SELECTIONS AND ORDERABILITY

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ABSTRACT. Let $X$ be a compact Hausdorff space. Then $X$ has a selection if and only if $X$ is orderable.

0. Introduction. Let $X$ be a compact Hausdorff space and let $2^X$ denote the hyperspace of nonempty closed subsets of $X$. A selection for $X$ is a continuous map $F: 2^X \to X$ such that $F(A) \in A$ for all $A \in 2^X$. Let $X(2)$ denote the 2-fold symmetric product of $X$, i.e. the subspace of $2^X$ consisting of all nonempty closed subspaces of $X$ containing at most two points. A weak selection for $X$ is a continuous map $s: X(2) \to X$ such that $s(A) \in A$ for all $A \in X(2)$. It is easy to see that $X$ has a weak selection if and only if there is a continuous map $s: X^2 \to X$ such that for all $x, y \in X$,

\begin{enumerate}
  \item $s(x, y) = s(y, x)$, and
  \item $s(x, y) \in \{x, y\}$.
\end{enumerate}

Such a map $s: X^2 \to X$ will also be called a weak selection.

Michael [M] showed that for a continuum $X$ the following statements are equivalent: (a) $X$ has a selection, (b) $X$ has a weak selection, and (c) $X$ is orderable. In [Y], Young claims, without giving a proof, that statements (a), (b), and (c) are also equivalent for compact zero-dimensional spaces $X$. In this paper we will show that, for compacta, statements (a), (b), and (c) are always equivalent.

1. The construction. Let $X$ be compact and let $s: X^2 \to X$ be a weak selection. For each $x \in X$ define

$$B_x = \{ y \in X \mid s(y, x) = y \},$$

and

$$A_x = \{ y \in X \mid s(y, x) = x \}.$$

Observe that both $A_x$ and $B_x$ are closed, that $A_x \cup B_x = X$ and that $A_x \cap B_x = \{x\}$.

1.1. Theorem. Let $X$ be a compact space. Then the following statements are equivalent:

(a) $X$ is orderable,

(b) $X$ has a weak selection,

(c) $X$ has a selection.
The implication (c) $\Rightarrow$ (b) is trivial and the implication (a) $\Rightarrow$ (c) is well known. Indeed, simply define $F: 2^X \to X$ by $F(A) = \min(A)$. An easy check shows that $F$ is a selection. It therefore suffices to prove that (b) $\Rightarrow$ (a). To this end, let $s: X^2 \to X$ be a weak selection for $X$ and, for each $x \in X$, let $A_x$ and $B_x$ be defined as above. Let $\prec$ be a wellordering on $X$. For every $x \in X$ we will construct closed sets $L_x$, $U_x \subseteq X$ such that

1. $L_x \cup U_x = X$ and $L_x \cap U_x = \{x\}$,
2. if $y \prec x$ and if $x \in L_y$ then $L_x \subseteq L_y \setminus \{y\}$,
3. if $y \prec x$ and if $x \in U_y$ then $U_x \subseteq U_y \setminus \{y\}$,
4. if $z \in L_x$ and if $z \not\in \cup \{L_y \mid y \prec x \land x \in U_y\}$ then $z \in B_x$,
5. if $z \in U_x$ and if $z \not\in \cup \{U_y \mid y \prec x \land x \in L_y\}$ then $z \in A_x$.

(In the total ordering on $X$ which we will construct in this proof, $L_x$ will be the set of all points smaller than or equal to $x$, and $U_x$ will be the set of all points larger than or equal to $x$.)

Let $x_0$ be the first element of $X$ and define $L_{x_0} = B_{x_0}$ and $U_{x_0} = A_{x_0}$. Assume that we have defined $L_y$ and $U_y$ for all $y \prec x$ satisfying (1) through (5). Let $E = \{y \prec x \mid x \not\in L_y\}$ and $F = \{y \prec x \mid x \not\in U_y\}$. Put

$$Z = X \setminus \left( \bigcup_{y \in E} L_y \cup \bigcup_{y \in F} U_y \right).$$

Let $\kappa = |E|$ and for each $\xi \leq \kappa$ define points $y_\xi \in E$ in the following way:

6. $y_0 = \min(E),
7. y_\xi = \min(\{x\} \cup \{y \in E \mid (\forall \mu < \xi \exists L_y)\}) \cup \{y \in E \mid (\forall \mu < \xi \exists L_y)\})$. Let $\xi < \kappa$ be the first ordinal for which $y_\xi = x$.

