AN EXTREMALLY DISCONNECTED DOWKER SPACE

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Abstract. We give an example of an extremely disconnected Dowker space. Our basic tool is that every $P$-space can be $C^*$-embedded in an extremely disconnected compactum.

0. Introduction. A Dowker space is a normal space $X$ for which $X \times I$ is not normal, where $I$ denotes the closed unit interval $[0, 1]$. Dowker spaces are hard to get. Under various set theoretic hypotheses, Dowker spaces with many additional properties have been constructed. In ZFC only one construction of a Dowker space is known, see Rudin [R].

Hardy and Juhász [HJ] asked whether extremely disconnected Dowker spaces exist, where a space $X$ is called extremely disconnected if the closure of each open subspace of $X$ is given again open. They also announced that Wage had constructed such a space; however that turned out to be incorrect. The aim of this note is to construct an extremely disconnected Dowker space in ZFC. The reader who hopes that we found a new way of constructing Dowker spaces in ZFC will be quite disappointed. What we do is simply modify Mary Ellen Rudin's [R] Dowker space so that it becomes extremely disconnected. Our technique is to show that every $P$-space can be $C^*$-embedded in some compact extremely disconnected space, thus generalizing results in [BSV and vD].

1. Preliminaries. Let $X$ be a compact space and let $RO(X)$ be the Boolean algebra of regular open subsets of $X$. The Stone space of $RO(X)$ is denoted by $EX$ and is called the projective cover of $X$. The function $\pi: EX \to X$ defined by

$$\{\pi(u)\} = \bigcap_{U \in u} U,$$

is easily seen to be continuous, onto and irreducible, i.e. if $A \subseteq EX$ is a proper closed subspace, then $\pi(A) \neq X$. Since $RO(X)$ is complete, $EX$ is extremely disconnected. If $h: X \to X$ is a homeomorphism, then the function $eh: EX \to EX$ defined by $eh(u) = \{h(U): U \in u\}$ is easily seen to be a homeomorphism such that $\pi \circ eh = h \circ \pi$. The reader is encouraged to check this, since we use this later. For a recent survey on projective covers, see Woods [W]. By a result of Efimov [E], every extremely disconnected compactum embeds in the Čech-Stone compactification $\beta X$.

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of some cardinal $\kappa$, where $\kappa$ is given the discrete topology. As usual, we call a space $X$ a $P$-space if every $G_\delta$ in $X$ is open. If $X$ is a Tychonoff space, then $\beta X$ denotes the Čech-Stone compactification of $X$. A subspace $Y \subseteq X$ is said to be $C^*$-embedded in $X$ provided that every map $f: Y \to I$ extends to a map $f: X \to I$. Our terminology is standard. $\omega(X)$ denotes the weight of a space $X$.

2. Embedding $P$-spaces in $\beta \kappa$. In this section we show that if $X$ is a $P$-space, then $\beta X$ can be embedded in the Čech-Stone compactification of some discrete space. Obviously, this is equivalent to the statement that every $P$-space can be $C^*$-embedded in some extremally disconnected compact space.

To this end, let $X$ be a $P$-space. Since $X$ is strongly zero-dimensional, we may assume that $\beta X \subseteq 2^\kappa$ for certain $\kappa$. Take $p \in 2^\kappa$. The map $g_p: 2^\kappa \to 2^\kappa$ defined by $g_p(x) = x + p$ lifts to a map $e g_p: E(2^\kappa) \to E(2^\kappa)$, see §1. The homeomorphism $e g_p$ will be called $h_p$ for short.

2.1. Lemma. If $U \in RO(2^\kappa)$ then there exist a countable collection $\{F_n: n \in \omega\}$ of finite subsets of $\kappa$ and elements $\delta_n \in 2^\kappa$, for $n \in \omega$, such that $\bigcup_{n \in \omega} \bigcap_{i \in F_n} \pi^{-}_i (\delta(i))$ is a dense subset of $U$ (where $\pi_i$ is the $i$th projection map).

