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SUPERCOMPACT SPACES

Eric van DOUWEN
Institute for Medicine and Mathematics, Ohio University, Athens, OH 45701, USA

Jan van MILL
Wiskundig Seminarium, Vrije Universiteit, Amsterdam, The Netherlands

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We prove the following theorem: Let Y be a Hausdorff space which is the continuous image of a supercompact Hausdorff space, and let K be a countably infinite subset of Y. Then (a) at least one cluster point of K is the limit of a nontrivial convergent sequence in Y (not necessarily in K), and (b) at most countably many cluster points of K are not the limit of some nontrivial sequence in Y. This theorem implies that spaces like $\beta\mathbb{N}$ and $\beta\mathbb{N}\backslash\mathbb{N}$ are not supercompact. Moreover we will give an example of a separable first countable compact Hausdorff space which is not supercompact.

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supercompact network $\beta\mathbb{N}$ Cantor tree

1. Introduction

A family $\mathscr{S}$ of subsets of a space X is a subbase for the closed subsets of X, *or a closed subbase for short,* if

$$\{\bigcap\{\bigcup\mathscr{F}:\mathscr{F}\in\mathscr{C}\}:\ \mathscr{C}\text{ a collection of finite subfamilies of }\mathscr{S}\}$$

is precisely the family of closed subsets of X. By Alexander's subbase lemma a space is compact if and only if it has a closed subbase every *centered* (=any finite subfamily has nonempty intersection) subfamily of which has nonempty intersection. In [15], de Groot defined a space to be *supercompact* if it has a closed subbase every *linked* (=any subfamily with at most two members has nonempty intersection) subfamily of which has nonempty intersection. (Such a closed subbase will be called *binary.*)

Examples of supercompact spaces are compact linearly orderable spaces (easy), compact tree-like spaces, [5, 16], and compact metrizable spaces, [24] (this is far from trivial), see also [6] and [19]. Every space, whether compact or not, has many "natural" supercompact extensions, called *superextensions,* see Verbeek's mono-

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graph [25]. Also, it is easy to see that a product of supercompact spaces is supercompact.

De Groot raised the question of whether all compact Hausdorff spaces are supercompact, [15]. (An easy example of a non-Hausdorff compact T_1-space which is not supercompact was constructed by Verbeek, [25, II.2.2(8)].) This question was answered in the negative by Bell, [1], who showed that if βX is supercompact, then X is pseudocompact. Consequently a space like $\beta\mathbb{N}$ is not supercompact. Our first Theorem implies Bell's results, and also implies that a space like $\beta\mathbb{N}\setminus\mathbb{N}$ is not supercompact.

1.1. Theorem. *Let Y be a Hausdorff space which is a continuous image of a supercompact Hausdorff space, and let K be a countably infinite subset of Y. Then*

(a) *at least one cluster point of K is the limit of nontrivial convergent sequence in Y (not necessarily in K), and*

(b) *at most countably many cluster points of K are not the limit of some nontrivial convergent sequence in Y.*

1.2. Corollary. *$\beta\mathbb{N}$, and $\beta\mathbb{N}\setminus\mathbb{N}$, and $\beta\mathbb{R}\setminus\mathbb{R}$, or, more generally, an infinite compact Hausdorff F-space,* [14], *or, yet more generally, an infinite compact Hausdorff space in which no sequence converges cannot be a continuous image of a supercompact Hausdorff space.* □

1.3. Corollary. *If βX is the continuous image of a supercompact Hausdorff space, then X is pseudocompact.*

Examples of compact Hausdorff spaces which are not supercompact, obtained from Theorem 1.1, are not first countable, and have cardinality at least $2^{\mathfrak{c}}$. This suggests two questions: are first countable compact Hausdorff spaces supercompact? and: are "small" compact Hausdorff spaces supercompact? These questions are answered in the negative by the following examples.

1.4. Example. *There is a separable first countable compact Hausdorff space which is not supercompact.*

1.5. Example. *There is a separable compact Hausdorff space with ω_1 points which is not supercompact.*

Example 1.5 also answers another natural question in the negative. As mentioned above, compact metrizable spaces, i.e. compact Hausdorff spaces with countable weight, are supercompact, [24]. This suggests the question whether one can show that compact Hausdorff spaces with weight less than $\mathfrak{c}$ are supercompact, without using the Continuum Hypothesis of course.

