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SUBBASE CHARACTERIZATIONS OF SUBSPACES OF COMPACT TREES

Jan van MILL and Evert WATTEL
Department of Mathematics, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

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In this paper we apply the methods of supercompactifications and normal subbases to characterize subspaces of compact treelike spaces. This characterization is related to the subbase characterizations of ordered spaces, of trees and of normally supercompact spaces described in [5, 1, 9, 10, 14].

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1. Basic definitions

In this paper all subbases are assumed to be subbases for the *closed* sets of a topological space. All spaces under consideration are assumed to be T_1.

1.1. Definition. A topological space is called a *treelike space* [6, 12] provided that T is connected and for every two points x and y in T there exists a point z and a partition $X \cup Y$ of $T\backslash\{z\}$ such that X and Y are clopen in $T\backslash\{z\}$ with $x \in X$ and $y \in Y$. A treelike space is called a *tree* whenever it is "rim-finite", i.e. every point has a local base consisting of sets with finite boundary.

In this paper we only consider subspaces of compact treelike spaces which are clearly rim-finite.

1.2. Definition. A point z in a tree T is called a *cutpoint* whenever $T\backslash\{z\}$ is disconnected. According to Kok [7] the components of $T\backslash\{z\}$ are all open. In this paper components of $T\backslash\{z\}$ will be called *cutpoint components* of T.

Since a compact tree is a compact Hausdorff space [7] it follows easily that the collection of all cutpoint components is an open subbase for T, and in this paper we are especially interested in subcollections of the collection of all complements of cutpoint components which are closed subbases for T.

1.3. Definition. A partial ordering $<$ on a set X is called a *tree ordering* when there exists a unique minimal element r, which is called the *root*, and has the property that if $x, y \in X$, $x \not< y$ and $y \not< x$, then for each $z \in X$ we have either $x \not< z$ or $y \not< z$.

1.4. Definition. Let $\mathcal{S}$ be a collection of subsets of a topological space X. Then $\mathcal{S}$ is called a T_1 *collection* [13] whenever for each member $S \in \mathcal{S}$ and each point $X \in x \setminus S$ there exists a member $R \in \mathcal{S}$ with $x \in R$ and $R \cap S = \emptyset$.

$\mathcal{S}$ is called *normal* [2] if for each two disjoint members S_1 and S_2 in $\mathcal{S}$ there exists a pair of members R_1 and R_2 in $\mathcal{S}$ such that $R_1 \cup R_2 = X$, $R_1 \cap S_2 = \emptyset$, and $R_2 \cap S_1 = \emptyset$. The members S_1 and S_2 are said to be *screened* by R_1 and R_2 and those sets constitute a *screening* of S_1 and S_2.

$\mathcal{S}$ is called *linked* [3] when every two members of $\mathcal{S}$ have a non-empty intersection.

$\mathcal{S}$ is called *binary* [4] if every linked system in $\mathcal{S}$ has a non-empty intersection. A topological space which admits a closed binary subbase is called *supercompact*.

$\mathcal{S}$ is called *connected* [9] iff no two non-empty disjoint members of $\mathcal{S}$ cover X.

2. The main theorem

In this section we give a characterization of those spaces which are embeddable in a compact tree. Our characterization is in terms of subbases. Our method of proof is the following. We enlarge a given space X to a space $\lambda(X, \mathcal{S})$, which is called the superextension of X relative to the subbase $\mathcal{S}$ (for definitions, see below) and then use a characterization of compact trees in [9].

Let $\mathcal{S}$ be a subbase for a space X. The superextension $\lambda(X, \mathcal{S})$ has as underlying set the set of all maximal linked systems in $\mathcal{S}$ with topology generated by taking the collection

$$\mathcal{S}^+ = \{S^+ \mid S \in \mathcal{S}\}$$

where

$$S^+ = \{\mathfrak{m} \mid \mathfrak{m} \in \lambda(X, \mathcal{S}) \ \& \ S \in \mathfrak{m}\},$$

as a (closed) subbase. The following facts are well known and easy to prove:

- $\mathcal{S}^+$ is binary, (as a consequence, $\lambda(X, \mathcal{S})$ is compact).
- If $\mathcal{S}$ is normal, then $\lambda(X, \mathcal{S})$ is Hausdorff.
- If $\mathcal{S}$ is a T_1 collection, then the function $i: X \to \lambda(X, \mathcal{S})$ defined by $i(x) = \{S \in \mathcal{S} \mid x \in S\}$ is an embedding.
- $\mathcal{S}$ is connected iff $\mathcal{S}^+$ is connected.

