Topology and its Applications 13 (1982) 59–68
North-Holland Publishing Company

ON AN INTERNAL PROPERTY OF ABSOLUTE RETRACTS, II

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Received 6 October 1980

Let X be a (metrizable) space. A mixer for X is, roughly speaking, a map $\mu : X^3 \to X$ such that $\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x$ for all $x, y \in X$. We show that each AR has a mixer and that a finite dimensional path connected space with a mixer is an AR. Our main result is that each separable space with a mixer and having an open cover by sets contractible within the whole space, is LEC.

AMS Subj. Class. (1980): 54F40, 54F56

mixer   local mixer   AR   ANR   LEC

1. Introduction

Let X be a compact metric space. A map $\mu : X^3 \to X$ which has the property that $\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x$ for all $x, y \in X$, is called a *mixer.* It is known, [5], that each compact AR has a mixer and that a continuum with a mixer is C^∞ and LC^∞. Consequently, for finite dimensional continua, having a mixer is equivalent to being an AR.

We have tried for some time to prove that each continuum with a mixer is an AR. If this were true, then this would yield a nice 'internal' characterization of compact AR's. In the process of trying to solve this problem we found that each continuum X with a mixer having an open cover by sets contractible within X, is LEC (for definitions, see Section 2). Since it is unknown whether an LEC continuum is an ANR, this does not solve our problem but it shows that our question is relevant.

In this paper we study mixers for noncompact spaces as well. We therefore have to adapt the definition of a mixer stated above. Our main result is that each (separable metric) space X with a (local) mixer having an open cover by sets contractible within X is LEC. This generalizes some of the results in [5].

Throughout this paper, all spaces are metrizable.

2. Preliminaries

Let X be compact and let $\mu : X^3 \to X$ be a mixer. The compactness of X easily implies the following [5, Lemma 1.2]:

(*) *if x_n, y_n, z_n ($n \in \mathbb{N}$) are points of X such that the sequences $(x_n)_n$ and $(y_n)_n$ both converge to $a \in X$, then the sequences $(\mu(x_n, y_n, z_n))_n$, $(\mu(x_n, z_n, y_n))_n$ and $(\mu(z_n, x_n, y_n))_n$ converge to a.*

As is clear from the proof of [5, Theorem 1.3], this property of mixers is of crucial importance. In the presence of compactness (*) is automatically true but in the noncompact case this need not be the case. We therefore are forced to include (*) in the definition of a mixer for arbitrary (metric) spaces.

Definition 2.1. Let X be a space. A *mixer for* X is a map $\mu : X^3 \to X$ which satisfies (*).

Notice that if $\mu : X^3 \to X$ is a mixer, then $\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x$ for all $x, y \in X$. A *symmetric mixer* is a mixer $\mu : X^3 \to X$ which has the additional property that $\mu(x, y, z) = \mu(x, z, y) = \mu(z, x, y) = \cdots$ for all $x, y, z \in X$.

Lemma 2.2. *Let X be a space and let $\mu : X^3 \to X$ be a mixer. If $x \in X$ and if U is a neighborhood of x in X, then there is a neighborhood V of x such that*

$$\mu((V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V)) \subset U.$$

Proof. If not, then for all $n \in \mathbb{N}$ we can find $a_n, b_n \in U(x, 1/n)$ and $c_n \in X$ (here $U(x, 1/n) = \{z \in X : d(x, z) < 1/n\}$) such that, without loss of generality,

$$\mu(a_n, b_n, c_n) \notin U. \tag{1}$$

Since $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = x$, by (*) $\lim_{n\to\infty} \mu(a_n, b_n, c_n) = x$, which contradicts (1).

In the above lemma we have identified another important property of mixers and it leads us to the definition of a local mixer.

