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THERE IS NO COMPACTIFICATION THEOREM FOR THE SMALL INDUCTIVE DIMENSION

Jan van MILL*
Subfaculteit Wiskunde, Vrije Universiteit, De Boelelaan 1081, Amsterdam, The Netherlands

Teodor C. PRZYMUSIŃSKI
Instytut Matematyczny PAN, Sniadeckich 8, 00-950 Warsaw, Poland

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We give an example of a perfectly normal first countable space X^* with $\operatorname{ind} X^* = 1$ such that if Z is a Lindelöf space containing X^*, then $\operatorname{ind} Z = \dim Z = \infty$. Under CH, there is a perfectly normal, hereditarily separable and first countable such space.

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1. Introduction

All spaces under discussion are Tychonoff. For all undefined notions see [2].

It is well known that $\dim X = \dim \beta X$ for any space X and that $\operatorname{Ind} X = \operatorname{Ind} \beta X$ if X is normal. Unfortunately, $\operatorname{ind} X$ need not be equal to $\operatorname{ind} \beta X$. Smirnov [6] constructed an example of a normal space Z such that $\operatorname{ind} Z = 0$ and $\dim Z = \infty$, hence, $\dim \beta Z = \operatorname{ind} \beta Z = \infty$. It is clear however, that every space Y with $\operatorname{ind} Y = 0$ has a compactification γY with $\operatorname{ind} \gamma Y = 0$. It seems therefore natural to ask whether any space Y has a compactification γY such that $\operatorname{ind} Y = \operatorname{ind} \gamma Y$. Unfortunately, this is not the case. Applying a technique in van Douwen and Przymusiński [1], and using an example in Pol and Pol [4], we will construct a perfectly normal first countable space X^* with $\operatorname{ind} X^* = 1$ such that if Z is a Lindelöf space containing X^*, then $\operatorname{ind} Z = \dim Z = \infty$. Under the Continuum Hypothesis (CH), there even exists a perfectly normal, hereditarily separable and first countable such space.

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2. The construction

The aim of this section is prove Lemma 2.1 of which all the results announced in the introduction will be easy consequences (if the right input is used of course). The proof of Lemma 2.1 is inspired by [1].

2.1. Lemma. *For every space X there exists a space X^* with the following properties:*

(a) $\operatorname{ind} X^* \leq \operatorname{ind} X + 1$;

(b) *if Z is a Lindelöf space containing X^*, then* $\dim X \leq \dim Z \leq \operatorname{ind} Z$;

(c) *if X is normal (perfectly normal, first countable, (hereditarily) separable etc., respectively), then so is X^*.*

2.2. Corollarly. *There exists a perfectly normal first countable space X^* with* $\operatorname{ind} X^* = 1$ *such that if Z is a Lindelöf space containing X^*, then* $\operatorname{ind} Z = \dim Z = \infty$.

Proof. There exists a perfectly normal and first countable space X such that $\operatorname{ind} X = 0$ and $\dim X = \infty$, [4]. □

2.3. Corollary (CH). *There exists a perfectly normal, hereditarily separable and first countable space X^* as above.*

Proof. *CH* implies that there is a hereditarily separable space X as above, [3]. □

2.4. Proof of Lemma 2.1. Fix X and $\theta \notin X$. There exists a countable family $\mathscr{F} = \{f_i : i < \omega\}$ of continuous functions $f_i : X \to I$, where I denotes the interval [0, 1], such that if $k < \dim X$, then for some subfamily of $k+1$ functions from $\mathscr{F}$, say $f_{i_0}, f_{i_1}, \ldots, f_{i_k}$, the following holds:

$$\text{if } K_j \text{ is a zero-set partition between } f_{i_j}^{-1}(0) \text{ and } f_{i_j}^{-1}(1), \quad \text{then } \bigcap_{j=0}^{k} K_j \neq \emptyset. \tag{1}$$

Denote by C the Cantor set and by $Q \subset C$ the set of rationals in C. Let $\{Q_\alpha : \alpha < \mathfrak{c}\}$ enumerate the family of all dense subsets of Q. Decompose C into dense subsets $A_{\alpha,i}$, for $\alpha < \mathfrak{c}$, $i < \omega$, i.e. $C = \bigcup \{A_{\alpha,i} : \alpha < \mathfrak{c}, i < \omega\}$. For every $t \in C$, if $t \in A_{\alpha,i}$, choose a sequence $\{q_n(t)\}_{n<\omega}$ of points of Q_α so that

$$0 < |q_n(t) - t| < 1/n.$$

Let $B(t, \varepsilon) = \{s \in C : |t - s| < \varepsilon\}$. Define

$$X^* = (C \times \{\theta\}) \cup (Q \times X) \subset C \times (\{\theta\} \cup X).$$

Basic neighborhoods of points $\langle q, x \rangle$, where $q \in Q$ and $x \in X$, are of the form $\{q\} \times U$, where U is open in X and contains x. Suppose that $t \in C$, $t \in A_{\alpha,i}$ and $n < \omega$. Basic neighborhoods of $\langle t, \theta\rangle$ are of the form

$$U(t, n) = (B(t, 1/n) \times (X \cup \{\theta\})) \setminus ((\{t\} \times X) \cup (\{q_j(t)\}_{j<\omega} \times f_i^{-1}([1/n, 1]))).$$

One easily checks that X^* is a Tychonoff space and that $\operatorname{ind} X^* \leq \operatorname{ind} X + 1$. Clearly if X is first countable or (hereditarily) separable, then so is X^*.