Examples 1.4 and 1.5 are also of interest because they are quite close to being supercompact: In both examples the subspace consisting of the non-isolated points is supercompact, and both examples have a closed subbase $\mathcal{S}$ such that if $\mathcal{A} \subset \mathcal{S}$ is any subfamily such that $A_1 \cap A_2 \cap A_3 \neq \emptyset$ for any (not necessarily distinct) A_1, A_2, $A_3 \in \mathcal{A}$, then $\bigcap \mathcal{A} \neq \emptyset$. See [4] for more information about this type of weakening of supercompactness.

Theorem 1.1. also suggests some questions we cannot answer.

1.6. Question. Let Y be a Hausdorff continuous image of a supercompact Hausdorff space (or just a supercompact Hausdorff space). If K is a countable subset of Y, then is *every* cluster point of K the limit of a nontrivial convergent sequence in Y? Equivalently, is a point of Y the limit of a nontrivial convergent sequence iff it is a cluster point of a countable subset of Y?

1.7. Question. In an earlier version of this paper we asked if there is a non-supercompact Hausdorff space which is a continuous image of a supercompact Hausdorff space. This question has been answered affirmatively in [21]. However, two special cases remain open. We don't know whether or not a retract of a supercompact Hausdorff space is again supercompact, and in fact we don't even know if the factors of a supercompact product are supercompact. It also is unknown whether or not dyadic spaces are supercompact. (See (B) in Section 4 for a partial answer.)

In this connection we mention that Theorem 1.1 is valid under the (formally weaker) assumption that Y is a continuous image of a closed neighborhood retract of a supercompact Hausdorff space, and that Examples 1.4 and 1.5 cannot be embedded as a neighborhood retract in a supercompact Hausdorff space.

We frequently use the following facts without explicit reference. The easy proofs are omitted.

(1) A space has a binary closed subbase iff it has a binary closed subbase which is closed under arbitrary intersection.

(2) Let $\mathcal{S}$ be a closed subbase which is closed under finite intersection, for a compact space X. If $F \subset X$ is closed, $U \subset X$ is open and $F \subset U$, then there is a finite $\mathcal{A} \subset \mathcal{S}$ such that $F \subset \bigcup \mathcal{A} \subset U$. In particular each *clopen* (=closed and open) subset of X is the union of some finite subcollection of $\mathcal{S}$.

2. Theorem 1.1: proof and consequences

We will derive Theorem 1.1 from a more technical result. We first need a definition.

2.1. Definition. If T is a subspace of Y, a family $\mathcal{A}$ of subsets of Y is called a network for T in Y, if for each $p \in T$ and each neighborhood U of p in Y there is an $A \in \mathscr{A}$ with $p \in A \subset U$. (So if $T = Y$, then $\mathscr{A}$ simply is a network for Y.)

2.2. Lemma. *Let Y be a Hausdorff space which is a continuous image of a supercompact Hausdorff space. If K is any countably infinite subset of Y, then the subspace*

$$E = \{y \in Y: y \in \mathrm{Cl}_Y(K\setminus\{y\}), \text{ and no nontrivial sequence in } Y \text{ converges to } y\}$$

of Y has a countable network in Y.

Before we prove the Lemma, we show how to prove Theorem 1.1 from it. (Note that conversely Lemma 2.2 is a trivial consequence of part (b) of Theorem 1.1.)

2.3. *Proof of the Theorem from the Lemma.* Let Y and K be as in the Theorem, let E be as in the Lemma. We first show that E is countable. Let $\mathscr{A}$ be a countable network for E in K. In order to show that E is countable it suffices to find for each $p \in E$ a finite $\mathscr{F}_p \subset \mathscr{A}$ with $\bigcap \mathscr{F}_p = \{p\}$, since $\mathscr{A}$ has only countably many finite subfamilies.

Let $p \in E$ be arbitrary. List $\{A \in \mathscr{A}: p \in A\}$ as $\{A_n: n \in \omega\}$. We claim that $\bigcap_{i \leq n} A_i = \{p\}$ for some $n \in \omega$. For assume not. Then we can pick for each $n \in \omega$ an $a_n \in (\bigcap_{i \leq n} A_i)\setminus\{p\}$. Since each neighborhood of p in Y contains some A_n, it follows that the sequence $\langle a_n \rangle_{n \in \omega}$ converges to p. Since $a_n \neq p$ for all n, this contradicts $p \in E$.