Definition 2.2. A *local mixer of a space* X is a map $\mu : U \to X$ where U is a neighborhood of the diagonal $\Delta(X)$ in X^3 such that if x_n, y_n, z_n ($n \in \mathbb{N}$) are points of X such that the sequences $(x_n)_n$ and $(y_n)_n$ both converge to $a \in X$, then there is an $m \in \mathbb{N}$ so that the sequences $(\mu(x_n, y_n, z_n))_{n \geq m}$, $(\mu(x_n, z_n, y_n))_{n \geq m}$ and $(\mu(z_n, x_n, y_n))_{n \geq m}$ converge to a.

It should be clear what we mean by a *symmetric local mixer.*

As usual, $\mathrm{dom}(f)$ denotes the *domain* of a function f.

Lemma 2.3. *Let X be space and let μ be a local mixer for X. If $x \in X$ and if U is a neighborhood of x in X, then there is a neighborhood V of x such that*

$$E(V) = (V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V) \subset \mathrm{dom}(\mu)$$

while moreover $\mu(E(V)) \subset U$.

Proof. Use the same technique as in the proof of Lemma 2.2.

Corollary 2.4. *Let X be a space and let μ be a local mixer for X. There exists an open cover $\mathcal{V}$ of X such that for all $V \in \mathcal{V}$ we have that*

$$(V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V) \subset \mathrm{dom}(\mu).$$

This leads us to our first nontrivial result.

Theorem 2.5. *Let X be a space having an open cover by path-connected sets. If X has a (local) mixer, then X is* (LC^∞) C^∞.

Proof. We will first show that X is locally path-connected. To this end, let $x \in X$ and let U be a neighborhood of x in X. Let V be a path-connected neighborhood of x. In addition, let μ be a (local) mixer for X. By Lemma 2.3 there exists a neighborhood W of x such that

$$\mu((W \times W \times X) \cup (W \times X \times W) \cup (X \times W \times W)) \subset U \cap V. \tag{2}$$

Take $a, b \in W$ and let $f : I \to V$ be a path with $f(0) = a$ and $f(1) = b$. Define $g : I \to X$ by

$$g(t) = \mu(a, b, f(t)).$$

Since $g(0) = \mu(a, b, f(0)) = \mu(a, b, a) = a$ and $g(1) = \mu(a, b, f(1)) = \mu(a, b, b) = b$, g is a path connecting a and b. In addition, by (2), $g(I) \subset U$. This proves that X is locally path-connected. That X is LC^∞, and C^∞ in case $\mathrm{dom}(\mu) = X^3$, can now be proved in precisely the same way as in [5, Theorem 1.3].

We will now show that each A(N)R has a (local) mixer (even a symmetric one). In the compact case this is easy [5]; the general result needs some justification.

Let X be a space and let d be a metric for X. Without loss of generality, $\mathrm{diam}(X) \leqslant 1$. Let Z be the set of all bounded continuous real valued functions on X and put, as usual,

$$\rho(f_1, f_2) = \sup_{x \in X} |f_1(x) - f_2(x)|.$$

It is well known that Z is a normed linear space and that the function $\varphi : X \to Z$ defined by

$$\varphi(x)(y) = d(x, y)$$

is an isometry (see [1, p. 79]).

Define a function $\mu : Z^3 \to Z$ by

$$\mu(f_1, f_2, f_3)(x) = \text{the middle one of } f_1(x), f_2(x) \text{ and } f_3(x).$$

It is clear that μ is well defined and continuous.