Proof. As is well known, $2^\kappa$ is ccc (families of pairwise disjoint open sets are countable) and the collection $\mathcal{B} = \{ \bigcap_{i \in F} \pi^{-}_i (\delta(i)): F \subseteq \kappa$ is finite and $\delta \in 2^\kappa \}$ is a base for the topology of $2^\kappa$. Choose a maximal cellular collection, $\mathcal{C} \subseteq \mathcal{B}$, of subsets of $U$. Clearly $\mathcal{C}$ is countable and $\bigcup \mathcal{C}$ is dense in $U$. Take $\{F_n: n \in \omega\}$, finite subsets of $\kappa$, and $\delta_n \in 2^\kappa$, for $n \in \omega$, so that $\mathcal{C} = \{ \bigcap_{i \in F_n} \pi^{-}_i (\delta_n(i)): n \in \omega \}$. \square

If $U \in RO(2^\kappa)$ and $\{F_n: n \in \omega\}$ is chosen as in 2.1, then we say that $U$ is determined by $D = \bigcup F_n$. The following lemma follows trivially from the definition of $g_p$ for $p \in 2^\kappa$.

2.2. Lemma. If $i \in \kappa$ and $\pi_i(p) = \pi_i(q)$ for $p, q \in 2^\kappa$ then $g_p(\pi^{-}(\delta)) = g_q(\pi^{-}(\delta))$ for $\delta \in \{0, 1\}$.

2.3. Lemma. If $U \in RO(2^\kappa)$, $U$ is determined by $D$ and $p, q \in 2^\kappa$ are such that $p \upharpoonright D = q \upharpoonright D$, then $g_p(U) = g_q(U)$.

Proof. Let $\{F_n: n \in \pm \omega\}$ and $\{\delta_n: n \in \omega\}$ with $D = \bigcup_{n \in \omega} F_n$ be as in 2.1. From 2.2, it follows that $g_p(\bigcap_{i \in F_n} \pi^{-}_i (\delta_n(i))) = g_q(\bigcap_{i \in F_n} \pi^{-}_i (\delta_n(i)))$ for each $n \in \omega$, and therefore

$$g_p\left( \bigcup_{n \in \omega} \left( \bigcap_{i \in F_n} \pi^{-}_i (\delta_n(i)) \right) \right) = g_q\left( \bigcup_{n \in \omega} \left( \bigcap_{i \in F_n} \pi^{-}_i (\delta_n(i)) \right) \right).$$

Since the image under $g_p$ and $g_q$ of a dense subset of $U$ is the same, $g_p(U) = g_q(U)$. \square

Take a point $u_0 \in \pi^{-}(\theta)$, where $\theta$ denotes the identity of $2^\kappa$. If $p \in X$, let $u_p = h_p(u_0)$. Observe that

$$\pi(u_p) = \pi(h_p(u_0)) = g_p(\pi(u_0)) = g_p(\theta) = p.$$
whence $u_p \in \pi^-(p)$. If $U \in RO(2^*)$ then $\overline{h_p(\pi^-(u))} = \pi^-(g_p(U))$ and from this it follows that $u_p = \{g_p(U): U \in u_0\}$. Note also that since $g_p \circ g_p = \text{id}$, $u_p = \{U: g_p(U) \subseteq u_0\}$. Let $P = \{u_p: p \in X\}$.