Suppose that e.g. X is perfectly normal and let U be an open subset of X^*. For every $\langle t, \theta\rangle \in U$ there exists a $U(t, n_t) \subset \overline{U(t, n_t)} \subset U$ and countably many of these sets cover $U \cap (C \times \{\theta\})$. Similarly, there exists a countable family of open subsets of X^* contained with their closures in U and covering $U \cap (Q \times X\}$, thus X^* is perfectly normal.

It remains to prove (b). Suppose that $Z \supset X^*$ is Lindelöf and $k = \dim Z$ is smaller than $\dim X$. Let $\{f_{i_j}\}_{j=0}^{k}$ be as in (1). For every $q \in Q$ let $Z_q = \overline{\{q\} \times X}$ (the closure is taken in Z). Thus Z_q is a Lindelöf extension of X. We say that q *separates* f_i if

$$\overline{\{q\} \times f_i^{-1}(0)} \cap \overline{\{q\} \times f_i^{-1}(1)} = \emptyset.$$

Note first that

$$\text{if } t \in A_{\alpha,i}, \text{ then there exists an } m(t) \in \omega, \text{ so that if } l \geq m(t), \text{ then } q_l(t) \in Q_\alpha \text{ and } q_l(t) \text{ separates } f_i. \tag{2}$$

Indeed, if U is open in Z such that $U \cap X^* = U(t, 1)$, then there exists an $m(t)$ with $\overline{U(t, m(t))} \subset U$. It is easily seen that $m(t)$ is as required (this is the same technique as in [1, 2.2]).

For $s = 0, 1, \ldots, k$, let $T_s = \{q \in Q\colon q \text{ separates } f_{i_j} \text{ for all } j = 0, 1, \ldots, s\}$. We shall show by induction that T_s is dense in Q. Indeed, density of T_0 follows from (2) and the density of A_{0,i_0}. Suppose that T_s is dense for certain $s < k$. Then $T_s = Q_\alpha$ for some α and the density of T_{s+1} follows from (2) and the density of $A_{\alpha,i_{s+1}}$.

Let $q \in T_k$. Then q separates all f_{i_j}'s, for $j = 0, 1, \ldots, k$. since Z_q is a closed subspace of the Lindelöf (hence normal) space Z, we have that $\dim Z_q \leq k$. Therefore there exist zero-set partitions K_j in Z_q between $\overline{\{q\} \times f_{i_j}^{-1}(0)}$ and $\overline{\{q\} \times f_{i_j}^{-1}(1)}$ with $\bigcap_{j=0}^{k} K_j = \emptyset$, which contradicts (1).

This proves that $\dim X \leqq \dim Z$. That $\dim Z \leqq \operatorname{ind} Z$ follows from the fact that Z is Lindelöf, [2, 7.2.7] □

3. Remarks

Clearly if $\operatorname{ind} X = 0$, then X^* is a countable union of zero-dimensional closed subspaces. If one collapses $C \times \{\theta\}$ to a point then the resulting space X^{**} has similar properties, but is the union of the zero-dimensional space $Q \times X$ (with Q discrete) and a point.

One naturally wonders whether a metrizable space M exists such that $\operatorname{ind} M = 1$ and if Z is a Lindelöf (or compact) space containing M, then $\operatorname{ind} Z > 1$. If M exists then $\dim M > 1$. That can be seen as follows. If $\dim M \leq 1$, then by the Katětov–

Morita Theorem, [2, 7.3.2], $\operatorname{Ind} M \leqslant 1$ which implies that $\operatorname{ind} \beta M \leqslant \operatorname{Ind} \beta M \leq 1$. We don't know whether M exists. Notice that there is a metrizable space Δ with $\operatorname{ind} \Delta = 0$ and $\dim \Delta > 0$, [5].

We have seen that there is no compactification theorem for the small inductive dimension. In fact we have shown that there are spaces which have no ind preserving Lindelöf extension. This suggests the question whether there is a Lindelöf space L with $\operatorname{ind} L = 2$ but if γL is any compactification of L, then $\operatorname{ind} \gamma L > 2$. We don't know the answer to this question. Observe that if L is Lindelöf and $\operatorname{ind} L = 1$, then $\operatorname{Ind} L = 1$, [2, 7.2.9], and thus $1 \leqslant \operatorname{ind} \beta L \leqslant \operatorname{Ind} \beta L = 1$.

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