Lemma 2.6. *μ is a symmetric mixer.*

Proof. Clearly, μ is symmetric. Suppose that $\lim_{n\to\infty} f_n = g$, $\lim_{n\to\infty} h_n = g$ and take $k_n \in Z$ $(n \in \mathbb{N})$ arbitrarily. We have to show that

$$\lim_{n\to\infty} \mu(f_n, h_n, k_n) = g. \tag{3}$$

Let $\varepsilon > 0$ and find $m \in \mathbb{N}$ so that $\rho(f_n, g) < \varepsilon$ and $\rho(h_n, g) < \varepsilon$ for all $n \geqslant m$. Take $x \in X$. Without loss of generality assume that $f_n(x) \leqslant h_n(x)$. Since $\mu(f_n, h_n, k_n)(x) \in [f_n(x), h_n(x)]$ it is clear that

$$|g(x) - \mu(f_n, h_n, k_n)(x)| < 2\varepsilon.$$

Since x was arbitrary, $\rho(g, \mu(f_n, h_n, k_n)) \leqslant 2\varepsilon$ for all $n \geqslant m$. This proves (3).

Define $Y = \mu(\varphi(X)^3)$.

Lemma 2.7. *$\varphi(X)$ is closed in Y.*

Proof. First observe that, since μ is a mixer, $\varphi(X) \subset Y$. Now take $x_n \in X$ and assume that $\lim_{n\to\infty} \varphi(x_n) = f \in Y$. Assume that $f \notin \varphi(X)$. We will derive a contradiction. There are points $p_0, p_1, p_2 \in X$ such that

$$f = \mu(\varphi(p_0), \varphi(p_1), \varphi(p_2)).$$

Since $f \notin \varphi(X)$ and since μ is a mixer, $p_i \neq p_j$ whenever $i \neq j$. Choose $\delta > 0$ such that whenever $x \in X$ then there exist two distinct $i, j \in \{0, 1, 2\}$ with $d(x, p_i) > \delta$ and $d(x, p_j) > \delta$. Let $m \in \mathbb{N}$ be such that

$$\rho(\varphi(x_n), f) < \delta$$

for all $n \geqslant m$. Without loss of generality, $d(x_m, p_1) \geqslant d(x_m, p_0) > \delta$. Since $\rho(\varphi(x_m), f) < \delta$ it follows that

$$|\varphi(x_m)(x_m) - f(x_m)| < \delta,$$

and consequently, $|f(x_m)| < \delta$. By definition of μ,

$$f(x_m) = \text{middle one of } d(p_0, x_m), d(p_1, x_m) \text{ and } d(p_2, x_m).$$

Since $f(x_m) \in [d(x_m, p_0), d(x_m, p_1)]$ we conclude that $f(x_m) > \delta$. This is a contradiction.

As is well known, $\varphi(X)$ is closed in its convex hull. Simple examples show that $\mu(\varphi(X)^3)$ need not be contained in the convex hull of $\varphi(X)$. This explains why we have to obtain Lemma 2.7.

Theorem 2.8. *Let X be an $A(N)R$. Then X has a symmetric (local) mixer.*

Proof. Define φ, μ, Z and Y as above. Since X is an ANR, there is a neighborhood U of $\varphi(X)$ in Y and a retraction $r: U \to \varphi(X)$. Let $V = \mu^{-1}(U) \cap \varphi(X)^3$. Notice that V is a neighborhood of $\Delta(\varphi(X))$ in $\varphi(X)^3$.

Define a function $\xi: V \to \varphi(X)$ by $\xi(f, g, h) = r(\mu(f, g, h))$. Since μ is a mixer and since r is continuous, ξ is a local mixer. In the same way it is easily seen that each AR has a mixer.

Corollary 2.9. *Let X be a finite dimensional space. If X has an open cover by path-connected sets, then X is an ANR iff X has a local mixer. In addition, if X is path-connected, then X is an AR iff X has a mixer.*

Proof. This is a direct consequence of Theorems 2.5 and 2.8 and Dugundji [2].

A *local equiconnecting function* for a space Y is a map $\lambda: U \times I \to Y$ where U is a neighborhood of the diagonal in $Y \times Y$, such that $\lambda(y_0, y_1, i) = y_i$ ($i \in \{0, 1\}$), and $\lambda(y, y, t) = y$ for every $y_0, y_1, y \in Y$, $t \in I$. An equiconnecting function for a space Y is a local equiconnecting function the domain of which is $Y \times Y \times I$. We say that Y is EC (LEC) if it admits an equiconnecting function (a local equiconnecting function).