For details, see [13]. Superextensions were introduced by de Groot [3].

Since in our characterization we do not require the subbases to be normal or T_1, we first indicate how a subbase with certain properties can be replaced by a subbase with the same properties which in addition is normal and T_1. We then can use superextensions to obtain the desired results.

2.1. Lemma. *Let X be a Hausdorff space with a closed subbase satisfying the following condition:*

(*) *For every S and R in $\mathcal{S}$ we have that*

$$S \cap R = \emptyset \quad or \quad S \subset R \quad or \quad R \subset S \quad or \quad S \cup R = X.$$

Then there exists a T_1 normal subbase for X which satisfies condition (*).

(In certain papers in discrete mathematics a collection of sets which satisfies condition (*) is called "crossfree".)

Proof. First of all we extend $\mathcal{S}$ to a larger subbase $\mathcal{S}^{\mathrm{t}}$ by taking:

$$\mathcal{S}^{\mathrm{t}} = \mathcal{S} \cup \{\{p\} \mid p \in X\}$$

(i.e. we add all singletons to the subbase). In this case $\mathcal{S}^{\mathrm{t}}$ still satisfies (*) because $\{p\} \cap \{q\} = \emptyset$ for all $p \neq q$ and either $\{p\} \cap S = \emptyset$ or $\{p\} \subset S$ for each $S \in \mathcal{S}$. Clearly the subbase $\mathcal{S}^{\mathrm{t}}$ is a T_1 collection.

Next, for each clopen $S \in \mathcal{S}^{\mathrm{t}}$, we add the complement of S and obtain

$$\mathcal{S}^{\mathrm{n}} = \mathcal{S}^{\mathrm{t}} \cup \{X \backslash S \mid S \in \mathcal{S}^{t} \text{ and } S \text{ is clopen}\}.$$

Also $\mathcal{S}^{\mathrm{n}}$ is a T_1 collection satisfying condition (*) since if $S, R \in \mathcal{S}^{\mathrm{t}}$, then

$$S \subset R \quad \text{implies} \quad X\backslash S \supset X\backslash R \text{ and } X\backslash S \cup R = X,$$

$$R \subset S \quad \text{implies} \quad X\backslash S \subset X\backslash R \text{ and } X\backslash S \cap R = \emptyset,$$

$$R \cap S = \emptyset \quad \text{implies} \quad X\backslash S \cup X\backslash R = X \text{ and } R \subset X\backslash S,$$

$$R \cup S = X \quad \text{implies} \quad X\backslash S \cap X\backslash R = \emptyset \text{ and } X\backslash S \subset R.$$

We now show that $\mathcal{S}^{\mathrm{n}}$ not only satisfies (*) but is moreover normal.

Let R and S be two disjoint members of $\mathcal{S}^{\mathrm{n}}$. If S is clopen, then also $X\backslash S$ is in $\mathcal{S}^{\mathrm{n}}$ and we obtain a screening between S and R by S and $X\backslash S$, and the same holds for R. If neither S nor R is clopen, then we can find a point $r \in R$ and a point $s \in S$ such that $r \in \mathrm{Cl}_X(X\backslash R)$ and $s \in \mathrm{Cl}_X(X\backslash S)$.

Next we will derive a screening of $\{s\}$ and $\{r\}$ by means of two subbase members. Since X is Hausdorff we can find two basic closed subsets B_s and B_r such that $B_s \cup B_r = X$, $r \notin B_s$ and $s \notin B_r$. Moreover, B_r is a finite union of subbase members $F_{r1}, \ldots, F_{rn}$, and B_s is a finite union of $F_{s1}, \ldots, F_{sm}$.