We next show that (a) holds. Suppose not. Then $\mathrm{Cl}_Y K = K \cup E$, hence $\mathrm{Cl}_Y K$ is countable. But each compact countable Hausdorff space is metrizable, hence each cluster point of K is the limit of a nontrivial sequence of points in K. Contradiction. □

2.4. *Proof of the Lemma.* Let X be a supercompact Hausdorff space which admits a continuous map, f say, onto Y. Let $\mathscr{S}$ be a binary closed subbase of X which is closed under intersection. For $A \subset X$ define $I(A) \subset X$ by

$$I(A) = \bigcap\{S \in \mathscr{S}: A \subset S\}.$$

Note that $\mathrm{Cl}_X A \subset I(A)$, since sets of $\mathscr{S}$ are closed, that $I(I(A)) = I(A)$, and that $I(A) \subset I(B)$ if $A \subset B$, for all $A, B \subset X$.

Fact 1. Let $p \in X$. If U is a neighborhood of p and if A is a subset of X with $p \in \mathrm{Cl}_X A$, then there is a subset B of A with $p \in \mathrm{Cl}_X B$ and $I(B) \subset U$.

Proof of Fact 1. Since X is regular, p has a neighborhood V such that $\mathrm{Cl}_X V \subset U$. There is a finite $\mathscr{F} \subset \mathscr{S}$ with $\mathrm{Cl}_X V \subset \bigcup \mathscr{F} \subset U$. Now $\mathscr{F}$ is finite, and $A \cap V \subset \bigcup \mathscr{F}$, and $p \in \mathrm{Cl}_X (A \cap V)$, hence there is an $S \in \mathscr{F}$ with $p \in \mathrm{Cl}_X(A \cap V \cap S)$. Let $B = A \cap V \cap S$. Then $p \in \mathrm{Cl}_X B$, and $B \subset A$, and $I(B) \subset S \subset \bigcup \mathscr{F} \subset U$. □

Choose any countable subset J of X such that $f[J]=K$. Since J has only countably many finite subsets, the family

$$\mathscr{A}=\{f[I(F)]\colon F \text{ is a finite subset of } J\}$$

is countable. We claim that it is a network for E in Y.

Let $y\in E$ be arbitrary, and let U be any neighborhood of y in Y, and let $J^*=J\backslash f^{\leftarrow}[\{y\}]$.

Since f is a closed map (Y is Hausdorff), and $f[J^*]=K\backslash\{y\}$, and $y\in \mathrm{Cl}_Y(K\backslash\{y\})$, there is an $x\in\mathrm{Cl}_X J^*$ with $f(x)=y$. Then Fact 1 implies that there is a $B\subset J^*$ such that $x\in\mathrm{Cl}_X B$ and $I(B)\subset f^{\leftarrow}[U]$. We will show that there is a finite $F\subset B$ such that $y=f(x)\in f[I(F)]$. Since y and U are arbitrary, and $f[I(F)]\subset f[I(B)]\subset U$, it would follow that $\mathscr{A}$ is a network for E in Y.

Enumerate B as $\{b_k\colon k\in\omega\}$, and for each $n\in\omega$ define Z_n and T_n by

$$Z_n=\Big[\bigcap_{k\leqslant n} I(\{x,b_k\})\Big]\cap I(\{b_k\colon k\leqslant n\}),$$

$$T_n=\Big[\bigcap_{k\leqslant n} I(\{x,b_k\})\Big]\cap I(B).$$

The existence of F is an easy consequence of the following

Claim. There is an n_0 such that $f[Z_n]=\{y\}$ for all $n\geqslant n_0$.

Indeed, just put $F=\{b_k\colon k\leqslant n_0\}$. Before we proceed to the proof of the claim, we prove one more fact.

Fact 2. $\bigcap_{b\in B} I(\{x,b\})=\{x\}$.

