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AN EXTREMALLY DISCONNECTED DOWKER SPACE

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ABSTRACT. We give an example of an extremally disconnected Dowker space. Our basic tool is that every P-space can be C^*-embedded in an extremally disconnected compactum.

0. Introduction. A *Dowker space* is a normal space X for which $X \times I$ is not normal, where I denotes the closed unit interval $[0, 1]$. Dowker spaces are hard to get. Under various set theoretic hypotheses, Dowker spaces with many additional properties have been constructed. In ZFC only one construction of a Dowker space is known, see Rudin [**R**].

Hardy and Juhász [**HJ**] asked whether extremally disconnected Dowker spaces exist,[1] where a space X is called *extremally disconnected* if the closure of each open subspace of X is given again open. They also announced that Wage had constructed such a space; however that turned out to be incorrect. The aim of this note is to construct an extremally disconnected Dowker space in ZFC. The reader who hopes that we found a new way of constructing Dowker spaces in ZFC will be quite disappointed. What we do is simply modify Mary Ellen Rudin's [**R**] Dowker space so that it becomes extremally disconnected. Our technique is to show that every P-space can be C^*-embedded in some compact extremally disconnected space, thus generalizing results in [**BSV** and **vD**].

1. Preliminaries. Let X be a compact space and let $RO(X)$ be the Boolean algebra of regular open subsets of X. The Stone space of $RO(X)$ is denoted by EX and is called the *projective cover* of X. The function $\pi\colon EX \to X$ defined by

$$\{\pi(u)\} = \bigcap_{U \in u} \overline{U},$$

is easily seen to be continuous, onto and irreducible, i.e. if $A \subseteq EX$ is a proper closed subpsace, then $\pi(A) \neq X$. Since $RO(X)$ is complete, EX is extremally disconnected. If $h\colon X \to X$ is a homeomorphism, then the function $eh\colon EX \to EX$ defined by $eh(u) = \{h(U)\colon U \in u\}$ is easily seen to be a homeomorphism such that $\pi \circ eh = h \circ \pi$. The reader is encouraged to check this, since we use this later. For a recent survey on projective covers, see Woods [**W**]. By a result of Efimov [**E**], every extremally disconnected compactum embeds in the Čech-Stone compactification $\beta\kappa$

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[1]Actually it appears that this question was first asked by R. G. Woods.

of some cardinal κ, where κ is given the discrete topology. As usual, we call a space X a *P-space* if every G_δ in X is open. If X is a Tychonoff space, then βX denotes the Čech-Stone compactification of X. A subspace $Y \subseteq X$ is said to be C^*-embedded in X provided that every map $f\colon Y \to I$ extends to a map $\bar{f}\colon X \to I$. Our terminology is standard. $w(X)$ denotes the weight of a space X.

2. Embedding P-spaces in $\beta\kappa$. In this section we show that if X is a P-space, then βX can be embedded in the Čech-Stone compactification of some discrete space. Obviously, this is equivalent to the statement that every P-space can be C^*-embedded in some extremally disconnected compact space.

To this end, let X be a P-space. Since X is strongly zero-dimensional, we may assume that $\beta X \subseteq 2^\kappa$ for certain κ. Take $p \in 2^\kappa$. The map $g_p\colon 2^\kappa \to 2^\kappa$ defined by $g_p(x) = x + p$ lifts to a map $eg_p\colon E(2^\kappa) \to E(2^\kappa)$, see §1. The homeomorphism eg_p will be called h_p for short.

2.1. LEMMA. *If $U \in RO(2^\kappa)$ then there exist a countable collection $\{F_n\colon n \in \omega\}$ of finite subsets of κ and elements δ_n of 2^{F_n}, for $n \in \omega$, such that $\bigcup_{n\in\omega}(\bigcap_{i\in F_n}\pi_i^{\leftarrow}(\delta(i)))$ is a dense subset of U (where π_i is the ith projection map).*

PROOF. As is well known, 2^κ is ccc (families of pairwise disjoint open sets are countable) and the collection $\mathcal{B} = \{\bigcap_{i\in F}\pi_i^{\leftarrow}(\delta(i))\colon F \subseteq \kappa \text{ is finite and } \delta \in 2^F\}$ is a base for the topology of 2^κ. Choose a maximal cellular collection, $\mathcal{C} \subseteq \mathcal{B}$, of subsets of U. Clearly $\mathcal{C}$ is countable and $\bigcup\mathcal{C}$ is dense in U. Take $\{F_n\colon n \in \omega\}$, finite subsets of κ, and $\delta_n \in 2^{F_n}$, for $n \in \omega$, so that $\mathcal{C} = \{\bigcap_{i\in F_n}\pi_i^{\leftarrow}(\delta_n(i))\colon n \in \omega\}$. □

If $U \in RO(2^\kappa)$ and $\{F_n\colon n \in \omega\}$ is chosen as in 2.1, then we say that U *is determined by* $D = \bigcup F_n$. The following lemma follows trivially from the definition of g_p for $p \in 2^\kappa$.

