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SIXTEEN TOPOLOGICAL TYPES IN $\beta\omega-\omega$

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In this paper we describe sixteen topological types in $\beta\omega$–ω. Among others, we show that there is a weak P-point $x\in\beta\omega-\omega$ which is a limit point of some ccc subset of $\beta\omega-\omega-\{x\}$ and that there is a point $y\in\beta\omega-\omega$ which is a limit point of some countable subset of $\beta\omega-\omega-\{y\}$ but not of any countable discrete subset of $\beta\omega-\omega-\{y\}$.

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βX independent linked family weak P-point topological type

0. Introduction

The aim of this paper is to construct sixteen distinct topological types in $\beta\omega-\omega$. We are interested in points $x\in\beta\omega-\omega$ which are a limit point of certain type of subset of $\beta\omega-\omega-\{x\}$ but not of any subset of $\beta\omega-\omega-\{x\}$ of another type. Call a space *π-homogeneous* provided that all nonempty open subspaces have the same π-weight. Let

$$A_1=\{x\in\beta\omega-\omega:\exists \text{ countable discrete } D\subset\beta\omega-\omega-\{x\} \text{ with } x\in\bar{D}\},$$

$$A_2=\{x\in\beta\omega-\omega:\exists \text{ countable } \pi\text{-homogeneous dense in itself set } D\subset\beta\omega-\omega-\{x\} \text{ of countable } \pi\text{-weight with } x\in\bar{D}\},$$

$$A_3=\{x\in\beta\omega-\omega:\exists \text{ countable } \pi\text{-homogeneous dense in itself set } D\subset\beta\omega-\omega-\{x\} \text{ of } \pi\text{-weight } \omega_1 \text{ with } x\in\bar{D}\},$$

and

$$A_4=\{x\in\beta\omega-\omega:\exists \text{ locally compact ccc nowhere separable } D\subset\beta\omega-\omega-\{x\} \text{ with } x\in\bar{D}\}.$$

Theorem 0.1. *For every subset* $F\subset\{1,2,3,4\}$, $\bigcap_{i\in F}A_i-\bigcup_{i\notin F}A_i\neq\emptyset$ *(by definition,* $\bigcap_{i\in\emptyset}A_i=\beta\omega-\omega$ *and* $\bigcup_{i\in\emptyset}A_i=\emptyset$*).*

It is known [6] that there are 2^{2^ω} types in $\beta\omega-\omega$, but none of these types is described in topological terms. It is also known [9] that there are weak P-points in $\beta\omega-\omega$, i.e. points which are not a limit point of any countable set. We heavily rely on the technique used in the proof of this result. In addition, we were forced to prove that $\omega\times 2^{\omega_1}$ has a remote point (see Section 1), in fact, we will show that any nonpseudocompact space which is a product of at most ω_1 spaces of countable π-weight has a remote point. This is not a very shocking generalization of results in [3] and [4], but is of crucial importance for the proof of our theorem. Interestingly, we also use the recent result due to Bell [2] that there is compactification $\gamma\omega$ of ω with $\gamma\omega-\omega$ ccc but not separable.

This paper is organized as follows: in Sections 1 and 2 we prove general results which have interest in their own rights and which will be the tools in proving our theorem in Section 3.

Some of the results in this paper are also to be found, in a preliminary form, in the (unpublished) reports [11, 12].

1. Remote filters

All spaces are completely regular and X^* *denotes* $\beta X-X$. If X_n $(n<\omega)$ is a sequence of spaces, then $\sum_{n<\omega}X_n$ denotes the disjoint topological sum of the X_n's. Whenever we write $\sum_{n<\omega}X_n$, for convenience we will assume that the spaces X_n are pairwise disjoint.

The aim of this section is to prove that if $X=\sum_{n<\omega}X_n$, where each X_n is a product of at most ω_1 compact spaces of countable π-weight[1], then there is a collection $\mathscr{F}$ of closed subsets of X such that

(1) if $D\subset X$ is nowhere dense, then there is some $F\in\mathscr{F}$ with $F\cap D=\emptyset$.

(2) if $\mathscr{G}\subset\mathscr{F}$ is finite, then $|\{n<\omega: X_n\cap\bigcap\mathscr{G}=\emptyset\}|<\omega$.

This implies that X has a *remote point*, i.e. there is a point $x\in X^*$ such that $x\notin \mathrm{cl}_{\beta X}D$ for any nowhere dense $D\subset X$, but also that the set of remote points of X is big enough to be manipulated later on.

A collection of sets is called *κ-centered* if every nonempty subfamily with at most κ members has nonempty intersection. A collection of closed subsets $\mathscr{A}$ of a space X is called *remote* if for each nowhere dense set $D\subset X$ there is some $A\in\mathscr{A}$ which does not intersect D. Finally, a continuous surjection $f:X\to Y$ is called *quasi-open* provided that for each nonempty open $U\subset X$ we have that $\mathrm{int}_Y f(U)\neq\emptyset$.