Proof of Fact 2. Evidently $x\in I(\{x,b\})$ for all $b\in B$. Let $t\in X\backslash\{x\}$ be arbitrary. By Fact 1 there is a $C\subset B$ such that $x\in\mathrm{Cl}_X C$ and $I(C)\subset X\backslash\{t\}$. Choose any $b\in C$. Then $t\notin I(\{x,b\})$, since $\{x,b\}\subset \mathrm{Cl}_X C\subset I(C)$, which implies that $I(\{x,b\})\subset I(I(C))=I(C)$. □

Proof of Claim. Since $x\in\mathrm{Cl}_X B\subset I(B)$, it follows from Fact 2 that $\bigcap_{n\in\omega} T_n=\{x\}$. But $Z_n\subset T_n$ for each $n\in\omega$, and $\{T_n\colon n\in\omega\}$ is a decreasing collection of closed sets in a compact space, hence

(∗) If V is any neighborhood of x in X, then there is an m_0 such that $Z_k\subset V$ for all $k\geqslant m_0$.

Now assume the claim to be false. Then for each $k\in\omega$ there is a $z(k)\geqslant k$ with $f[Z_{z(k)}]\neq\{p\}$. But $Z_n\neq\emptyset$ for all $n\in\omega$ since $\mathscr{S}$ is binary. (This is the only point in the proof where we use the fact that $\mathscr{S}$ is binary.) Consequently we can choose for each $k\in\omega$ a $y_k\in f[Z_{z(k)}]\backslash\{y\}$. Then $\langle y_k\rangle_{k\in\omega}$ converges to y. Indeed, let U be any neighborhood of $y=f(x)$. Then there is an m_0 such that $Z_k \subset f^{\leftarrow}[U]$ for all $k \geqslant m_0$. Since $z(k) \geqslant k$ for all k, it follows that $y_k \in U$ for all $k \geqslant m_0$. Since $y_k \neq y$ for all $k \in \omega$, this contradicts $y \in E$. □

This completes the proof of the Lemma.

2.5. *Proof of Corollary* 1.3. We give two proofs.

First proof. If X is not pseudocompact, then $\beta\mathbb{N}$ can be embedded in βX as a neighborhood retract, [7]. It easily follows that if βX were a continuous image of (a closed neighborhood retract of) a supercompact Hausdorff space, then $\beta\mathbb{N}$ would be a neighborhood retract of a supercompact Hausdorff space. This contradicts Theorem 1.1.

Second proof. Since X is not pseudocompact there is a countably infinite relatively discrete C-embedded set D in X. Then $\bar{D}$ is homeomorphic to $\beta\mathbb{N}$, hence has no nontrivial convergent sequences; also, then $\bar{D} \cap \bar{A} = \emptyset$ for every countable $A \subset \beta X$ such that $A \cap \bar{D} = \emptyset = \bar{A} \cap D$, [13, p. 706], hence there is no countable $A \subseteq \beta X - \bar{D}$ such that $\bar{A} - A \subseteq \bar{D} - D$. It follows that no cluster point of D is the limit of a nontrivial convergent sequence.

Corollary 1.3 generalizes the fact that X is pseudocompact if βX is dyadic. (Recall that a dyadic space is a Hausdorff continuous image of some product of a family of two-point discrete spaces.) Corollary 1.2 was also (essentially) known for dyadic spaces, cf. [12, footnote 2], see also [10, Theorem 1.5]. This suggests the question of which other theorems on dyadic spaces generalize. None of the theorems on dyadic spaces recorded in [9], [10] or [12] (not necessarily the original paper) which are not related to Corollary 1.2 or 1.3 can be generalized for Hausdorff continuous images of supercompact Hausdorff spaces, see the examples below, with the possible exception of the theorem that closed G_δ-subspaces of dyadic spaces are dyadic, [12]. In view of this we asked in an earlier version of this paper if a closed G_δ-subspace of a supercompact Hausdorff space is supercompact, or is at least a continuous image of a supercompact Hausdorff space. Bell has answered both questions in the negative by finding a supercompact Hausdorff space in which our Example 1.4 embeds as a G_δ, [3]. (However, not every compact Hausdorff space embeds as a G_δ in some supercompact Hausdorff space, [18].)

We now sketch the examples. Note that three of the examples are compact linearly orderable spaces, and all four examples are supercompact.

2.6. Example. The Alexandroff double arrow line A, i.e. $[0,1] \times \{0,1\} \setminus \{\langle 0,0\rangle, \langle 1,1\rangle\}$, topologized by the lexicographic order.

