COUNTABLY COMPACT SPACES ALL COUNTABLE SUBSETS
OF WHICH ARE SCATTERED
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Abstract: We give several examples of countably compact dense in itself spaces in which all countable subsets are scattered, thus answering a problem raised by M. G. Tkachenko in [5].

Key words: countably compact, scattered, F-space.

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0. Introduction. It is well-known, and easy to prove, that every compact dense in itself space \( X \) contains a countable dense in itself subset. Simply construct a closed subset of \( X \) which admits an irreducible map, say \( f \), onto the Cantor set and then proceed as follows. Choose a countable dense set \( \{ d_n : n < \omega \} \) of the Cantor set and pick, for each \( n < \omega \), a point \( x_n \in f^{-1}(d_n) \).

Then \( \{ x_n : n < \omega \} \) is a countable dense in itself subset of \( X \).

In view of this result the following question, due to M.G. Tkachenko [5] is quite natural. Does every countably compact space which is dense in itself and regular contain a countable dense in itself subspace? In this note we will answer this question in the negative. In fact, we will give several counterexamples, one of which is of \( \tau \)-weight \( 0 \), and one of which satisfies the countable chain condition.

All topological spaces under discussion are Tychonoff.
1. A Theorem. An $F$-space is a space in which cozero-sets are $C_0$-embedded. It is easy to show that a normal space $X$ is an $F$-space iff for any two $P_2$-subsets $A, B \subseteq X$ such that $A \cap B \neq \emptyset = \overline{B} \cap \overline{A}$ we have that $A \cap \overline{B} \neq \emptyset$. This result will be used frequently without explicit reference throughout the remaining part of this note. Observe that among familiar examples of $F$-spaces are the extremally disconnected spaces and all spaces of the form $\beta X - X$, where $X$ is any locally compact and $\sigma$-compact space, $[5,14,27]$.

A point $x$ of a space $X$ is said to be a weak $P$-point provided that $x \notin \overline{F}$ for any countable $F \subseteq X - \{x\}$.

1.1. Theorem: Let $X$ be a compact $F$-space with the property that it contains a dense set of weak $P$-points. Then $X$ contains a dense countably compact subset $C$ such that all countable subsets of $C$ are scattered.

Proof: For each $\alpha < \omega_1$ we will construct a subset $P_\alpha \subseteq X$ and for each $x \in P_\alpha - \bigcup_{\beta < \alpha} P_\beta$ a countable set $H(x,\alpha) \subseteq \bigcup_{\beta < \alpha} P_\beta$ such that

1. If $E \subseteq \bigcup_{\beta < \alpha} P_\beta$ is countably infinite, then $E$ has a limit point in $P_\alpha$,

2. If $x \in P_\alpha - \bigcup_{\beta < \alpha} P_\beta$ and if $x \notin \overline{F}$, where $F \subseteq X - \{x\}$ is countable, then $F \cap H(x,\alpha) \neq \emptyset$.

Put $P_0 = \emptyset$ and $P_1 = \{x \in X : x$ is a weak $P$-point$\}$ and let $H(x,1) \subseteq \emptyset$ for all $x \in P_1$. Now suppose that we have constructed for each $\beta \times \alpha < \omega_1$ the sets $P_\beta$ and for each $x \in P_\beta \subseteq \bigcup_{\gamma < \beta} P_\gamma$ the set $H(x,\beta)$. Define

$$ E = \{E \subseteq \bigcup_{\beta < \alpha} P_\beta : E$ is countably infinite and discrete$\}. $$
Take \( E \in E \) arbitrarily. Since \( X \) is a compact \( F \)-space and \( E \) is discrete, \( \bar{E} \cap \bigcap_{\alpha \in \mathbb{A}} P_\alpha \) is closed, \( \mathbb{A} \in [3, 14] \). Consequently, by a result of Kunen [4], we can find a point \( x_\varepsilon \in \bar{E} \) which is a weak \( P \)-point of \( \bar{E} \). Define

\[
P_\alpha = \bigcup_{\beta < \alpha} P_\beta \cup \{ x_\varepsilon : E \in E \}.
\]

Take \( x \in P_\alpha \) arbitrarily. Choose an \( E(x) \in E \) such that \( x = x_{E(x)} \) and, for each \( y \in E(x) \), let \( \gamma(y) = \min \{ \beta < \alpha : y \in P_\beta \} \). Define

\[
H(x, \alpha) = E(x) \cup \bigcup_{y \in E(x)} H(y, \gamma(y)).
\]

We claim that our inductive hypotheses are satisfied. For this we only need to check (2).