[1] A π-basis for a space X is a collection of nonempty open subsets $\mathscr{B}$ of X such that each nonempty open set in X contains some element of $\mathscr{B}$. The πweight, $\pi(A)$, of X is ω. min$\{|\mathscr{B}|: \mathscr{B}$ is a π-basis for $X\}$. A space X is said to be of countable π-weight in case $\pi(x)=\omega$.

Lemma 1.1. *Let X be a compact space, let $f: X \to Y$ be quasi-open and let $\mathcal{U}$ be a π-basis for X. For each n-centered $(1 < n < \omega)$ remote system $\mathcal{A}$ of Y and for each nowhere dense $D \subset X$ there is a finite $\mathcal{F} \subset \mathcal{U}$ with $(\bigcup \mathcal{F})^- \subset X - D$ while in addition $\{f^{-1}(A): A \in \mathcal{A}\} \cup \{\bigcup \mathcal{F}\}$ is n-centered.*

Proof. Let $\mathcal{E} = \{E \in \mathcal{U}: \bar{E} \cap D = \emptyset\}$. For each $E \in \mathcal{E}$ put $U(E) = \text{int}_Y f(E)$. Since $\bigcup \mathcal{E}$ is a dense in X and since f is quasi-open it easily follows that $\bigcup_{E \in \mathcal{E}} U(E)$ is dense in Y, or, equivalently, $Y - \bigcup_{E \in \mathcal{E}} U(E)$ is nowhere dense. Since $\mathcal{A}$ is remote, there is some $A \in \mathcal{A}$ which is contained in $\bigcup_{E \in \mathcal{E}} U(E)$. By the compactness of A there is a finite $\mathcal{F} \subset \mathcal{E}$ such that $A \subset \bigcup_{E \in \mathcal{F}} U(E)$. We claim that $\mathcal{F}$ is as required. Suppose, to the contrary, that this is not true. Then there is a subfamily $\mathcal{G}$ of $\mathcal{A}$ of $n-1$ elements such that

$$\bigcup \mathcal{F} \cap \bigcap_{G \in \mathcal{G}} f^{-1}(G) = \emptyset.$$

Then $A \cap \bigcap \mathcal{G} = \emptyset$ since $A \subset \bigcup_{E \in \mathcal{F}} U(E) \subset f(\bigcup \mathcal{F})$ and

$$f(\bigcup \mathcal{F}) \cap \bigcap \mathcal{G} = \emptyset,$$

which contradicts $\mathcal{A}$ being n-centered.

Let us notice that in [3] and [4] it was shown that each space of countable π-weight has, for each $1 < n < \omega$, a remote n-centered system. The proof of the following lemma makes use of ideas in [3] and [4].

Lemma 1.2. *Let X be a compact space of countable π-weight and let $f: X \to Y$ be quasi-open. Then for each remote n-centered system $\mathcal{A}$ $(2 < n < \omega)$ for Y there is a remote $(n-1)$-centered system $\mathcal{F}$ for X such that $\{f^{-1}(A): A \in \mathcal{A}\} \subset \mathcal{F}$.*

Proof. Let $\mathcal{U}$ be a countable π-basis for X which is closed under finite unions. Recall that $\emptyset \notin \mathcal{U}$. For each $2 \leq i \leq n$ define

$$\mathcal{E}(i) = \{U \in \mathcal{U}: \{\bar{U}\} \cup \{f^{-1}(A): A \in \mathcal{A}\} \text{ is } i\text{-centered}\}.$$

Let us notice that for each nowhere dense $D \subset X$ there is some $E \in \mathcal{E}(n)$ such that $\bar{E} \subset X - D$ (Lemma 1.1). Since $f \restriction V: V \to f(V)$ is quasi-open for each regular closed set $V \subset X$ the reader can easily verify, by using Lemma 1.1, the following fact.

Fact 1. *For each $2 < j \leq n$ and $E \in \mathcal{E}(j)$ and for each nowhere dense set $D \subset X$ there is some $F \in \mathcal{E}(j-1)$ such that $\bar{F} \subset E - D$.*

Enumerate $\mathcal{E}(i)$ as $\{E_k^i: k < \omega\}$. Let $D \subset X$ be some nowhere dense set. For each $2 \leq i \leq n$ define

$$H(D, i) = \{m < \omega: \overline{E_m^i} \cap D = \emptyset\}.$$

Define integers $\kappa(D,m)$ $(2\leqslant m\leqslant n)$ as follows

$$\kappa(D,n)=\min H(D,n),$$

and

$$\kappa(D,m)=\min\{i<\omega\colon \text{for all } s\leqslant\kappa(D,m+1) \text{ there is some } j\leqslant i \text{ with } j\in H(D,m) \text{ and } E_j^m\subset E_s^{m+1}\}.$$

Put $F(D)=\bigcup_{i=2}^{n}\bigcup\{\overline{E_j^i}\colon j\leqslant\kappa(D,i) \text{ and } j\in H(D,i)\}$. Notice that $F(D)$ is closed and does not intersect D. Let $\mathscr{D}$ denote the collection of nowhere dense subsets of X.