If $\pi : A \to [0,1]$ is the "projection", then π is a continuous surjection, yet there is no (closed) metrizable $M \subset A$ with $\pi[M]=[0,1]$, cf. [12, Corollary on p. 56). Also, A is a nonmetrizable supercompactification of a metrizable space (any countable dense subspace), cf. [12, Appendix], and A is first countable but not second countable, cf. [9, Theorem 4].

2.7. Example. ω_1+1, the space of all ordinals $\leq\omega_1$.

The point ω_1 is not the limit of a nontrivial convergent sequence in ω_1+1, cf. [10, Corollary to Theorem 1.5].

Note however that Theorem 1.1 is a partial generalization of the theorem that every non-isolated point of a dyadic space is the limit of a nontrivial convergent sequence.

2.8. Example. An Aronszajn line. An Aronszajn line, L, can be constructed from an Aronszajn tree in the same way one constructs a Souslin line from a Souslin tree, cf. [22].

It is known that there is a collection $\{U_\alpha: \alpha<\omega_1\}$ of dense open subsets of L such that $U_\alpha\supset U_\beta$ if $\alpha<\beta$, and $\bigcap_{\alpha<\omega_1} U_\alpha=\emptyset$. So [9, Theorem 3] does not generalize.

2.9. Example. The Alexandroff double D of the product $P=\{0,1\}^{\mathfrak{c}}$, [11]: the underlying set of D is $P\times\{0,1\}$. Points of $P\times\{0\}$ are isolated in D. A basic neighborhood of $\langle x,1\rangle$ has the form $U\times\{0,1\}\setminus\{\langle x,0\rangle\}$, where U is a neighborhood of x in P.

It is a straightforward exercise to show that D is supercompact. Let B be any closed subspace without isolated points of P which is not the continuous image of a supercompact Hausdorff space, e.g. a homeomorph of $\beta\mathbb{N}\setminus\mathbb{N}$. Then $B\times\{0,1\}$ is the closure of the open subset $B\times\{1\}$ of the supercompact space D, yet it is not supercompact, not even the continuous image of a supercompact Hausdorff space, since the "natural" map from $B\times\{0,1\}$ to B is continuous, cf. [10, Theorem 13].

3. Construction of the examples

We first fix some notation. The domain of a function f is $\mathrm{dom}(f)$. If A and B are sets, ${}^AB$ is the set of functions from A to B; recall that each $f\in{}^AB$ is a sub*set* of $A\times B$. So if f and g are functions, then $f\subset g$ means $f=g\restriction\mathrm{dom}(f)$, the restriction of g to $\mathrm{dom}(f)$.

We will be interested in ${}^\alpha2$, for ordinals $\alpha\leq\omega$. An element of ${}^\alpha2$ can be seen as an α-sequence of 0's and 1's. As usual we denote $\bigcup_{n<\omega}{}^n2$, the set of finite sequences of 0's and 1's, by ${}^{\underline{\omega}}2$. For each $f\in{}^\omega2$ we define

$$I(f)=\{f\restriction n: n\in\omega\}(=\{g\in{}^{\underline{\omega}}2: g\subset f\}),$$

the set of initial sequences of f; $I(f)$ can be seen as the set of finite approximations to f. It is clear that

(1) if $f,g\in{}^\omega2$ are distinct, then $I(f)\cap I(g)$ is finite.

In other words, $\{I(f): f\in{}^\omega2\}$ is an *almost disjoint* collection of subsets of the countable set ${}^{\underline{\omega}}2$.

The set $T = {}^{\underline{\omega}}2 \cup {}^{\omega}2$ is a tree, partially ordered by $\subset$, the so-called *Cantor tree*, cf. [23]. We give T the usual tree topology by using the set of all open intervals as base. To be specific: points of ${}^{\underline{\omega}}2$ are isolated, and a basic neighborhood of $f \in {}^{\omega}2$ contains f and all but finitely many points of $I(f)$. T is first countable, and every subspace is locally compact, by (1). Example 1.2 will be a compactification of T, Example 1.3 will be the one-point compactification of a subspace of T.

The set ${}^{\omega}2$ can be viewed as a product of countably many two-point discrete spaces. Under the product topology ${}^{\omega}2$ is nothing but the Cantor Discontinuum, a basis for this topology is

$$\{\{f \in {}^{\omega}2 : f \supset g\} : g \in {}^{\underline{\omega}}2\}$$

as the reader should make clear to himself or herself.