Define

$$\mathcal{F} = \{F_{si}\} \cup \{F_{rj}\} \text{ and } \mathcal{F}_s = \{F_{sj} \mid s \in F_{sj}\},$$

then for F_{si} and $F_{sj} \in \mathscr{F}_s$ we have that

$$s \in F_{si} \cap F_{sj} \quad \text{and} \quad r \notin F_{si} \cup F_{sj},$$

hence either $F_{si} \subset F_{sj}$ or $F_{sj} \subset F_{si}$ and so there exists a largest member $F_s = \bigcup \mathscr{F}_s \in \mathscr{F}$. In the same way there is a maximal F_r in $\mathscr{F}$ which contains r. We have now two cases. If $F_s \cup F_r = X$, then we have obtained our screening with two members of $\mathscr{S}$.

In the other case we can find a point x in $X \backslash (F_s \cup F_r)$. Let F_x be the maximal member of $\mathscr{F}$ containing x. Since

$$r \notin F_x \cup F_s, \qquad s \in F_s \backslash F_x \quad \text{and} \quad x \in F_x \backslash F_s,$$

we have $F_x \cap F_s = \emptyset$ and similarly $F_x \cap F_r = \emptyset$ and $F_s \cap F_r = \emptyset$. Consequently, we obtain a partition of the space into three disjoint closed parts: F_s, F_r and $\bigcup \{F_x \mid x \notin F_s \cup F_r\}$. (The last collection is closed since it is the union of a finite collection because $\mathscr{F}$ is finite.) This means that F_s is clopen and $X \backslash F_s$ is in $\mathscr{S}^{\mathrm{n}}$.

Anyway we obtain a screening of s and r by means of two subbase members, call them F'_s and F'_r. Now S does not contain a neighborhood of s and F'_r is closed and does not contain s and hence $S \cup F'_r \neq X$. Moreover, $s \in S \backslash F'_r$ and $r \in F'_r \backslash S$ and therefore $F'_r \cap S = \emptyset$ and similarly $F'_s \cap R = \emptyset$. Since $F'_s \cup F'_r = X$ we have $R \subset F'_r$ and $S \subset F'_s$ and we obtained a screening of R and S.

2.2. Remark. In the previous lemma the Hausdorff property cannot be omitted since in an infinite space with the cofinite topology the collection of all singletons is a T_1 subbase satisfying (*), but it cannot have a T_1 normal subbase since a space with a T_1 normal subbase is completely regular (cf. [4]).

2.3. Lemma. *Let X be a space and let $\mathscr{S}$ be a closed subbase of X with the following properties*:

(a) *$\mathscr{S}$ is a T_1 collection.*

(b) *$\mathscr{S}$ is normal.*

(*) *For every S and R in $\mathscr{S}$ we have that*

$$S \cap R = \emptyset \quad \text{or} \quad S \subset R \quad \text{or} \quad R \subset S \quad \text{or} \quad S \cup R = X.$$

Then X can be embedded in a compact tree T.

Proof. *Case* 1. *X is compact and connected.*

If we prove that $\mathscr{S}$ is binary and connected, then we can conclude that X is a tree from [9, Theorem 4.3] (cf. [8, Theorem 1.3.21]). That $\mathscr{S}$ is connected follows from the connectivity of X itself. So we only have to show that $\mathscr{S}$ is binary, i.e. every linked system in $\mathscr{S}$ has a non-empty intersection.

Suppose not. Let $\mathfrak{m}$ be a maximal linked system in $\mathscr{S}$ which is not centered and suppose that $M_1, \ldots, M_n$ is a minimal subcollection of $\mathfrak{m}$ which has an empty intersection. Then $M_i \cap M_j \neq \emptyset$ and M_i is not contained in M_j or conversely. So $M_i \cup M_j = X$. Now $M_i \cup M_n = X$ for $0 < i < n$ and hence $M_n \cup \langle \bigcap_{0<i<n} M_i \rangle = X$.

Moreover, $M_n \cap \langle \bigcap_{0<i<n} M_i \rangle = \emptyset$ which implies that M_n is clopen, contradicting that X is connected.

Case 2. $\mathcal{S}$ *is a connected subbase satisfying* (a), (b) and (∗).

In this case the subbase $\mathcal{S}^+$ for $\lambda(X, \mathcal{S})$ is binary and also satisfies the requirements (a), (b) and (∗); we conclude from [9, Theorem 4.3], that X is a subspace of the treelike space $\lambda(X, \mathcal{S})$.