So let \( x \in P_\alpha \) and take a countable \( F \subset X \) with \( x \in \bar{F} \). We obviously may assume that \( F \cap E(x) = \emptyset \) and also, since \( x \) is a weak \( P \)-point of \( \bar{E} \cap \bigcap_{\alpha \in \mathbb{A}} P_\alpha \), that \( F \cap \bar{E} \cap E(x) = \emptyset \). Now if \( \bar{F} \cap E(x) = \emptyset \) then, since \( X \) is an \( F \)-space, \( \bar{F} \cap \bar{E} \cap E(x) = \emptyset \), which is a contradiction since \( x \in \bar{F} \cap \bar{E} \cap E(x) \). Therefore, \( \bar{F} \cap E(x) \neq \emptyset \) and we get what we wanted because of the definition of \( H(x, \alpha) \) and our inductive assumptions. This completes the induction.

Put \( D = \bigcup_{\alpha \in \mathbb{A}} P_\alpha \). Then \( D \) is clearly countably compact and dense in \( X \). It remains to be shown that all countable subsets of \( D \) are scattered which will follow if we show that every countable subset of \( D \) has an isolated point. Let \( F \subset D \) be countable and define

\[
\alpha = \min \{ \beta < \omega_1 : F \cap P_\beta \neq \emptyset \}.
\]

Take \( x \in P_\alpha \cap F \). If \( x \notin \bar{F} \) then \( \bar{F} \cap \bigcap_{\alpha \in \mathbb{A}} P_\alpha \neq \emptyset \) and since

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\( H(x, a) \in U_{\delta \alpha} \beta \); this contradicts the minimality of \( a \). Therefore, \( x \) is an isolated point of \( F \).

2. Examples: As was remarked in the proof of Theorem 1.1, Kunen [4] has shown that \( Bu \cdot w \) contains a dense set of weak P-points. Since \( Bu \cdot w \) has no isolated points, in view of Theorem 1.1 this gives us our first example.

It is natural to ask whether under MA one could actually find a dense in itself countably compact subspace of \( Bu \cdot w \) with the property that all subsets of cardinality less than \( 2^\omega \) are scattered. This we do not know, however the next example shows that this will not be satisfied automatically. Let \( X = (\omega_1 + 1)^{\omega} \). It is easily seen that \( X \) is a compact nowhere ccc dense in its own space of weight \( \omega_1 \). Hence the projective cover \( EX \) of \( X \) is a compact nowhere ccc F-space (in fact, extremally disconnected) without isolated points. Clearly, \( EX \) has \( \tau \)-weight \( \omega_1 \). By [2, 3.1], every nowhere ccc compact F-space contains a dense set of weak P-points. Therefore, \( EX \) contains a dense set \( D \) which is countably compact and which has the property that all of its countable subsets are scattered (Theorem 1.1). Since \( D \) has also \( \tau \)-weight \( \omega_1 \), \( D \) has a dense in itself subspace of size \( \omega_1 \).

We can obtain other interesting examples in the following way. Dow [1] proved that the projective cover \( E \) of the Cantor cube of weight \( (2^\omega)^+ \) contains a dense set of weak P-points. Applying Theorem 1.1 again gives us a countably compact, dense in itself ccc space all countable subsets of which are scattered.

The following interesting problem remains open: does there exist a cardinal \( \kappa \) such that every dense in itself regular countably compact space has a dense in itself subspace of size \( \kappa \)? C.F. Mills claims to have constructed a consistent example of a sequentially compact 0-dimensional space which is dense in
itself and which has the additional property that every subspace of size $2^{\aleph_0}$ is scattered. Thus such a $\kappa$ must be greater than $2^{\aleph_0}$.

References:


(Oblatum 26.5.1981)