The construction of the Examples depends on the following Lemma, the proof of which is postponed.

3.1. Lemma. *Let $L \subset {}^{\omega}2$ be uncountable. Then no compactification of the subspace ${}^{\underline{\omega}}2 \cup L$ of T is supercompact.*

Construction of Example 1.5. Choose any subset L of ${}^{\omega}2$ with cardinality ω_1. Then the subspace $S = {}^{\underline{\omega}}2 \cup L$ of T is a locally compact space with ω_1 points, hence the one-point compactification of S has all properties required.

Construction of Example 1.4. As indicated above, we will construct a first countable compactification of T. The basic idea is to identify the points of the subset ${}^{\omega}2$ of T with the isolated points of the Alexandroff Double, [11], of the Cantor Discontinuum, in the "natural" way. It will be technically convenient to change the underlying set of T to $\{0\} \times {}^{\omega}2 \cup \{1\} \times {}^{\omega}2$, and the underlying set of the Cantor Discontinuum to $\{2\} \times {}^{\omega}2$, if only to tell the two ${}^{\omega}2$'s apart.

Let K be $\{0\} \times {}^{\omega}2 \cup \{1, 2\} \times {}^{\omega}2$. We topologize K by assigning each $x \in K$ a neighborhood base $\{U(x, n) : n \in \omega\}$. For $\langle i, f\rangle \in K$ define

$$U(\langle i, f\rangle, n) = \begin{cases} \{\langle i, f\rangle\} & \text{if } i = 0, \\ \{\langle i, f\rangle\} \cup \{\langle 0, f \restriction k\rangle : k \geq n\} & \text{if } i = 1, \\ \{\langle j, g\rangle \in K : j \in 3, f \restriction n \subset g\} \setminus U(\langle 1, f\rangle, 0) & \text{if } i = 2. \end{cases}$$

The straightforward check that this is a valid neighborhood assignment for a Hausdorff topology is left to the reader. Note that the subspace $\{1, 2\} \times {}^{\omega}2$ of K is the Alexandroff Double of the Cantor Discontinuum, and that $\{0\} \times {}^{\underline{\omega}}2 \cup \{1\} \times {}^{\omega}2$ is a dense subspace of K which is (homeomorphic to) T. Hence K cannot be supercompact.

It remains to show that K is compact. For $\langle i, f\rangle \in K$ let $n(i, f) \in \omega$ be arbitrary. We have to show that the open cover

$$\mathcal{U} = \{U(\langle i, f\rangle, n(i, f)) : \langle i, f\rangle \in K\}$$

of K has a finite subcover. Since the subspace $\{2\}\times{}^{\omega}2$, which is homeomorphic to the Cantor discontinuum, is compact, there are for some $p\in\omega$ functions $f_0,\ldots f_p\in{}^{\omega}2$ such that

$$\mathcal{U}_0=\{U(\langle 2,f_i\rangle,n(2,f_i)): 0\le i\le p\}$$

covers $\{2\}\times{}^{\omega}2$. Then $\mathcal{U}_0$ covers $\{1\}\times{}^{\omega}2$, with possible exception of the points $\langle 1,f_i\rangle$, $0\le i\le p$. Let

$$\mathcal{U}_1=\{U(\langle 1,f_i\rangle,n(1,f_i)): 0\le i\le p\}$$

and define m by

$$m=\max\{n(j,f_i): j=1 \text{ or } 2, 0\le i\le p\}.$$

A straightforward check shows that $\mathcal{U}_0\cup\mathcal{U}_1$ covers all points of K with possible exception of the points of the finite set $\bigcup_{k<m}{}^{k}2$. It follows that $\mathcal{U}$ has a finite subcover. $\square$

Before we proceed to the proof of Lemma 3.1 we prove a simple useful result on the almost disjoint family $\{I(f): f\in{}^{\omega}2\}$.

Fact. Let G be any uncountable subset of ${}^{\omega}2$. Then there are a $g\in G$ and an infinite $H\subset G\backslash\{g\}$ such that $I(h)\cap I(h')\subset I(g)$ for any two distinct h, $h'\in H$ (then also $(I(h)\cup\{h\})\cap(I(h')\cup\{h'\})\subset I(g)$).

