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COUNTABLY COMPACT SPACES ALL COUNTABLE SUBSETS OF WHICH ARE SCATTERED

I. JUHÁSZ, J. van MILL

Abstract: We give several examples of countably compact dense in itself spaces in which all countable subsets are scattered, thus answering a problem raised by M. G. Tkačenko in [5].

Key words: countably compact, scattered, F-space.

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0. *Introduction.* It is well-known, and easy to prove, that every compact dense in itself space X contains a countable dense in itself subset. Simply construct a closed subset of X which admits an irreducible map, say f, onto the Cantor set and then proceed as follows. Choose a countable dense set $\{d_n: n<\omega\}$ of the Cantor set and pick, for each $n<\omega$, a point $x_n \in f^{-1}(d_n)$. Then $\{x_n: n<\omega\}$ is a countable dense in itself subset of X.

In view of this result the following question, due to M.G. Tkačenko [5] is quite natural. *Does every countably compact space which is dense in itself and regular contain a countable dense in itself subspace ?* In this note we will answer this question in the negative. In fact, we will give several counterexamples, one of which is of π-weight ω_1 and one of which satisfies the countable chain condition.

All topological spaces under discussion are Tychonoff.

1. *A Theorem.* An F-space is a space in which cozero-sets are C^*-embedded. It is easy to show that a normal space X is an F-space iff for any two F_σ-subsets $A,B \subset X$ such that $\bar{A} \cap B = \emptyset = \bar{B} \cap A$ we have that $\bar{A} \cap \bar{B} = \emptyset$. This reresult will be used frequently without explicit reference throughout the remaining part of this note. Observe that among familiar examples of F-spaces are the extremally disconnected spaces and all spaces of the form $\beta X-X$, where X is any locally compact and σ-compact space, [3,14.27].

A point x of a space X is said to be a *weak* P-*point* provided that $x \notin \bar{F}$ for any countable $F \subset X-\{x\}$.

1.1. THEOREM: *Let* X *be a compact* F-*space with the property that it contains a dense set of weak* P-*points. Then* X *contains a dense countably compact subset* C *such that all countable subsets of* C *are scattered.*

PROOF: For each $\alpha < \omega_1$ we will construct a subset $P_\alpha \subset X$ and for each $x \in P_\alpha - \bigcup_{\beta<\alpha} P_\beta$ a countable set $H(x,\alpha) \subset \bigcup_{\beta<\alpha} P_\beta$ such that

(1) if $E \subset \bigcup_{\beta<\alpha} P_\beta$ is countably infinite, then E has a limit point in P_α,

(2) if $x \in P_\alpha - \bigcup_{\beta<\alpha} P_\beta$ and if $x \in \bar{F}$, where $F \subset X-\{x\}$ is countable, then $F \cap H(x,\alpha) \neq \emptyset$.

Put $P_0 = \emptyset$ and $P_1 = \{x \in X\colon x \text{ is a weak P-point}\}$ and let $H(x,1) = \emptyset$ for all $x \in P_1$. Now suppose that we have constructed for each $\beta < \alpha < \omega_1$ the sets P_β and for each $x \in P_\beta - \bigcup_{\gamma<\beta} P_\gamma$ the set $H(x,\beta)$. Define

$$\mathcal{E} = \{E \subset \bigcup_{\beta<\alpha} P_\beta\colon E \text{ is countably infinite and discrete}\}.$$

Take $E \in \mathcal{E}$ arbitrarily. Since X is a compact F-space and E is discrete, $\bar{E} \approx \beta E \approx \beta\omega$, [3,14N]. Consequently, by a result of Kunen [4], we can find a point $x_E \in \bar{E}-E$ which is a weak P-point of $\bar{E}-E$. Define

$$P_\alpha = \cup_{\beta<\alpha} P_\beta \cup \{x_E : E \in \mathcal{E}\}.$$

Take $x \in P_\alpha - \cup_{\beta<\alpha} P_\beta$ arbitrarily. Choose an $E(x) \in \mathcal{E}$ such that $x = x_{E(x)}$ and, for each $y \in E(x)$, let $\gamma(y) = \min\{\beta < \alpha : y \in P_\beta\}$. Define

$$H(x,\alpha) = E(x) \cup \cup_{y\in E(x)} H(y,\gamma(y)).$$

We claim that our inductive hypotheses are satisfied. For this we only need to check (2).