3. The main result

In this section we present our main result that each space X with a local mixer, having an open cover by sets contractible within X, is LEC. Let X be a space with local mixer μ. If A, B are subsets of X, then we write $A \subset_\mu B$ provided that

(a) $(A \times A \times X) \cup (A \times X \times A) \cup (X \times A \times A) \subset \mathrm{dom}(\mu)$;

(b) $\mu((A \times A \times X) \cup (A \times X \times A) \cup (X \times A \times A)) \subset B$.

If $\mathcal{U}$ and $\mathcal{V}$ are coverings of (subspaces of) X, then we write $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{U}$ refines $\mathcal{V}$ (i.e. if each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$). We shall write $\mathcal{U} \leq_\mu \mathcal{V}$ if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ with $U \subset_\mu V$. We say that $\mathcal{U}$ is a *μ-refinement* of $\mathcal{V}$. By Lemma 2.3 for any open cover $\mathcal{U}$ of an open subset of X there exists an open cover $\mathcal{V}$ with $\mathcal{V} \leq_\mu \mathcal{U}$.

We now come to our main result.

Theorem 3.1. *Let X be a separable metric space having an open cover by sets contractible within X. If X has a local mixer, then X is* LEC.

Proof. Let μ be a local mixer of X, and let $\mathcal{U}_0$ be an open cover of X such that for each $U \in \mathcal{U}_0$ there is a contraction

$$F_U: U \times [0, 1] \to X$$

of U onto some point $x_U \in X$. Let $\mathcal{U}$ be an arbitrary open cover of X. Then there clearly exists an open cover $\mathcal{V} = \{V(i): i = 1, 2, \ldots\}$ of X which is star finite and countable, Hanner [4], and such that

$$\bar{\mathcal{V}} = \{\bar{V}: V \in \mathcal{V}\}$$

is a common refinement of both $\mathcal{U}$ and $\mathcal{U}_0$. We also assume that $\overline{V(i)} \neq \overline{V(j)}$ if $i \neq j$ and that $\bar{\mathcal{V}}$ is star finite.

For each $i \in \mathbb{N}$ we can find an infinite sequence $(V(i, n))_{n=0}^{\infty}$ of open sets with the following properties

$$\overline{V(i)} \subset V(i, 0) \subset \overline{V(i, 0)} \subset \cdots \subset V(i, n) \subset \overline{V(i, n)} \subset V(i, n+1) \subset \cdots; \tag{1}$$

$$\overline{V(i)} \cap \overline{V(j)} = \emptyset \Rightarrow \forall\, n \geq 0: V(i, n) \cap V(j, n) = \emptyset; \tag{2}$$

$$\exists\, U \in \mathcal{U}\ \exists\, U_0 \in \mathcal{U}_0: \bigcup_{n=0}^{\infty} V(i, n) \subset U \cap U_0. \tag{3}$$

Let n_i be equal to

$$1 + \text{number of } V \in \mathcal{V} \text{ with } \bar{V} \cap \overline{V(i)} \neq \emptyset.$$

For each $1 \leq k \leq n_i$ let $\mathcal{W}_k(i)$ be an open cover of $V(i)$ such that

$$\mathcal{W}_{n_i}(i) \leq_\mu \mathcal{W}_{n_i - 1}(i) \leq_\mu \cdots \leq_\mu \mathcal{W}_1(i) \leq_\mu \{V(i)\}; \tag{4}$$

$$\mathcal{W}_k(i) \text{ is point finite;} \tag{5}$$

$$\text{if } W_k \in \mathcal{W}_k(i) \text{ meets } V(j, n), \text{ then } W_k \subset V(j, n+1). \tag{6}$$