Fact 2. *$\{F(D)\colon D\in\mathscr{D}\}$ is $(n-1)$-centered. In fact, whenever $\mathscr{L}$ is a subfamily of $\mathscr{D}$ of cardinality e, where $1\leqslant e\leqslant n-1$, then $\bigcap_{L\in\mathscr{L}}F(L)\supset E_l^{n-e+1}$ for some $l\leqslant\max\{\kappa(L,n-e+1)\colon L\in\mathscr{L}\}$.*

The proof of this fact is by induction on e. The case $e=1$ is trivial, so assume the fact to be proven for all $1\leqslant i<j$, where $j\leqslant n-1$. Let $\mathscr{L}$ be a subfamily of $\mathscr{D}$ of cardinality j. Put

$$\kappa=\max\{\kappa(L,n-j+2)\colon L\in\mathscr{L}\}$$

and take $L_0\in\mathscr{L}$ such that $\kappa=\kappa(L_0,n-j+2)$. Define $\mathscr{L}'=\mathscr{L}-\{L_0\}$. By induction hypothesis,

$$\bigcap_{L\in\mathscr{L}'}F(L)\supset E_l^{n-j+2}$$

for some $l\leqslant\max\{\kappa(L,n-j+2)\colon L\in\mathscr{L}'\}$. Since

$$l\leqslant\max\{\kappa(L,n-j+2)\colon L\in\mathscr{L}'\}\leqslant\kappa(L_0,n-j+2)$$

there is some $i\leqslant\kappa(L_0,n-j+1)$ such that $E_i^{n-j+1}\subset E_l^{n-j+2}$ and $i\in H(D,n-j+1)$. Therefore $\bigcap_{L\in\mathscr{L}}F(L)\supset E_i^{n-j+1}$ and since $i\leqslant\kappa(L_0,n-j+1)$ this completes the induction.

Fact 3. *The family $\{F(D)\colon D\in\mathscr{D}\}\cup\{f^{-1}(A)\colon A\in\mathscr{A}\}$ is $(n-1)$-centered.*

Let $\mathscr{D}_0$ be a subfamily of $\mathscr{D}$ of cardinality e_0 and let $\mathscr{A}_0$ be a subfamily of $\mathscr{A}$ of cardinality e_1 such that $e_0+e_1=n-1$. By Fact 2 we may assume that $e_1>0$. Also, by Fact 2, $\bigcap_{\mathscr{D}\in\mathscr{D}_0}F(D)$ contains some element of $\mathscr{E}(n-e_0+1)=\mathscr{E}(e_1+2)$. We may conclude that

$$\bigcap_{D\in\mathscr{D}_0}F(D)\cap\bigcap_{A\in\mathscr{A}_0}f^{-1}(A)\neq\emptyset.$$

Now define $\mathscr{F}=\{F(D)\colon D\in\mathscr{D}\}\cup\{f^{-1}(A)\colon A\in\mathscr{A}\}$. Then $\mathscr{F}$ is as required.

A *remote filter* on a space is a closed filter which is remote. We now come to the main result in this section.

Theorem 1.3. *Let $X=\sum_{n<\omega} X_n$ where each X_n is compact and a product of at most ω_1 spaces of countable π-weight. Then there is a remote filter $\mathscr{F}$ on X such that for each $F\in\mathscr{F}$ the set $\{n<\omega: F\cap X_n=\emptyset\}$ is finite.*

Proof. Let $X_n=\prod_{\alpha<\omega_1} X^n_\alpha$. For each $\beta\leq\omega_1$ let $X^\beta_n=\prod_{\alpha<\beta} X^n_\alpha$ and if $\kappa<\mu\leq\omega_1$ let $\pi^n_{\mu\kappa}: X^\mu_n\to X^\kappa_n$ be the projection.

Whenever $\mathscr{F}$ is a collection of nonempty sets let $o(\mathscr{F})=\sup\{i<\omega$: each subfamily of $\mathscr{F}$ of cardinality i has nonempty intersection$\}$.

By transfinite induction we will construct for each $\alpha<\omega_1$ and $n<\omega$ a remote system $\mathscr{F}^\alpha_n$ in X^α_n such that

(a) $o(\mathscr{F}^\alpha_n)\geq 2$ for each $\alpha<\omega_1$, $n<\omega$; and $o(\mathscr{F}^\alpha_n)\geq o(\mathscr{F}^\alpha_m)$ if $n\geq m$;

(b) $\sup\{o(\mathscr{F}^\alpha_n): n<\omega\}=\omega$ for each $\alpha<\omega_1$;

(c) whenever $\alpha<\beta$ then there is an $n<\omega$ such that

$$\{{\pi^i_{\beta\alpha}}^{-1}(F): F\in\mathscr{F}^\alpha_i\}\subset\mathscr{F}^\beta_i \quad \text{for each } i\geq n.$$

