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SHORTER NOTES

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$\beta\omega - \omega$ IS NOT FIRST ORDER HOMOGENEOUS

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ABSTRACT. We find a first order property shared by some but not all point of $\beta\omega - \omega$.

Our result. Throughout, cardinals carry the discrete topology, and X^* denotes $\beta X - X$. The purpose of this note is to point out the following consequence of known results.

THEOREM. *Some but not all points x of ω^* have the following property.*

$\mathcal{P}$: *There is a closed subspace Y of ω^* which is extremally disconnected and has x as its cluster point.*

Our property is simpler than previously known properties shared by some, but not by all, points, the simplest of which is

$\mathcal{Q}$: there is a countable $A \subseteq \omega^*$ with $x \in \bar{A} - A$,

see [K]; see also [F] and [R]. The reason that $\mathcal{P}$ is simpler is that it can be formulated in a much simpler language: in order to formulate $\mathcal{Q}$ one needs the notion "countable" (or an infinitely long expression), while $\mathcal{P}$ can be formulated with an expression of finite length which only uses the notion of a closed subset. (Since ω^* is a T_1-space we can discuss $x \in \omega^*$ by talking about $\{x\}$.) Properties of this sort are called *first order*; see [HJRT] for a more accurate description. To see that our property $\mathcal{P}$ is first order, note that for closed $Y \subseteq \omega^*$ and for $x \in \omega^*$.

Y is extremally disconnected iff $\forall$ closed $F, G \subseteq \omega^*$ $\exists$ closed $F', G' \subseteq \omega^*$ such that

$$[F \cup G = Y \Rightarrow ((F' \cup G' = Y) \wedge (F' \subseteq F) \wedge (G' \subseteq G) \wedge (F' \cap G' = \varnothing))]$$

and x is a cluster point of Y iff $\forall$ closed $F \subseteq \omega^*[F \cup \{x\} = Y \Rightarrow F = Y]$.

The fact that not all points of ω^* have the same first order properties answers a question of Hensel, Jockusch, Rubel and Takeuti, [HJRT, §10, Q9]. Actually they ask if every two points of ω^* have the same first order properties in $\beta\omega$ (this looks

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like a slip of the pen). Since for closed $Y \subseteq \beta\omega$ one has $Y \subseteq \omega^*$ iff Y contains no isolated points of $\beta\omega$, which is a first order property of Y in $\beta\omega$, our theorem implies that the answer is no.

REMARKS. (a) If not all points of a space X have the same first order properties then certainly X is not homogeneous. We mention without proof that the converse is false, even for zero-dimensional compact spaces.

(b) Let X denote e.g. the rationals or the irrationals. It was shown in [**vD**, 6.6] that some but not all points x of X^* have a property similar to $\mathcal{P}$, namely

$\mathcal{P}'$: there are disjoint open sets U and V in X^* with $x \in \overline{U} \cap \overline{V}$.

We leave it to the reader to verify that $\mathcal{P}'$ is first order.

The proof. Since the extremally disconnected space $\beta\omega$ can be embedded into ω^* [**GJ**, 6.10(a)] some points of ω^* satisfy $\mathcal{P}$. (In fact $\mathcal{Q}$ implies $\mathcal{P}$ since every separable subspace of $\beta\omega$ is extremally disconnected by [**GJ**, 9H.1 and 6.M2]. It is shown in [**vM**] that $\mathcal{P}$ does not imply $\mathcal{Q}$.)

For the proof that not every point of ω^* has $\mathcal{P}$ we need Kunen's κ-*OK*-points: a point p of a space is called a κ-*OK*-point if for every sequence $\langle U_n \rangle_{n<\omega}$ of neighborhoods of p there is a κ-sequence $\langle V_\alpha \rangle_{\alpha<\kappa}$ of neighborhoods of p such that for all $n < \omega$ and $F \subset \kappa$, if $|F| = n + 1$, then $\bigcap_{\alpha \in F} V_\alpha \subseteq U_n$. Kunen proved the important result that ω^* has a $\mathfrak{c}$-*OK*-point, where $\mathfrak{c} = 2^\omega$, [**K**]. Let x be a $\mathfrak{c}$-*OK*-point of ω^*, and suppose there is a closed extremally disconnected subspace Y of ω^* which has x as its cluster point. Clearly x is a $\mathfrak{c}$-*OK*-point of Y. But x is not a P-point of Y, for if X is an extremally disconnected space with $|X|$ not Ulam-measurable, then no nonisolated point of X is a P-point of X, [**GJ**, 12H.5]. It follows that Y has a disjoint open family of cardinality $\mathfrak{c}$, since, more generally, if X is regular then X has a disjoint open family of cardinality κ if it has a κ-*OK*-point that is not a P-point, by [**K**, Proof of 1.4]. Since Y is compact and extremally disconnected it follows that $\beta\mathfrak{c}$ embeds into Y, hence into $\beta\omega$. This is absurd, since $|\beta\kappa| = 2^{2^\kappa}$ for each $\kappa \geqslant \omega$, [**GJ**, 9.2].

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