Let $\alpha: \mathbb{N} \to \mathbb{N}$ be a function such that $\alpha(i) \leq n_i$ for each $i \in \mathbb{N}$. For each $x \in X$, we put $W(\alpha, x)$ equal to the intersection of all sets $W_{\alpha(i)} \in \mathcal{W}_{\alpha(i)}(i)$, $i \in \mathbb{N}$, with the property that there is a sequence of type

$$x \in W_{n_i} \subset_\mu W_{n_i - 1} \subset_\mu \cdots \subset_\mu W_{\alpha(i)}$$

with $W_k \in \mathcal{W}_k(i)$. Notice that $W(\alpha, x)$ is an open set, since $\mathcal{V}$ and $\mathcal{W}_{n_i}$ are point finite. This defines an open cover $\mathcal{W}(\alpha) = \{W(\alpha, x): x \in X\}$, which clearly is a μ-refinement of $\mathcal{V}$.

The equations

$$\alpha_0(i) = n_i \quad \text{for each } i \in \mathbb{N}; \tag{7}$$

$$\alpha_{k+1}(i) = \alpha_k(i) \quad \text{if } \overline{V(k+1)} \cap \overline{V(i)} = \emptyset; \tag{8}$$

$$\alpha_{k+1}(i) = \alpha_k(i) - 1 \quad \text{if } \overline{V(k+1)} \cap \overline{V(i)} \neq \emptyset \tag{9}$$

determine a sequence of functions $(\alpha_k : \mathbb{N} \to \mathbb{N})_{k=0}^{\infty}$. The only nontrivial fact to verify is that $\operatorname{dom}(\alpha_k) = \mathbb{N}$ for each $k \geq 0$. Let us assume that for some $k \geq 0$, $\alpha_k(i) = 1$. We claim that $\alpha_{k+1}(i) = 1$. Indeed, among $\overline{V(1)}, \ldots, \overline{V(k)}$ there exist already $n_i - 1$ sets meeting $\overline{V(i)}$. By definition of n_i, $\overline{V(k+1)} \cap \overline{V(i)} = \emptyset$, whence $\alpha_{k+1}(i) = \alpha_k(i) = 1$ (by (8)).

Put $\mathcal{W}_0 = \mathcal{W}(\alpha_0)$. We will show that for each space Y and for any two $\mathcal{W}_0$-close mappings $f_0, g_0 : Y \to X$ (i.e. for each $y \in Y$ there is a $W \in \mathcal{W}_0$ with $\{f_0(y), g_0(y)\} \subset W$) there is a $\mathcal{U}$-small homotopy joining f_0 and g_0. This obviously implies that X is LEC.

Recall that a homotopy $F : Y \times [0, 1] \to X$ is *$\mathcal{U}$-small* if for each $y \in Y$ there is a $U \in \mathcal{U}$ with $F(\{y\} \times [0, 1]) \subset U$. We add the following definition to this: two homotopies $F, G : Y \times [0, 1] \to X$ are called *jointly $\mathcal{U}$-small* if for each $y \in Y$ there is a $U \in \mathcal{U}$ with

$$F(\{y\} \times [0, 1]) \cup G(\{y\} \times [0, 1]) \subset U.$$

For each $i \in \mathbb{N}$, let $A(i) = f_0^{-1}(\overline{V(i)}) \cap g_0^{-1}(\overline{V(i)})$. Since $\mathcal{W}_0 \leqslant \bar{\mathcal{V}}$, and since f_0 and g_0 are $\mathcal{W}_0$-close, it follows that $\{A(i) : i \in \mathbb{N}\}$ is a closed cover of Y, which is locally finite since $\bar{\mathcal{V}}$ is star-finite. In order to simplify our inductive construction of a homotopy $f_0 \simeq g_0$ let us define $f_{-1} = f_0$, $g_{-1} = g_0$, $V(0) = \emptyset$, $A(0) = \emptyset$, and, $F_0 : f_{-1} \simeq f_0$, $G_0 : g_{-1} \simeq g_0$ are constant homotopies.

