Let $X$ denote e.g. the rationals or the irrationals. It was shown in [vD, 6.6] that some but not all points $x$ of $X^*$ have a property similar to $\mathcal{P}$, namely
\[
\mathcal{P}': \text{there are disjoint open sets } U \text{ and } V \text{ in } X^* \text{ with } x \in \overline{U} \cap \overline{V}.
\]
We leave it to the reader to verify that $\mathcal{P}'$ is first order.

**The proof.** Since the extremely disconnected space $\beta\omega$ can be embedded into $\omega^*$ [GJ, 6.10(a)] some points of $\omega^*$ satisfy $\mathcal{P}$. (In fact $\mathcal{Q}$ implies $\mathcal{P}$ since every separable subspace of $\beta\omega$ is extremely disconnected by [GJ, 9H.1 and 6.M2]. It is shown in [vM] that $\mathcal{Q}$ does not imply $\mathcal{Q}$.)

For the proof that not every point of $\omega^*$ has $\mathcal{P}$ we need Kunen’s $\kappa$-OK-points: a point $p$ of a space is called a $\kappa$-OK-point if for every sequence $\langle U_n \rangle_{n<\omega}$ of neighborhoods of $p$ there is a $\kappa$-sequence $\langle V_a \rangle_{a<\kappa}$ of neighborhoods of $p$ such that for all $n<\omega$ and $F \subseteq \kappa$, if $|F| = n+1$, then $\bigcap_{a \in F} V_a \subseteq U_n$. Kunen proved the important result that $\omega^*$ has a $\kappa$-OK-point, where $\kappa = 2^\omega$, [K]. Let $x$ be a $\kappa$-OK-point of $\omega^*$, and suppose there is a closed extremely disconnected subspace $Y$ of $\omega^*$ which has $x$ as its cluster point. Clearly $x$ is a $\kappa$-OK-point of $Y$. But $x$ is not a $P$-point of $Y$, for if $X$ is an extremely disconnected space with $|X|$ not Ulam-measurable, then no nonisolated point of $X$ is a $P$-point of $X$, [GJ, 12H.5]. It follows that $Y$ has a disjoint open family of cardinality $\kappa$, since, more generally, if $X$ is regular then $X$ has a disjoint open family of cardinality $\kappa$ if it has a $\kappa$-OK-point that is not a $P$-point, by [K, Proof of 1.4]. Since $Y$ is compact and extremely disconnected it follows that $\beta\mathcal{C}$ embeds into $Y$, hence into $\beta\omega$. This is absurd, since $|\beta\mathcal{C}| = 2^{2^\omega}$ for each $\kappa > \omega$, [GJ, 9.2].

**References**


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