To make the collections $\mathscr{F}^o_n$ $(n<\omega)$ is no problem ([3, 4], Lemma 1.2). So let us suppose that we have constructed the collections $\mathscr{F}^\alpha_n$ $(n<\omega)$ for all $\alpha<\beta<\omega_1$. In case β is not a limit ordinal we can apply Lemma 1.2. So assume that β is a limit. Let γ_n $(n<\omega)$ be a strictly increasing sequence of ordinals whose supremum is β. Let k_0 be the first integer for which $o(\mathscr{F}^{\gamma_0}_{k_0})\geq 3$ while in addition

$$\{{\pi^n_{\gamma_1\gamma_0}}^{-1}(F): F\in\mathscr{F}^{\gamma_0}_n\}\subset\mathscr{F}^{\gamma_1}_n$$

for each $n\geq k_0$. For each $n<k_0$ define $\mathscr{F}^\beta_n$ to be any remote system for X^β_n such that $o(\mathscr{F}^\beta_n)\geq 2$. Define $\mathscr{F}^\beta_{k_0}$ to be some remote system of $X^\beta_{k_0}$ such that

$$o(\mathscr{F}^\beta_{k_0})\geq o(\mathscr{F}^{\gamma_0}_{k_0})-1; \tag{1}$$

$$\{{\pi^{k_0}_{\beta\gamma_0}}^{-1}(F): F\in\mathscr{F}^{\gamma_0}_{k_0}\}\subset\mathscr{F}^\beta_{k_0} \tag{2}$$

(Lemma 1.2). Let $k_1\geq k_0$ be the first integer for which $o(\mathscr{F}^{\gamma_1}_{k_1})\geq 4$ while in addition $\{{\pi^n_{\gamma_2\gamma_1}}^{-1}(F): F\in\mathscr{F}^{\gamma_1}_n\}\subset\mathscr{F}^{\gamma_2}_n$ for each $n\geq k_1$. For each $k_0<n\leq k_1$ define $\mathscr{F}^\beta_n$ to be any remote system for X^β_n for which

$$o(\mathscr{F}^\beta_n)\geq o(\mathscr{F}^{\gamma_1}_n)-1; \tag{3}$$

$$\{{\pi^n_{\beta\gamma_1}}^{-1}(F): F\in\mathscr{F}^{\gamma_1}_n\}\subset\mathscr{F}^\beta_n \tag{4}$$

(Lemma 1.2). Proceeding in this way inductively we can define the families $\mathscr{F}^\beta_n$ $(n<\omega)$. It is clear that our inductive hypotheses are satisfied. For each $n<\omega$ put $\mathscr{F}_n=\bigcup_{\alpha<\omega_1}\{{\pi^n_{\omega_1\alpha}}^{-1}(F): F\in\mathscr{F}^\alpha_n\}$ and

$$\tilde{\mathscr{F}}=\{A\subset X: A \text{ is closed and } A\cap X_n\in\mathscr{F}_n \text{ for every } n<\omega\}.$$

We first claim that $\tilde{\mathscr{F}}$ is remote. Let $D\subset X$ be nowhere dense and closed. By [10, Lemma 2.1] there is an $\alpha<\omega_1$ such that $\bigcup_{n<\omega}\pi^n_{\omega_1\alpha}(D\cap X_n)$ is nowhere dense in $\sum_{n<\omega} X^\alpha_n$. For each $n<\omega$ take $F_n\in\mathscr{F}^\alpha_n$ which misses $\pi^n_{\omega_1\alpha}(D\cap X_n)$. Then $F=\bigcup_{n<\omega}{\pi^n_{\omega_1\alpha}}^{-1}(F_n)\in\tilde{\mathscr{F}}$ and misses D.

If $\mathcal{F}$ is the closed filter generated by $\tilde{\mathcal{F}}$, then, by (c), $\mathcal{F}$ is clearly as required.

As remarked before, a space X is said to have a remote point provided that there is a point $x \in X^*$ with the property that $x \notin \mathrm{cl}_{\beta X} D$ for any nowhere dense subset $D \subset X$. For more information concerning remote points see [3, 4, 5, 10].

Corollary 1.4. *Let X be a nonpseudocompact space which is a product of at most ω_1 spaces of countable π-weight. Then X has a remote point.*

Proof. Let $Z \subset X^*$ be a nonempty closed G_δ which misses X and put $Y = \beta X - Z$. Let $\{V_n : n < \omega\}$ be a sequence of compact nonempty regular closed subsets of Y such that

(1) $n < m$ implies that $V_n \cap V_m = \emptyset$,

(2) if $E \subset \omega$, then $\bigcup_{n \in E} V_n$ is closed in Y.

We may assume that for each $n < \omega$, $V_n = \mathrm{cl}_{\beta X} W_n$ where W_n is a product of at most ω_1 spaces of countable π-weight. Since $\beta Y = \beta X$ [7, 6.7] and since, by normality of Y, $\beta(\sum_{n<\omega} V_n) = \mathrm{cl}_{\beta X}(\sum_{n<\omega} V_n)$, it sufficies to show that $\sum_{n<\omega} V_n$ has a remote point.

For each $n < \omega$ let $f_n : \beta W_n \to V_n$ be a continuous surjection such that $f_n \restriction W_n = \mathrm{id}$. Since f_n is irreducible[2] the function $f : \sum_{n<\omega} \beta W_n \to \sum_{n<\omega} V_n$ defined by $f(x) = f_n(x)$ $(x \in \beta W_n)$ is irreducible. This easily implies that $\sum_{n<\omega} V_n$ has a remote point iff $\sum_{n<\omega} \beta W_n$ has a remote point.