We will now construct two sequences of maps

$$(f_i : Y \to X)_{i=0}^{\infty}, \quad (g_i : Y \to X)_{i=0}^{\infty}$$

and two sequences of homotopies

$$(F_i : f_{i-1} \simeq f_i)_{i=1}^{\infty}, \quad (G_i : g_{i-1} \simeq g_i)_{i=0}^{\infty}$$

with the following properties: for each $i \geqslant 0$

[1, i] F_i and G_i are jointly $\mathcal{W}(\alpha_i)$-small;

[2, i] if $y \in \bigcup_{j \leqslant i} A(j)$, then $f_i(y) = g_i(y)$;

[3, i] if $y \in A(j)$, then $F_i(\{y\} \times [0, 1]) \cup G_i(\{y\} \times [0, 1]) \subset V(j, i)$;

[4, i] if $y \in A(j)$, and if either $\overline{V(j)} \cap \overline{V(i)} = \emptyset$ or $j < i$, then F_i and G_i are constant at y.

These conditions are trivially true for $i = 0$. Assume these mappings have been properly constructed up to $i \geqslant 0$. By (1), we can fix a Urysohn map $h_{i+1} : X \to [0, 1]$ which is 1 on $V(i+1, i)$ and 0 outside $V(i+1, i+1)$. Fix $U \in \mathcal{U}_0$ with $\bigcup_{n=0}^{\infty} V(i+1, n) \subset U$, cf. (3), and define $H'_{i+1} : X \times [0, 1] \to X$ by

$$H'_{i+1}(x, t) = \begin{cases} F_U(x, h_{i+1}(x) \cdot t) & \text{if } x \in V(i+1, i+1), \\ x & \text{otherwise.} \end{cases}$$

H'_{i+1} is obviously well defined and continuous. Then define

$$F_{i+1}(y, t) = \mu(f_i(y), g_i(y), H'_{i+1}(f_i(y), t)) \quad \text{and} \quad f_{i+1} = F_{i+1}(-, 1),$$

$$G_{i+1}(y, t) = \mu(f_i(y), g_i(y), H'_{i+1}(g_i(y), t)) \quad \text{and} \quad g_{i+1} = G_{i+1}(-, 1).$$

Since F_i and G_i are jointly $\mathcal{W}(\alpha_i)$-small, we find that f_i and g_i are $\mathcal{W}(\alpha_i)$-close.

If $y \in Y$ is such that $\{f_i(y), g_i(y)\}$ is contained is some member of $\mathcal{W}(\alpha_i)$, then, since $\mathcal{W}(\alpha_i) \leqslant_\mu \mathcal{V}$, we find that

$$\{f_i(y)\} \times \{g_i(y)\} \times X \subset \operatorname{dom}(\mu).$$

This implies that F_{i+1} and G_{i+1} are well defined and continuous.

Verifications of $[1, i+1]$ *and of* $[4, i+1]$. Let $y \in Y$, and assume first that at least one of $f_i(y)$, $g_i(y)$ is in $V(i+1, i+1)$. As f_i and g_i are $\mathscr{W}(\alpha_i)$-close, there exists a $u \in X$ such that for each sequence of type

$$u \in W_{n_k} \subset_\mu W_{n_k-1} \subset_\mu \cdots \subset_\mu W_{\alpha_i(k)} \quad (W_l \in \mathscr{W}_l(k), k \in \mathbb{N}),$$

$f_i(y)$ and $g_i(y)$ are both in $W_{\alpha_i(k)}$.