**Claim 1.** If $\xi_0 < \xi$ then $\cup \{L_y \mid y \in E \land y < y_{\xi_0}\} = \cup_{\mu < \xi_0} L_y$.

Take $y \in \{z \in E \mid z < y_{\xi_0}\} \setminus \{y_{\mu_0} \mid \mu < \xi_0\}$ and let $\mu < \xi_0$ be the first ordinal for which $y < y_{\mu}$. Since $y_{\rho} < y$ for all $\rho < \mu$ (notice that $\mu \neq 0$) and since $y \neq y_{\mu}$, by (7), $y \in \cup_{\rho < \mu} L_{y_\rho}$. Choose $\rho < \mu$ such that $y \in L_{y_\rho}$. Since $y_{\rho} < y$, by (2),

$$L_y \subseteq L_{y_\rho} \subseteq \bigcup_{\delta < \xi_0} L_{y_\delta}.$$  

**Claim 2.** If $\mu_0 < \mu_1 < \xi$ then $L_{y_{\mu_0}} \subseteq L_{y_{\mu_1}} \setminus \{y_{\mu_0}\}$.

By (7), $y_{\mu_1} \not\in L_{y_{\mu_0}}$. Consequently, $y_{\mu_1} \in U_{y_{\mu_0}}$ and therefore, by (3), $U_{y_{\mu_1}} \subseteq U_{y_{\mu_0}} \setminus \{y_{\mu_0}\}$. Consequently, by (1), $L_{y_{\mu_0}} \subseteq L_{y_{\mu_1}} \setminus \{y_{\mu_1}\}$.

**Claim 3.** If $\mu_0 < \mu_1 < \xi$ then $L_{y_{\mu_0}} \setminus L_{y_{\mu_1}} \subseteq A_{y_{\mu_0}}$.

Take $t \in L_{y_{\mu_1}} \setminus L_{y_{\mu_0}}$. Since $t \in U_{y_{\mu_0}}$ and, by (5),

$$U_{y_{\mu_0}} \subseteq \bigcup \{U_y \mid y < y_{\mu_0} \land y_{\mu_0} \in L_y \} \cup A_{y_{\mu_0}},$$

we may assume, without loss of generality that $t \in U_z$ for certain $z < y_{\mu_0}$ with $y_{\mu_0} \in L_z$; we will reach a contradiction. Assume that $y_{\mu_1} \in L_z$. Since $y_{\mu_0} < y_{\mu_1}$ and since $z < y_{\mu_0}$ this implies by (2), that $L_{y_{\mu_1}} \subseteq L_z \setminus \{z\}$. Consequently, $t \in L_z \setminus \{z\}$ and $t \in U_z$, contradicting (1). This shows that $y_{\mu_1} \not\in L_z$ which implies that $y_{\mu_1} \in U_z$. Since $z < y_{\mu_1}$ by (3), $U_{y_{\mu_1}} \subseteq U_z$ and therefore $x \in U_z$. If also $x \in L_z$ then $x = z$ which is impossible since $z < x$. We conclude that $x \not\in L_z$ or equivalently,
$z \in E$. Let $\varepsilon \leqslant \mu_0$ be the smallest ordinal such that $z \leqslant y_\varepsilon$. Since $y_\delta < z$ for every $\delta < \varepsilon$ by (7), either $z = y_\varepsilon$ or $z \in L_\gamma$ for certain $\delta < \varepsilon$. If $z = y_\varepsilon$ then $y_{\mu_0} \in L_\gamma$, which contradicts $z < y_{\mu_0}$ (Claim 2). Therefore, $z \in L_{\gamma_0}$ for certain $\delta < \varepsilon$. Then $z \in L_{\gamma_0} \subseteq L_{\gamma_0} \setminus \{y_{\mu_0}\}$. Since $z < y_{\mu_0}$ and since $y_{\mu_0} \in L_\gamma$, by (2), we also have that

$$L_{\gamma_0} \subseteq L_\gamma \setminus \{z\},$$

which implies that $z \in L_{\gamma_0} \subseteq L_\gamma \setminus \{z\}$, a contradiction.