Consider G to be a subspace of the Cantor Discontinuum ${}^{\omega}2$. Then G is an uncountable separable metrizable space, hence we can find a nonisolated g in G. Basic neighborhoods of g in G have the form

$$\{h\in G: g\restriction n\subset h\},\quad n\in\omega$$

hence we can find $H=\{h_n: n\in\omega\}\subset G\backslash\{g\}$ such that

$$\min\{k: g(k)\ne h_n(k)\}<\min\{k: g(k)\ne h_{n+1}(k)\}$$

for all $n\in\omega$. Then g and H are as required. $\square$

Proof of Lemma 3.1. Denote the subspace ${}^{\underline{\omega}}2\cup L$ of T by Z. Let bZ be any (Hausdorff) compactification of Z. Let $\mathcal{S}$ be any closed subbase for bZ which is closed under intersection.

For each $f\in L$ the set $I(f)\cup\{f\}$ is open in Z and compact, hence it is clopen in bZ. Therefore $I(f)\cup\{f\}$ is the union of some finite subfamily of $\mathcal{S}$. It follows that for each $f\in L$ we can choose an $S(f)\in\mathcal{S}$ such that

(2) $S(f)\subset I(f)\cup\{f\}$, $S(f)\cap{}^{\underline{\omega}}2$ is infinite.

Since L is uncountable and ${}^{\underline{\omega}}2$ is countable, it follows that for some $p\in{}^{\underline{\omega}}2$ the set

$$G=\{f\in L: p\in S(f)\}$$

is uncountable. By the Fact there is a $g \in L$ (even $g \in G$) and an infinite $H \subset G \setminus \{g\}$ such that

(3) $(I(h) \cup \{h\}) \cap (I(h') \cup \{h'\}) \subset I(g)$ for distinct $h, h' \in H$.

Since $(I(a) \cup \{a\}) \cap (I(b) \cup \{b\})$ is finite for distinct $a, b \in {}^{\omega}2$, it follows from (2) and (3) that

(4) $\{S(h) \setminus (I(g) \cup \{g\}) \colon h \in H\}$ is a disjoint collection of nonempty subsets of bZ.

Since $I(g) \cup \{g\}$ is a clopen subset of bZ, so is its complement in bZ. Hence $bZ \setminus (I(g) \cup \{g\})$ is the union of a finite subfamily of $\mathcal{S}$. It now follows from (4) that there is an $S \in \mathcal{S}$ with

(5) $S \cap (I(g) \cup \{g\}) = \emptyset$,

such that there are distinct $h, h' \in H$ such that S intersects both $S(h)$ and $S(h')$. But $S(h)$ and $S(h')$ intersect since $p \in S(h) \cap S(h')$, consequently $\{S, S(h), S(h')\}$ is linked. However, it follows from (2), (3) and (5) that

$$\begin{aligned} S \cap S(h) \cap S(h') &\subset S \cap (I(h) \cup \{h\}) \cap (I(h') \cup \{h'\}) \\ &\subset S \cap I(g) = \emptyset. \end{aligned}$$

Consequently $\mathcal{S}$ is not binary. □

Remark. This lemma is similar to the proof in [1], and was discovered independently, but after learning that not every compact Hausdorff is supercompact.

We now show that Examples 1.4 and 1.5 are close to being supercompact. Note that if X is compact, then any *base* for X consisting of clopen sets is a closed *subbase* for X.

Proposition 3.2. *Let E be either Example* 1.4 *or Example* 1.5, *and let I be the (countable) set of isolated points of E. Then*

(a) *$E \setminus I$ is supercompact*

(b) *E has a base $\mathcal{B}$ consisting of clopen sets such that for any $\mathcal{A} \subset \mathcal{B}$ if $A_1 \cap A_2 \cap A_3 \neq \emptyset$ for any $A_1, A_2, A_3 \in \mathcal{A}$ then $\bigcap \mathcal{A} \neq \emptyset$.*

We prove this for Example 1.4, and leave the proof for Example 1.5 to the reader.

Proof of (a). $E \setminus I$ is the one-point compactification $D \cup \{p\}$ of a discrete space D. Clearly

$$\{\{x\} \colon x \in D\} \cup \{(E \setminus I) \setminus F \colon F \subset D \text{ finite}\}$$

is a binary subbase for $E \setminus I$.