For each $n < \omega$ let γW_n be a compactification of W_n which is a product of at most ω_1 spaces of countable π-weight. By similar arguments as above, $\sum_{n<\omega} \gamma W_n$ has a remote point iff $\sum_{n<\omega} \beta W_n$ has a remote point. However, by Theorem 1.3, $\sum_{n<\omega} \gamma W_n$ has a remote point.

Corollary 1.5. *$\omega \times 2^{\omega_1}$ has a remote point.*

Remark 1.6. For a slightly more general result see [11].

2. Embedding projective covers in X^*

A closed subset $A \subset X$ is called κ-OK ([9, 1.2]) provided that for each sequence $\{U_n : n < \omega\}$ of neighborhoods of A in X there are neighborhoods $\{A_\alpha : \alpha < \kappa\}$ of A such that for each $n \geq 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$:

$$\bigcap_{1 \leq i \leq n} A_{\alpha_i} \subset U_n.$$

[2] A continuous surjection $f : X \to Y$ is called irreducible if $f(A) \neq Y$ for any proper closed subset $A \subset X$.

Observe that the property of being κ-OK gets stronger as κ gets bigger. It is easy to show, [9, 1.2], that each ω_1-OK set is a weak P-set (a subset $A\subset X$ is called a *weak P-set* whenever $\bar{F}\cap A=\emptyset$ for each countable $F\subset X-A$).

As in [13] for our purposes it will be convenient to slightly change the definition of a κ-OK set in the special case of Čech–Stone remainders. A closed subset $A\subset X^*$, where X is locally compact and σ-compact, is called κ-OK provided that for each sequence $\{U_n: n<\omega\}$ of neighborhoods of A in X^*, there is sequence $\{A_\alpha: \alpha<\kappa\}$ of closed subsets of X such that $A\subset\bigcap_{\alpha<\kappa}A_\alpha^*$, and for each $n\geq 1$ and $\alpha_1<\alpha_2<\cdots<\alpha_n<\kappa$:

$$\bigcap_{1\leq i\leq n} A_{\alpha_i}^*\subset U_n.$$

(as usual, if $B\subset X$ is closed, $B^*=(\mathrm{cl}_{\beta X}B)-B$).

The following lemma shows why we are interested in κ-OK sets. The proof is straightforward and is similar to, but not the same as, [9, 1.4].

Lemma 2.1. *Let X be locally compact and σ-compact and let $A\subset X^*$ be* ω_1-OK. *If $B\subset X^*-A$ is* ccc, *then $\bar{B}\cap A=\emptyset$.*

Proof. Assume, to the contrary, that $\bar{B}\cap A\neq\emptyset$ and put $C=\bar{B}\cap A$. Then C is a nonempty nowhere dense subset of $\bar{B}$. Consequently, C is not a P-set[3] of $\bar{B}$, i.e. there are countable many neighborhoods U_n $(n<\omega)$ of C in $\bar{B}$ such that $C\not\subset\mathrm{int}_{\bar{B}}(\bigcap_{n<\omega}\bar{U}_n)$. Put $V_n=U_n\cup(X^*-\bar{B})$. Then V_n is a neighborhood of A for all $n<\omega$. Since A is ω_1-OK we can find closed sets $\{A_\alpha: \alpha<\omega_1\}$ in X such that for each $n\geq 1$ and $\alpha_1<\alpha_2<\cdots<\alpha_n<\omega_1$:

$$\bigcap_{1\leq i\leq n} A_{\alpha_i}^*\subset U_n$$

while moreover $A\subset\bigcap_{\alpha<\omega_1}A_\alpha^*$. By [14, 5.1] each set of the form D^*, where $D\subset X$ is closed, is a P-set in X^*. Consequently, if $W_\alpha=A_\alpha^*\cap\bar{B}$, then W_α is a P-set of $\bar{B}$, and since $\bar{B}$ is ccc, W_α is clopen. Consequently, W_α is a neighborhood (in $\bar{B}$) of C. The proof is now completed by precisely the same argument as in [9, 1.4]. Since $C\not\subset\mathrm{int}_{\bar{B}}(\bigcap_{n<\omega}\bar{U}_n)$, for each α there is an n such that $W_\alpha-\bar{U}_n\neq\emptyset$. Therefore, we can find an $n<\omega$ so that $E=\{\alpha<\omega_1: W_\alpha-\bar{U}_n\neq\emptyset\}$ is uncountable. For $\alpha\in B$, let $S_\alpha=W_\alpha-\bar{U}_n$. Then the S_α's are nonempty, but any n of them have empty intersection, which contradicts the fact that $\bar{B}$ is ccc.

Let $X=\sum_{n<\omega}X_n$, where each X_n is a compact space of weight at most 2^ω. A closed filter $\mathscr{F}$ on X is called *nice*, [13], provided that for all $F\in\mathscr{F}$ the set $\{n<\omega: F\cap X_n=\emptyset\}$ is finite and $\bigcap\mathscr{F}=\phi$. There are very nontrivial nice filters (see e.g. Theorem 1.3).

[3] A subset B of a space X is called a P-set provided that the intersection of countably many neighborhoods of B is again a neighborhood of B.