2.4. Lemma. The function $\pi^+ P: P \to X$ is a homeomorphism.

Proof. For convenience, put $f = \pi^+ P$. Then $f$ is clearly one-to-one, onto and continuous. It therefore suffices to show that $f$ is open. Basic open sets of $P$ are of the form $\hat{U}$, where $U \in RO(2^*)$ and $\hat{U} = \{u_p \in P: U \in u_p\}$. Choose $p \in f(\hat{U})$ and let $U$ be determined by $D$. Let $Z = \{q \in X: p \upharpoonright D = q \upharpoonright D\}$. By 2.3, $g_p(U) = g_q(U)$ and, therefore, $u_q \in \hat{U}$ by the above remarks, for each $q \in Z$. Now $Z = X \cap \bigcap_{i \in D} \pi^+(\pi_i(p))$ is a $G_\delta$ set of $X$ and therefore open in $X$. Since $p \in Z$ and $Z \subseteq f(\hat{U})$, we conclude that $f(\hat{U})$ is a neighborhood of $p$. $\square$

The closure of $P$ in $E(2^*)$ is a compactification of $P$ which is clearly homeomorphic to $\beta X$ since $\beta X$ is the largest compactification of $X$. This completes the proof, since by Efimov’s result (§1), $E(2^*)$ can be embedded in the Čech-Stone compactification of a discrete space.

The reader can easily verify that in fact we have shown that if $X$ is a $P$-space of weight $\kappa$ then $\beta X$ can be embedded in $\beta(2^*)$ (here $2^*$ has the discrete topology of course).

3. The example. The Dowker space $R$ constructed in Rudin [R] is a $P$-space. By the results in §2, $\beta R$ embeds in $\beta \kappa$ for certain $\kappa$. Since $\beta \kappa$ embeds in $\beta \kappa - \kappa$, we may assume that $\beta R \subseteq \beta \kappa - \kappa$. Put $X = \kappa \cup R$. Since each dense subspace of an extremally disconnected space is extremally disconnected, $X$ is extremally disconnected. Also, $R$ is closed in $X$ which implies that $X \times I$ is not normal since $R \times I$ is not normal. Since $\kappa$ is discrete, a moment’s reflection shows that $X$ is normal iff disjoint closed subsets of $R$ have disjoint neighborhoods in $X$. Let $A, B \subseteq R$ be closed and disjoint. Since the closure of $R$ in $\beta \kappa$ is $\beta R$, $A$ and $B$ have disjoint closures in $\beta R$, hence they have disjoint neighborhoods in $\beta \kappa$. We conclude that $X$ is normal and consequently that $X$ is an extremally disconnected Dowker space.

Observe that our example, in particular, is an example of a normal extremally disconnected space which is not paracompact. Such a space was earlier constructed by Kunen [K].

4. Remarks. (1) The technique used in §2 is a modification of a technique due to Balcar, Simon and Vojtás [BSV] and, independently, Kunen, and Shelah. They observe that if $p_\alpha$ is the point of $2^*$ with value 1 only in the point $\alpha$ then the set $\{u_{p_\alpha}: \alpha < \kappa\} \subseteq E(2^*)$ is discrete and each neighborhood of $u_0$ contains all but countably many points of $\{u_{p_\alpha}: \alpha < \kappa\}$ (the notation is as in §2).

(2) van Douwen [vD] used the technique described in (1) to prove the important result that every $P$-space embeds in $\beta \kappa$ for certain $\kappa$. His proof goes as follows. Let $X = \{u_p: p \in 2^*\}$. Then $X$ considered to be a subspace of $E(2^*)$ with the $G_\delta$ topology, is homeomorphic to $2^*$ with the $G_\delta$ topology. Moreover, $E(2^*)$ with the $G_\delta$ topology embeds in $E(2^*)$. Consequently, $2^*$ with the $G_\delta$ topology embeds in $E(2^*)$ and hence in $\beta(2^*)$. If $P \subseteq 2^*$ is a $P$-space, then $P$ is homeomorphic to $P$ considered
to be a subspace of $2^*$ with the $G_δ$ topology. Consequently, $P$ embeds in $β(2^*)$. Our results in §2 were motivated by these ideas but our construction is much simpler and proves more since our embeddings of $P$-spaces are embeddings of $C^*$-embedded subspaces of $E(2^*)$ and this made our construction work.

References


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