Claim 4. If $t \in \text{Cl}_X(\bigcup_{y \in E} L_y) \setminus \bigcup_{y \in E} L_y$ then $t$ is a cluster point of the net \( \{y_\mu \mid \mu < \xi\} \).

Suppose not and take a closed neighborhood $C$ of $t$ which misses

$$\text{Cl}_X(\{y_\mu \mid \mu < \xi\}).$$

From Claim 1 it is clear that there is a cofinal subset $G \subseteq \xi$ with the property that for each $\mu \in G$ there exists a point $c_\mu \in C \cap L_{\gamma_0}$ such that

$$\mu = \min \{\delta < \xi \mid c_\mu \in L_{\gamma_0}\}.$$

Take $\mu \in G$. We claim that $c_\mu \in B_{\gamma_0}$. If not, then by (4) there is a $y < y_\mu$ such that $c_\mu \in L_y$ and $y \in U_y$. Since $y < y_\mu$ and $y \in U_y$, by (3), $U_y \subseteq U_y \setminus \{y\}$ which implies that $L_y \subseteq L_{\gamma_0}$. Consequently, $x \notin L_{\gamma_0}$, since $x \notin L_{\gamma_0}$, or equivalently, $y \in E$. By Claim 1 we can find $\delta < \mu$ such that $c_\mu \in L_{\gamma_0}$, which is a contradiction since

$$\mu = \min \{\delta < \xi \mid c_\mu \in L_{\gamma_0}\}.$$ This implies that for all $\mu \in G$ we have that $s(c_\mu, y_\mu) = c_\mu$.

Let $(c, y)$ be a cluster point of the net \( \{(c_\mu, y_\mu)\}_{\mu \in G} \). Then $c \in C$ and $y \notin C$, and since $s(c_\mu, y_\mu) = c_\mu \in C$ for all $\mu \in G$ it is clear that $s(c, y) = c$. Next take $\mu \in G$ arbitrarily. For all $\delta > \mu$ we have by Claim 3 that $s(y_\mu, c_\delta) = y_\mu$. Hence $s(c, y_\mu) = s(y_\mu, c) = y_\mu$. This would imply that $s(c, y) = y$, and since $y \neq c$ this is a contradiction.

Claim 5. If both $t$ and $u$ are cluster points of the net \( \{y_\mu \mid \mu < \xi\} \) then $t = u$.

Let $C$ and $D$ be closed and disjoint neighborhoods of, respectively, $t$ and $u$. There is clearly a cofinal subset $G \subseteq \xi$ and for each $\mu \in G$ points

$$c_\mu \in C \cap \{y_\lambda \mid \lambda < \xi\} \quad \text{and} \quad d_\mu \in D \cap \{y_\lambda \mid \lambda < \xi\}$$

such that if $\mu, \delta \in G$ and $\mu < \delta$ then

$$c_\mu < d_\mu < c_\delta.$$

Let $(t', u')$ be a cluster point of the net \( \{(c_\mu, d_\mu)\}_{\mu \in G} \), then $t' \in C$ and $u' \in D$. By Claim 3, $s(c_\mu, d_\mu) = c_\mu$ and consequently, $s(u', t') = t'$. Fix $\mu \in G$. For each $\delta > \mu$ it is clear that $s(d_\mu, c_\delta) = d_\mu$ (Claim 3). Since $t' \in \text{Cl}_X(\{c_\delta \mid \delta > \mu\})$ this implies that

$$s(d_\mu, t') = d_\mu.$$

Since $(u', t') \in \text{Cl}_X(\{(d_\mu, t') \mid \mu \in G\})$ this implies that $s(u', t') = u'$. Since $u' \neq t'$, this is a contradiction.

Claim 6. $\bigcup_{y \in E} L_y$ has at most one boundary point.

Follows immediately from Claims 4 and 5.