Let $\mathscr{F}$ be a nice filter on $X=\omega\times Z$, where Z is compact. The aim of this section is to show that whenever $f:\omega^*\to Y$ is a continuous surjection, then there is a continuous surjection $g:X^*\to Y$ and a closed set $E\subset\bigcap_{F\in\mathscr{F}}\mathrm{cl}_{\beta X}F$ such that

(a) E is a 2^ω-OK set of X^*;

(b) $g\restriction E$ is irreducible.

This shows that many projective covers of spaces can be embedded as 2^ω-OK sets in Čech–Stone remainders. It will come as no surprise that our method of proof is similar to Kunen's, however, we have to overcome a new difficulty.

For the remaining part of this section, let $X=\sum_{n<\omega}X_n$ where each X_n is a compact space of weight at most 2^ω.

Whenever I is a set and κ is a cardinal,

$$[I]^\kappa=\{A\subset I: |A|=\kappa\}$$

and

$$[I]^{<\kappa}=\{A\subset I: |A|<\kappa\}.$$

The following is a generalization of [9, 2.1].

Definition 2.2. Let $\mathscr{F}$ be a closed filter on X and assume that no $F\in\mathscr{F}$ is compact. In addition, let $f:X^*\to Y$ be a continuous surjection. If $1\leq n<\omega$, an indexed family $\{A_i: i\in I\}$ of closed subsets of X is *precisely n-linked* w.r.t. $\langle\mathscr{F},f\rangle$ if for all $\sigma\in[I]^n$ and $F\in\mathscr{F}$

$$f\Big(\bigcap_{i\in\sigma}A_i^*\cap F^*\Big)=Y,$$

but for all $\sigma\in[I]^{n+1}$, $\bigcap_{i\in\sigma}A_i$ is compact.

An indexed family $\{A_{in}: i\in I, 1\leq n<\omega\}$ is a *linked system* w.r.t. $\langle\mathscr{F},f\rangle$ if for each n, $\{A_{in}: i\in I\}$ is precisely n-linked w.r.t. $\langle\mathscr{F},f\rangle$ and, for each n and i, $A_{in}\subset A_{i,n+1}$.

An indexed family $\{A_{in}^j: i\in I, 1\leq n<\omega, j\in J\}$ is an *I by J independent linked family* w.r.t. $\langle\mathscr{F},f\rangle$ if for each $j\in J$, $\{A_{in}^j: i\in I, 1\leq n<\omega\}$ is a linked system w.r.t. $\langle\mathscr{F},f\rangle$, and

$$f\Big(\bigcap_{j\in\tau}\Big(\bigcap_{i\in\sigma_j}A_{in_j}^j\Big)^*\cap F^*\Big)=Y$$

whenever $\tau\in[J]^{<\omega}$, and for each $j\in\tau$, $1\leq n_j<\omega$ and $\sigma_j\in[I]^{n_j}$ and $F\in\mathscr{F}$.

Let $f:\omega\to\omega$ be a finite to one function. The *Stone extension* of f is denoted by βf and $\bar{f}=\beta f\restriction\omega^*$. The filter of cofinite subsets of ω is denoted by $\mathscr{CF}$.

The independent linked family described in the following lemma is the same as in [9, 2.2].

Lemma 2.3. *There is a finite to one surjection $\xi:\omega\to\omega$ and a 2^ω by 2^ω independent linked family w.r.t. $\langle\mathscr{CF},\bar{\xi}\rangle$.*

Proof. Let $S=\{\langle k,f\rangle: k\in\omega \,\&\, f\in\mathcal{P}\mathcal{P}(k)^{\mathcal{P}(k)}\}$ and let $\pi:\omega\to S$ be a bijection. Define $g:S\to\omega$ by

$$g(\langle k,f\rangle)=k$$

and define $\xi:S\to S$ to be the composition of g and π. The required family (defined on S) will be of the form $\{A^Y_{Xn}: X\in\mathcal{P}(\omega),\ 1\leq n<\omega,\ Y\in\mathcal{P}(\omega)\}$ where

$$A^Y_{Xn}=\{\langle k,f\rangle\in S: |f(Y\cap k)|\leq n \,\&\, X\cap k\in f(Y\cap k)\}.$$

We now present the main result in this section. The proof is an adaptation of [9, 3.1]. Due to the complexity of the proof we will give all details.

Theorem 2.4. *Let $X=\omega\times Z$ where Z is a compact space of weight at most 2^ω and suppose that $\mathcal{F}$ is a nice filter on X. If Y is a continuous image of ω^*, then there is a continuous surjection $g:X^*\to Y$ and a closed 2^ω-OK set $A\subset X^*$ such that $A\subset\bigcap_{F\in\mathcal{F}}F^*$ and $g\restriction A$ is irreducible.*