Consider a sequence of type

$$u \in W_{n_k} \subset_\mu W_{n_k-1} \subset_\mu \cdots \subset_\mu W_{\alpha_{i+1}(k)}. \tag{10}$$

As $\alpha_{i+1}(k) \leqslant \alpha_i(k)$, we find that $f_i(y)$ and $g_i(y)$ are also in $W_{\alpha_{i+1}(k)}$, whence

$$W_{\alpha_{i+1}(k)} \cap V(i+1, i+1) \neq \emptyset.$$

But $W_{\alpha_{i+1}(k)} \subset V(k) \subset V(k, 0) \subset V(k, i+1)$, whence by (2),

$$\overline{V(k)} \cap \overline{V(i+1)} \neq \emptyset,$$

and consequently $\alpha_{i+1}(k) = \alpha_i(k) - 1$. In the above sequence (10), the set preceding $W_{\alpha_{i+1}(k)}$ is therefore of type $W_{\alpha_i(k)} \in \mathscr{W}_{\alpha_i(k)}(k)$ and it contain $f_i(y)$ and $g_i(y)$ by assumption. As μ maps $W^2_{\alpha_i(k)} \times X$ into $W_{\alpha_{i+1}(k)}$, we find that

$$F_{i+1}(\{y\} \times [0, 1]) \cap G_{i+1}(\{y\} \times [0, 1]) \subset W_{\alpha_{i+1}}(k).$$

This proves that F_{i+1} and G_{i+1} are jointly $\mathscr{W}(\alpha_{i+1})$-small at this y. Assume next that neither $f_i(y)$ nor $g_i(y)$ are in $V(i+1, i+1)$. Then, by construction

$$H'_{i+1}(f_i(y), t) = f_i(y), \quad H'_{i+1}(g_i(y), t) = g_i(y),$$

and the homotopies F_{i+1}, G_{i+1} are constant at y. As f_i and g_i are $\mathscr{W}(\alpha_i)$-close, and since $\mathscr{W}(\alpha_i) \leqslant \mathscr{W}(\alpha_{i+1})$, we find again that F_{i+1} and G_{i+1} are jointly small of order $\mathscr{W}(\alpha_{i+1})$ at y, completing the proof of $[1, i+1]$.

If $y \in A(j)$ and if $\overline{V(j)} \cap \overline{V(i+1)} = \emptyset$, then by $[4, i]$, $f_i(y)$ and $g_i(y)$ are in $V(j, i)$ and hence in $V(j, i+1)$, whence by (1) and (2), $f_i(y)$ and $g_i(y)$ are not in $V(i+1, i+1)$. Applying the above argument, the homotopies F_{i+1} and G_{i+1} are constant at y.

If, on the other hand, $j < i+1$, then $j \leqslant i$ and $f_i(y) = g_i(y)$ by $[2, i]$. Again, the homotopies F_{i+1} and G_{i+1} are constant at y since μ is a mixer.

Verification of $[2, i+1]$. Let $y \in \bigcup_{j \leqslant i+1} A(j)$. If $y \in A(j)$ with $j \leqslant i$, then $f_i(y) = g_i(y)$ by $[2, i]$, and by construction, $f_{i+1}(y) = g_{i+1}(y)$. If $y \in A(i+1)$, then by $[3, i]$ $f_i(y)$ and $g_i(y)$ are both in $V(i+1, i)$.

On this set, h_{i+1} equals 1, whence

$$H'_{i+1}(f_i(y), 1) = F_U(f_i(y), 1) = x_U = F_U(g_i(y), 1) = H'_{i+1}(g_i(y), 1).$$

It follows that

$$f_{i+1}(y) = \mu(f_i(y), g_i(y), x_U) = g_{i+1}(y).$$

Verification of $[3, i+1]$. Let $y \in A(j)$. Then $f_i(y)$ and $g_i(y)$ are in $V(j, i)$ by $[3, i]$. As F_{i+1} is $\mathscr{W}(\alpha_{i+1})$-small, we can find a $W_{\alpha_{i+1}(k)} \in \mathscr{W}_{\alpha_{i+1}(k)}(k)$ containing the set

$F_{i+1}(\{y\}\times[0, 1])$. Hence,

$$W_{\alpha_{i+1}}(k)\cap V(j, i)\neq\emptyset$$

and by (6),

$$F_{i+1}(\{y\}\times[0, 1])\subset W_{\alpha_{i+1}(k)}\subset V(j, i+1).$$

A similar argument works for G_{i+1}.