So let $x \in P_\alpha - \cup_{\beta<\alpha} P_\beta$ and take a countable $F \subset X-\{x\}$ with $x \in \bar{F}$. We obviously may assume that $F \cap E(x) = \emptyset$ and also, since x is a weak P-point of $\overline{E(x)}-E(x)$, that $F \cap (\overline{E(x)}-E(x)) = \emptyset$. Now if $\bar{F} \cap E(x) = \emptyset$ then, since X is an F-space, $\bar{F} \cap \overline{E(x)} = \emptyset$, which is a contradiction since $x \in \bar{F} \cap \overline{E(x)}$. Therefore, $\bar{F} \cap E(x) \neq \emptyset$ and we get what we want because of the definition of $H(x,\alpha)$ and our inductive assumptions. This completes the induction.

Put $D = \cup_{\alpha<\omega_1} P_\alpha$. Then D is clearly countably compact and dense in X. It remains to be shown that all countable subsets of D are scattered which will follow if we show that every countable subset of D has an isolated point. Let $F \subset D$ be countable and define

$$\alpha = \min\{\beta < \omega_1 : F \cap P_\beta \neq \emptyset\}.$$

Take $x \in P_\alpha \cap F$. If $x \in \overline{F-\{x\}}$ then $(F-\{x\}) \cap H(x,\alpha) \neq \emptyset$ and since

$H(x,\alpha) \subset \bigcup_{\beta<\alpha} P_\beta$, this contradicts the minimality of α. Therefore, x is an isolated point of F.$\square$

2. *Examples:* As was remarked in the proof of Theorem 1.1, Kunen [4] has shown that $\beta\omega-\omega$ contains a dense set of weak P-points. Since $\beta\omega-\omega$ has no isolated points, in view of Theorem 1.1 this gives us our first example.

It is natural to ask whether under MA one could actually find a dense in itself countably compact subspace of $\beta\omega-\omega$ with the property that all subsets of cardinality less than 2^ω are scattered. This we do not know, however the next example shows that this will not be satisfied automatically. Let $X = (\omega_1 + 1)^\omega$. It is easily seen that X is a compact nowhere ccc dense in itself space of weight ω_1. Hence the projective cover EX of X is a compact nowhere ccc F-space (in fact, extremally disconnected) without isolated points. Clearly, EX has π-weight ω_1. By [2,3.1], every nowhere ccc compact F-space contains a dense set of weak P-points. Therefore, EX contains a dense set D which is countably compact and which has the property that all of its countable subsets are scattered (Theorem 1.1). Since D has also π-weight ω_1, D has a dense in itself subspace of size ω_1.

We can obtain other interesting examples in the following way. Dow [1] proved that the projective cover E of the Cantor cube of weight $(2^\omega)^+$ contains a dense set of weak P-points. Applying Theorem 1.1 again gives us a countably compact, dense in itself ccc space all countable subsets of which are scattered.

The following interesting problem remains open: *does there exist a cardinal κ such that every dense in itself regular countably compact space has a dense in itself subspace of size κ ?* C.F. Mills claims to have constructed a consistent example of a sequentially compact 0-dimensional space which is dense in itself and which has the additional property that every subspace of size $\le 2^{\omega}$ is scattered. Thus such a κ must be greater that 2^{ω}.

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Mathematical Institute of the Hungarian Academy of Sciences,
1053 Budapest V., Realtanoda u. 13-15,
Hungary.

Subfaculteit Wiskunde,
Vrije Universiteit,
De Boelelaan 1081,
Amsterdam,
The Netherlands.

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