Proof. Let $\pi:\omega\times Z\to\omega$ be the projection and let $\xi:\omega\to\omega$ be as in Lemma 2.3. In addition, let $\{A^\beta_{\alpha n}: \alpha<2^\omega, 1\leq n<\omega, \beta<2^\omega\}$ be an independent linked family w.r.t. $\langle\mathscr{C}\mathcal{F},\bar{\xi}\rangle$ (Lemma 2.3). For each $\alpha<2^\omega$, $1\leq n<\omega$, $\beta<2^\omega$ put $E^\beta_{\alpha n}=A^\beta_{\alpha n}\times Z$ and observe that

$$\{E^\beta_{\alpha n}: \alpha<2^\omega, 1\leq n<\omega, \beta<2^\omega\}$$

is an independent linked family w.r.t. $\langle\mathcal{F},\bar{\eta}\rangle$, where $\mathcal{F}$ is any nice filter in X, $\bar{\eta}=\beta\eta\restriction X^*$ and $\eta=\xi\circ\pi$. Let $f:\omega^*\to Y$ be a continuous surjection and define $g:X^*\to Y$ by $g=f\circ\bar{\eta}$. Notice that the family $\{E^\beta_{\alpha n}: \alpha<2^\omega, 1\leq n<\omega, \beta<2^\omega\}$ is also an independent linked family w.r.t. $\langle\mathcal{F},g\rangle$. We claim that g is as required and it suffices to construct A.

Let $\{B_\mu: \mu<2^\omega \,\&\, \mu$ is even$\}$ enumerate all nonempty closed G_δ's of X (there are clearly only 2^ω closed G_δ's). Let $\{\langle C_{\mu n}: n<\omega\rangle: \mu<2^\omega \,\&\, \mu$ is odd$\}$ enumerate all sequences of closed nonempty G_δ's satisfying $C_{\mu,n+1}\subset\operatorname{int}C_{\mu n}-(n\times Z)$ for each $n<\omega$. Furthermore we assume that each sequence is listed cofinally often.

By induction on μ we construct $\mathcal{F}_\mu$ and K_μ so that

(1) $\mathcal{F}_\mu$ is a closed filter on X, $K_\mu\subset 2^\omega$, and $\{E^\beta_{\alpha n}: \alpha<2^\omega, 1\leq n<\omega, \beta\in K_\mu\}$ is an independent linked family w.r.t. $\langle\mathcal{F}_\mu, g\rangle$;

(2) $K_0=2^\omega$ and $\mathcal{F}_0=\mathcal{F}$;

(3) $\nu<\mu$ implies $\mathcal{F}_\nu\subset\mathcal{F}_\mu$ and $K_\nu\supset K_\mu$;

(4) if μ is a limit ordinal, $\mathcal{F}_\mu=\bigcup_{\nu<\mu}\mathcal{F}_\nu$ and $K_\mu=\bigcap_{\nu<\mu}K_\nu$;

(5) for each μ, $K_\mu-K_{\mu+1}$ is finite;

(6) if μ is even, either $B_\mu\in\mathcal{F}_{\mu+1}$ or $g(B^*_\mu\cap F^*)\neq Y$ for some $F\in\mathcal{F}_{\mu+1}$;

(7) if μ is odd and each $C_{\mu n}\in\mathcal{F}_\mu$, then there are $D_{\mu\alpha}\in\mathcal{F}_{\mu+1}$ for $\alpha<2^\omega$ such that for all $n\geq 1$ and all $\alpha_1<\alpha_2<\cdots<\alpha_n<2^\omega$ we have that $(D_{\mu\alpha_1}\cap\cdots\cap D_{\mu\alpha_n})-C_{\mu n}$ has compact closure.

Let us assume for a moment that this construction can indeed be carried out. For each $\mu<2^\omega$ put $A_\mu=\bigcap_{F\in\mathscr{F}_\mu}F^*$. Clearly $g(A_\mu)=Y$. Therefore, by compactness of X^*, if we put $A=\bigcap_{\mu<2^\omega}A_\mu$, then $g(A)=Y$. We claim that A is as required. First observe that $A\subset\bigcap_{F\in\mathscr{F}}F^*$ and that, by (7), A is 2^ω-OK, So the only remaining thing to check is that $g\restriction A$ is irreducible. Suppose that $B\subset A$ is a proper closed set. For some $\mu<2^\omega$, $B\subset B_\mu^*$ while $A-B_\mu^*\neq\emptyset$. Consequently, $B_\mu\notin\mathscr{F}_{\mu+1}$ since $\mathscr{F}_{\mu+1}\subset\bigcup_{\mu<2^\omega}\mathscr{F}_\mu$. Therefore, by (6), for some $F\in\mathscr{F}_{\mu+1}$ we have $g(B_\mu^*\cap F^*)\neq Y$. Since

$$g(B)\subset g(B_\mu^*\cap F^*),$$

this implies that $g(B)\neq Y$. We conclude that $g\restriction A$ is irreducible. Fix $\mu<2^\omega$ and assume that the $\mathscr{F}_\nu$, K_ν have been constructed for $\nu\leqslant\mu$. We will construct $\mathscr{F}_{\mu+1}$ and $K_{\mu+1}$.