Claim 7. If $t \in Z$ and $\mu < \xi$ then $t \in A_{\gamma_0}$. 
Since \( t \notin L_{y_\xi} \), clearly \( t \in U_{y_{\xi}} \). Therefore by (5), if \( t \notin A_y \) then \( t \in U_y \) for certain \( y<y_\mu \) with \( y_\mu \in L_y \). If \( x \in L_y \) then \( x \notin A_y \) since \( x \neq y \) in which case \( Z \cap U_y = \emptyset \) which contradicts \( t \in Z \cap U_y \). Therefore \( y \in E \). By Claim 1

\[
\bigcup_{\delta<\mu} \{ L_y \mid y \in E \& y<y_\mu \} = \bigcup_{\delta<\mu} L_{y_\delta}.
\]

Therefore \( y_\mu \in L_{y_\delta} \) for certain \( \delta<\mu \) which contradicts (7).

Formally we have to consider two cases, namely that \( \xi \) is a successor or that \( \xi \) is a limit ordinal. Those two cases can be treated analogously and since the case that \( \xi \) is a limit is more complicated we will assume from now on that \( \xi \) is a limit.

Since \( L_{y_\xi} \setminus \{ y_\mu \} \) is open for each \( \mu<\xi \), by Claims 1 and 2, \( \bigcup_{y \in E} L_y \) must have a limit point, say \( a \), and by Claim 6 we see that \( a \) is unique. By using precisely the same technique as above and again restricting our attention to the limit case we can find a limit ordinal \( \eta \) and for each \( \mu<\eta \) a point \( z_\mu \in F \) such that

(8) if \( \mu<\delta \) then \( U_{z_\mu} \subset U_{z_\delta} \)

(9) \( \bigcup_{\mu<\eta} U_{z_\mu} = \bigcup_{y \in E} U_y \) and

(10) if \( t \in Z \) and \( \mu<\eta \) then \( t \in B_{z_\mu} \).

Again we find that \( \bigcup_{y \in E} U_y \) has a unique boundary point, say \( b \), and that this point is a cluster point of the net \( \{ z_\mu \mid \mu<\eta \} \).

(Note that, by (1), (2) and (3), \( y \in E \) and \( y' \in F \) implies that \( L_y \cap U_{y'} = \emptyset \).)

Case 1. \( a = b \). We then claim that \( Z = \{ x \} = \{ a \} = \{ b \} \). For assume that \( t \in Z \). By Claim 7, \( s(y_\mu, t) = y_\mu \) for all \( \mu<\xi \) and consequently \( s(a, t) = a \) since \( a \) is a limit point of \( \{ y_\mu \} \). On the other hand, by (10), \( s(t, z_\mu) = t \) for all \( \mu<\eta \). By the same argument \( s(t, a) = s(t, b) = t \). Hence \( t = a \).

We therefore conclude that \( a = b = x \) and that \( Z = \{ x \} \). Now define

\[
L_x = \bigcup_{y \in E} L_y \cup \{ x \} \quad \text{and} \quad U_x = \bigcup_{y \in F} U_y \cup \{ x \}.
\]

An easy check shows that our inductive hypotheses are satisfied.

Case 2. \( a \neq b \) and \( x \notin \{ a, b \} \). Define \( L_x = \bigcup_{y \in E} L_y \cup (Z \cap B_x) \) and \( U_x = \bigcup_{y \in F} U_y \cup (Z \cap A_x) \). Observe that both \( L_x \) and \( U_x \) are closed since \( a \in Z \cap B_x \) and \( b \in Z \cap A_x \). Again an easy check shows that our inductive hypotheses are satisfied.

Case 3. \( x = a \) and \( a \neq b \). Define \( L_x = \bigcup_{y \in E} L_y \cup \{ x \} \) and \( U_x = \bigcup_{\mu<\xi} U_{y_\mu} \).

Case 4. \( x = b \) and \( a \neq b \). Similar to Case 3.

Now define \( x \leq y \) iff \( x \in L_y \). Then \( \leq \) is a linear order which generates the topology of \( X \) since \( X \) is compact and since for each \( x \in X \) the sets \( \{ y \in X \mid y \leq x \} \) and \( \{ y \in X \mid x < y \} \) are both closed. It is easily seen that whenever \( X \) is a KOTS then the function \( s: X^2 \to X \) defined by \( s(x, y) = \min \{ x, y \} \) is a weak selection. This suggests the following question:
Question. Let $X$ be a space. Is $X$ a KOTS if and only if $X$ admits a weak selection?

The technique used in the proof of our theorem is not applicable to answer this question since certain transfinite sequences of points need not have limit points.

References


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