This completes the inductive construction of our sequences of maps, and we now proceed with the construction of a suitable homotopy between f_0 and g_0. First, define

$$H': Y\times[0, 1]\rightarrow X$$

as follows: if $y\in A(j)$ and if $t<1$, then fix an $n\in\mathbb{N}$ with

$$\frac{n-1}{n}\leqslant t\leqslant\frac{n}{n+1}.$$

Then we put

$$H'(y, t)=F_n\left(y, n(n+1)\left(t-\frac{n-1}{n}\right)\right).$$

For $t=1$, we put

$$H'(y, 1)=f_j(y).$$

If $y\in A(j)\cap A(k)$ with e.g. $j<k$, then by $[4, j+1], \ldots, [4, k]$, we find that the homotopies $F_{j+1}, \ldots, F_k$ are constant, and hence $f_j(y)=f_k(y)$. This shows that H' is well defined on Y.

H' is continuous on $A(j)\times[0, 1]$. Indeed, we only need to look after $t=1$. Let O be a neighborhood of $H'(y, 1)=f_j(y)$. Let P be a neighborhood of y in $A(j)$ with $f_j(P)\subset O$. Then for all $t>j/(j+1)$ and for all $y'\in P$, we find that

$$H'(y', t)=f_j(y')\in O,$$

using the conditions $[4, l]$, $l\geqslant j+1$. As the family of all $A(j)$ is a locally finite closed cover of Y, we find that H' is continuous.

In a similar way we construct a homotopy

$$H'': Y\times[0, 1]\rightarrow X$$

by $H''(y, t)=G_n(y, n(n+1)(t-(n-1)/n))$ if $(n-1)/n\leqslant t\leqslant n/(n+1)$, and $H''(y, 1)=g_j(y)$ if $y\in A_j$.

It is easy to see from $[2, n]$, $n\in\mathbb{N}$, that $H'(-, 1)=H''(-, 1)$. We then construct the desired homotopy

$$H: Y\times[0, 1]\rightarrow X$$

by putting

$$H(y,t)=H'(x,2t) \qquad \text{if } 0\leq t\leq \tfrac{1}{2},$$

$$H(y,t)=H'(x,2-2t) \quad \text{if } \tfrac{1}{2}\leq t\leq 1.$$

As

$$H(\{y\}\times[0,1])=\bigcup_{n=1}^{\infty} F_n(\{y\}\times[0,1])\cup\bigcup_{n=1}^{\infty} G_n(\{y\}\times[0,1])$$

we find by [3, n], $n\in\mathbb{N}$ that (assuming $y\in A(j)$)

$$H(\{y\}\times[0,1])\subset\bigcup_{n=0}^{\infty} V(j,n),$$

whereas by (3), there is a $U\in\mathcal{U}$ containing the right hand set. This shows that H is a $\mathcal{U}$-small homotopy joining f_0 with g_0.

4. Concluding remarks

The problem whether the ANR property is equivalent to the existence of a local mixer is still far from being solved. The construction of eventual counterexamples, in view of Theorem 3.1, promises to be rather difficult. 'Classical' counterexamples in ANR theory seem to be unuseful, cf. e.g. Borsuk's example of a (contractible) locally contractible metric continuum which is not an (AR) ANR [1].

By [3], this space does not admit a local equiconnecting function, and hence by Theorem 3.1, it cannot carry a local mixer.

As is clear from Section 2, certain spaces carry a very 'natural' mixer. We don't know whether every Banach space has a 'natural' mixer (of course, by Theorem 2.8, every Banach space has a mixer).

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