Proof of (b). For $f \in L$ and $n \in \omega$ let

$$B(f, n) = \{f\} \cup f \restriction (\omega \setminus n)$$

and let

$$\mathcal{T} = \{B(f, n) \colon f \in L, n \in \omega\}.$$

Let

$$\mathcal{U} = \{E \setminus \bigcup \{B(f, 0) \colon f \in F\} \colon F \subset L \text{ finite}\}.$$

Evidently $\mathscr{U}$ is a neighborhood base for the point p at infinity. Consequently $\mathscr{B} = \mathscr{U} \cup \mathscr{T}$ is a base for E. Clearly the members of $\mathscr{B}$ are clopen.

Let $\mathscr{A}$ be any subfamily of $\mathscr{B}$ such that $A_1 \cap A_2 \cap A_3 \neq \emptyset$ for any $A_1, A_2, A_3 \in \mathscr{A}$. Define F and $\mathscr{F}$ by

$$F = \{f \in L: \exists n \in \omega(B(f, n) \in \mathscr{A})\},$$

$$\mathscr{F} = \mathscr{A} \cap \mathscr{T}.$$

Case 1. $F = \emptyset$. Then $\mathscr{A} \subset \mathscr{U}$, hence $p \in \bigcap \mathscr{A}$.

Case 2. $|F| = 1$. Let $F = \{f\}$. Clearly, if $U \in \mathscr{U}$, $g \in L$ and $g \notin U$, then $B(g, n) \cap U = \emptyset$ for all $n \in \omega$. It follows that $f \in \bigcap \mathscr{A}$.

Case 3. $|F| > 1$. We claim that

(*) there are $B(a, p)$ and $B(b, q) \in \mathscr{F}$ such that $B(a, p) \cap B(b, q) = \bigcap \mathscr{F}$.

For any $f, g \in {}^{\omega}2$ we can define $d(f, g) \leq \omega$ by

$$d(f, g) = \max\{\alpha \leq \omega: f \restriction \alpha = g \restriction \alpha\}.$$

Let $B(f, m)$ and $B(g, n)$ be any two members of $\mathscr{F}$ with $f \neq g$. Then for any $h \in {}^{\omega}2$, if $j \geq d(f, g)$, then $B(h, j)$ can not intersect both $B(f, m)$ and $B(g, n)$. Since any two members of $\mathscr{F}$ intersect, it follows that

$$p = \max\{n \in \omega: \exists h \in F(B(h, n) \in \mathscr{F})\}$$

exists. Choose any $a \in F$ such that $B(a, p) \in \mathscr{F}$. Let

$$s = \min\{n \in \omega: \exists h \in F(h \neq a \text{ and } d(a, h) = n)\}$$

and choose any $B(b, q) \in \mathscr{F}$ such that $d(a, b) = s$. Since $q \leq p$ one easily verifies that $B(a, p) \cap B(b, q) \subset \bigcap \mathscr{F}$. This completes the proof of (*).

Let $j = d(a, b)$. Then $a \restriction j \in B(a, p) \cap B(b, q)$, and if $f \in B(a, p) \cap B(b, q)$, then $f = a \restriction i$ for some $i \leq j$. It is clear from the form of the members of $\mathscr{U}$ that if $U \in \mathscr{U}$ and $a \restriction j \notin U$, then $a \restriction i \notin U$ for any $i \leq j$. Since $A_1 \cap A_2 \cap A_3 \neq \emptyset$ for any A_1, A_2, $A_3 \in \mathscr{A}$, it follows from (*) that $a \restriction j \in \cap \mathscr{A}$. □

4. Further developments

During the long time refereeing of this paper took, and during the long time we took to make a few revisions, there have been several developments in addition to the fact that several of the questions we raised in an earlier version have been answered.

(A) There are more results like Theorem 1.1 which show that supercompact spaces have easy to check properties not shared by all compact spaces, see [2], [17]. These new results involve cardinal functions.

(B) Mills has found a new large class of supercompact Hausdorff spaces by proving that all compact groups are supercompact, [20]. Since compact groups are dyadic spaces, this can be seen as a partial answer to the question of whether dyadic spaces are supercompact.

(C) One of us has found an example of a compact Hausdorff space for which the proof that it is not supercompact is particularly simple because it requires only a trivial observation on binary subbases, [8].

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