Case 3. $\mathcal{S}$ *is not connected.*

In this case we extend X to a space Y and $\mathcal{S}$ to a subbase $\mathcal{S}^\sim$ in such a way that $\mathcal{S}^\sim$ is a connected subbase for Y, and since $\lambda(Y, \mathcal{S}^\sim)$ contains X as a subspace we have that X is a subspace of a compact tree.

Let $\{\langle H_\alpha, K_\alpha\rangle \mid \alpha \in A\}$ enumerate all the pairs $\langle H, K\rangle \in \mathcal{S} * \mathcal{S}$ such that $K = X\backslash H$ (in such a way that $\langle H, K\rangle$ and $\langle K, H\rangle$ do not both occur). Let $\mathcal{H} = \{H_\alpha \mid \alpha \in A\}$ and $\mathcal{K} = \{K_\alpha \mid \alpha \in A\}$. Define

$$Y = X \cup (I * A),$$

where I is the open unit interval $(0, 1)$. For $\alpha \in A$ we define

$$A_0(\alpha) = \{\beta \in A\backslash\{\alpha\} \mid H_\beta \subset H_\alpha \text{ or } K_\beta \subset H_\alpha\},$$

and

$$A_1(\alpha) = \{\beta \in A\backslash\{\alpha\} \mid H_\beta \supset H_\alpha \text{ or } K_\beta \supset H_\alpha\},$$

Thus $A = A_0(\alpha) \cup A_1(\alpha) \cup \{\alpha\}$. For $\alpha \in A$ define

$$H_\alpha^\sim = H_\alpha \cup (I * A_0(\alpha)), \qquad K_\alpha^\sim = K_\alpha \cup (I * A_1(\alpha)).$$

Then for $r \in I$ we define

$$H_{\alpha r}^\sim = H_\alpha^\sim \cup ((0, r] * \{\alpha\}) \quad \text{and} \quad K_{\alpha r}^\sim = K_\alpha^\sim \cup ([r, 1) * \{\alpha\}).$$

For each $S \in \mathcal{S}\backslash(\mathcal{H} \cup \mathcal{K})$, let

$$A(S) = \{\alpha \in A \mid H_\alpha \subset S \text{ or } K_\alpha \subset S\};$$

then let

$$S^\sim = S \cup (I * A(S)).$$

Finally, set

$$\mathcal{S}^\sim = \{S^\sim \mid S \in \mathcal{S}\backslash(\mathcal{H} \cup \mathcal{K})\} \cup \{H_{\alpha r}^\sim \mid \langle r, \alpha\rangle \in I * A\} \cup \{K_{\alpha r}^\sim \mid \langle r, \alpha\rangle \in I * A\}.$$

The conditions (a), (b) and (∗) are easily verified for the space Y with subbase $\mathcal{S}^\sim$ from the related conditions for X and $\mathcal{S}$. Moreover, $\mathcal{S}^\sim$ is a connected subbase.

2.4. Remark. Note that we can define a tree ordering on X if we choose an arbitrary member $r \in X$ to be the root and if we put $x < y$ iff every member of $\mathcal{S}$ containing y and r also contains x. This procedure yields a tree ordering which is compatible with the tree structure of the superspace T in Lemma 2.3 whenever $\mathcal{S}$ satisfies condition (∗).

2.5. Theorem. *Let X be a Hausdorff space. Then the following requirements are equivalent*:

(a) *X can be embedded in a compact tree.*

(b) *X has a subbase satisfying* (a), (b) *and* (∗) *of Lemma* 2.3.

(c) *X has a subbase satisfying* (∗) *of Lemma* 2.1.

Proof. (b) ⇒ (a). This case is proved in Lemma 2.3.

(a) ⇒ (b). This follows if we notice that the collection of all complements of cutpoint components of a compact tree is a subbase for that tree which satisfies the requirements of Lemma 2.3 (cf. [9]) and these requirements stay valid if we restrict ourselves to a subspace.

(b) ⇒ (c) This follows from the fact that a T_1 space with a T_1 normal subbase is completely regular and hence Hausdorff, cf. [4].

(c) ⇒ (b) This case is proved in Lemma 2.1.

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