2.2. LEMMA. *If $i \in \kappa$ and $\pi_i(p) = \pi_i(q)$ for $p, q \in 2^\kappa$ then $g_p(\pi^{\leftarrow}(\delta)) = g_q(\pi^{\leftarrow}(\delta))$ for $\delta \in \{0, 1\}$.*

2.3. LEMMA. *If $U \in RO(2^\kappa)$, U is determined by D and $p, q \in 2^\kappa$ are such that $p \restriction D = q \restriction D$, then $g_p(U) = g_q(U)$.*

PROOF. Let $\{F_n\colon n \in \pm\omega\}$ and $\{\delta_n\colon n \in \omega\}$ with $D = \bigcup_{n\in\omega}F_n$ be as in 2.1. From 2.2, it follows that $g_p(\bigcap_{i\in F_n}\pi_i^{\leftarrow}(\delta_n(i))) = g_q(\bigcap_{i\in F_n}\pi_i^{\leftarrow}(\delta_n(i)))$ for each $n \in \omega$, and therefore

$$g_p\left(\bigcup_{n\in\omega}\left(\bigcap_{i\in F_n}\pi_i^{\leftarrow}(\delta_n(i))\right)\right) = g_q\left(\bigcup_{n\in\omega}\left(\bigcap_{i\in F_n}\pi_i^{\leftarrow}(\delta_n(i))\right)\right).$$

Since the image under g_p and g_q of a dense subset of U is the same, $g_p(U) = g_q(U)$. □

Take a point $u_0 \in \pi^{\leftarrow}(\mathbf{0})$, where $\mathbf{0}$ denotes the identity of 2^κ. If $p \in X$, let $u_p = h_p(u_0)$. Observe that

$$\pi(u_p) = \pi(h_p(u_0)) = g_p(\pi(u_0)) = g_p(\mathbf{0}) = p,$$

whence $u_p \in \pi^{\leftarrow}(p)$. If $U \in RO(2^\kappa)$ then $\overline{h_p(\pi^{\leftarrow}(u))} = \overline{\pi^{\leftarrow}(g_p(U))}$ and from this it follows that $u_p = \{g_p(U): U \in u_0\}$. Note also that since $g_p \circ g_p = \text{id}$, $u_p = \{U: g_p(U) \in u_0\}$. Let $P = \{u_p: p \in X\}$.

2.4. LEMMA. *The function $\pi \restriction P: P \to X$ is a homeomorphism.*

PROOF. For convenience, put $f = \pi \restriction P$. Then f is clearly one-to-one, onto and continuous. It therefore suffices to show that f is open. Basic open sets of P are of the form $\tilde{U}$, where $U \in RO(2^\kappa)$ and $\tilde{U} = \{u_p \in P: U \in u_p\}$. Choose $p \in f(\tilde{U})$ and let U be determined by D. Let $Z = \{q \in X: p \restriction D = q \restriction D\}$. By 2.3, $g_p(U) = g_q(U)$ and, therefore, $u_q \in \tilde{U}$ by the above remarks, for each $q \in Z$. Now $Z = X \cap \bigcap_{i \in D} \pi_i^{\leftarrow}(\pi_i(p))$ is a G_δ-set of X and therefore open in X. Since $p \in Z$ and $Z \subseteq f(\tilde{U})$, we conclude that $f(\tilde{U})$ is a neighborhood of p. □

The closure of P in $E(2^\kappa)$ is a compactification of P which is clearly homeomorphic to βX since βX is the largest compactification of X. This completes the proof, since by Efimov's result (§1), $E(2^\kappa)$ can be embedded in the Čech-Stone compactification of a discrete space.

The reader can easily verify that in fact we have shown that if X is a P-space of weight κ then βX can be embedded in $\beta(2^\kappa)$ (here 2^κ has the discrete topology of course).

3. The example. The Dowker space R constructed in Rudin [**R**] is a P-space. By the results in §2, βR embeds in $\beta\kappa$ for certain κ. Since $\beta\kappa$ embeds in $\beta\kappa - \kappa$, we may assume that $\beta R \subseteq \beta\kappa - \kappa$. Put $X = \kappa \cup R$. Since each dense subspace of an extremally disconnected space is extremally disconnected, X is extremally disconnected. Also, R is closed in X which implies that $X \times I$ is not normal since $R \times I$ is not normal. Since κ is discrete, a moment's reflection shows that X is normal iff disjoint closed subsets of R have disjoint neighborhoods in X. Let $A, B \subseteq R$ be closed and disjoint. Since the closure of R in $\beta\kappa$ is βR, A and B have disjoint closures in βR, hence they have disjoint neighborhoods in $\beta\kappa$. We conclude that X is normal and consequently that X is an extremally disconnected Dowker space.

Observe that our example, in particular, is an example of a normal extremally disconnected space which is not paracompact. Such a space was earlier constructed by Kunen [**K**].

4. Remarks. (1) The technique used in §2 is a modification of a technique due to Balcar, Simon and Vojtáš [**BSV**] and, independently, Kunen, and Shelah. They observe that if p_α is the point of 2^κ with value 1 only in the point $\{\alpha\}$ then the set $\{u_{p_\alpha}: \alpha < \kappa\} \subseteq E(2^\kappa)$ is discrete and each neighborhood of u_0 contains all but countably many points of $\{u_{p_\alpha}: \alpha < \kappa\}$ (the notation is as in §2).

(2) van Douwen [**vD**] used the technique described in (1) to prove the important result that every P-space embeds in $\beta\kappa$ for certain κ. His proof goes as follows. Let $X = \{u_p: p \in 2^\kappa\}$. Then X considered to be a subspace of $E(2^\kappa)$ with the G_δ topology, is homeomorphic to 2^κ with the G_δ topology. Moreover, $E(2^\kappa)$ with the G_δ topology embeds in $E(2^\kappa)$. Consequently, 2^κ with the G_δ topology embeds in $E(2^\kappa)$ and hence in $\beta(2^\kappa)$. If $P \subseteq 2^\kappa$ is a P-space, then P is homeomorphic to P considered

to be a subspace of 2^κ with the G_δ topology. Consequently, P embeds in $\beta(2^\kappa)$. Our results in §2 were motivated by these ideas but our construction is much simpler and proves more since our embeddings of P-spaces are embeddings of C^*-embedded subspaces of $E(2^\kappa)$ and this made our construction work.

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