If μ is even, let $\mathscr{T}$ be the closed filter generated by $\mathscr{F}_\mu$ and B_μ. If $\mathscr{T}$ has no compact elements and if $\{E_{\alpha n}^\beta:\alpha<2^\omega, 1\leqslant n<\omega, \beta\in K_\mu\}$ is independent w.r.t. $\langle\mathscr{T},g\rangle$ we set $\mathscr{F}_{\mu+1}=\mathscr{T}$ and $K_{\mu+1}=K_\mu$. If not, then we can find $E\in\mathscr{F}_\mu$ such that

$$g\Big(B_\mu^*\cap E^*\cap\bigcap_{\beta\in\tau}\Big(\bigcap_{\alpha\in\sigma_\beta}E_{\alpha n_\beta}^\beta\Big)^*\Big)\neq Y$$

for some $\tau\in[K_\mu]^{<\omega}$, $n_\beta\in\omega$, and $\sigma_\beta\in[2^\omega]^{n_\beta}$. Then let $K_{\mu+1}=K_\mu-\tau$, and let $\mathscr{F}_{\mu+1}$ be the filter generated by $\mathscr{F}_\mu$ and

$$\bigcap_{\beta\in\tau}\Big(\bigcap_{\alpha\in\sigma_\beta}E_{\alpha n_\beta}^\beta\Big).$$

Clearly $\mathscr{F}_{\mu+1}$ and $K_{\mu+1}$ are as required.

If μ is odd and some $C_{\mu n}$ is not in $\mathscr{F}_\mu$, put $\mathscr{F}_{\mu+1}=\mathscr{F}_\mu$ and $K_{\mu+1}=K_\mu$. In case $C_{\mu n}\in\mathscr{F}_\mu$ for each $n<\omega$, then fix $\beta\in K_\mu$ and let $K_{\mu+1}=K_\mu-\{\beta\}$. Let $\mathscr{F}_{\mu+1}$ be the closed filter generated by $\mathscr{F}_\mu$ and the collection $\{D_{\mu\alpha}:\alpha<2^\omega\}$, where

$$D_{\mu\alpha}=\bigcup_{1\leqslant n<\omega}E_{\alpha n}^\beta\cap C_{\mu n}.$$

First observe that $D_{\mu\alpha}$ is closed since $C_{\mu n}\subset(\omega-n)\times Z$ for all $n<\omega$. To verify condition (7), let $\alpha_1<\alpha_2<\cdots<\alpha_n<2^\omega$, and put

$$Y=(D_{\mu\alpha_1}\cap\cdots\cap D_{\mu\alpha_n})-C_{\mu n}.$$

If $n=1$, then $Y=\emptyset$. On the other hand, if $n>1$, then

$$Y\subset E_{\alpha_1,n-1}^\beta\cap\cdots\cap E_{\alpha_n,n-1}^\beta,$$

which has compact closure since these $E_{\alpha,n-1}^\beta$ are precisely $(n-1)$-linked. Finally, to verify (1), observe that $D_{\mu\alpha}\supset C_{\mu n}\cap E_{\alpha n}^\beta$ for each n.

Remark 2.5. The reader will undoubtfully notice that the above proof, except for some easy adaptations, is the same as [9, 3.1].

Corollary 2.6. *Let Z be a compact space of weight at most 2^ω and put $X=\omega\times Z$. If Y is a continuous image of ω^* which satisfies the countable chain condition, then the projective cover EY[4] of Y embeds in X^* as a 2^ω-OK set.*

Proof. By Theorem 2.4 there is a closed 2^ω-ok set $A\subset X^*$ which admits an irreducible map onto Y. Since X^* is an F-space [7], A is an F-space and since, by irreducibility, A is ccc, A is extremally disconnected. Consequently $A\approx EY$.

3. Construction of the points

In this section we will construct for each subset $F\subset\{1,2,3,4\}$ a point

$$x_F\in\bigcap_{i\in F}A_i-\bigcup_{i\notin F}A_i,$$

where the A_i's are defined as in the introduction. Recall our convention that $\bigcap_{i\in\emptyset}A_i=\omega^*$ and $\bigcup_{i\in\emptyset}A_i=\emptyset$.

3.1. *Construction of $x_\emptyset$*

By [9, 3.1] there is a point $x\in\omega^*$ such that $\{x\}$ is a 2^ω-OK set (this also follows from Corollary 2.6 since the projective cover of a one point space is a one point space). Such a point is called a 2^ω-OK point. By [9, 1.4] (see also Lemma 2.1), if $A\subset\omega^*-\{x\}$ is ccc, then $x\notin\bar{A}$. Put $x_\emptyset=x$.

3.2. *Construction of $x_{\{1\}}$*

Since $\beta\omega$ is clearly a continuous image of ω^*, by Corollary 2.6, $E\beta\omega$ embeds in ω^* as a 2^ω-OK set. Since $\beta\omega$ is extremally disconnected, $E\beta\omega\approx\beta\omega$. We can therefore find points $x_n\in\omega^*$ $(n<\omega)$ so that if $D=\{x_n: n<\omega\}$, then $\bar{D}\approx\beta\omega$ is a 2^ω-OK set in ω^*. Let $x\in\bar{D}-D$ be a 2^ω-OK point of $\bar{D}-D$. Assume that $A\subset\omega^*-\{x\}$ has no isolated points and is ccc. We claim that $x\notin\bar{A}$. By Lemma 2.1, $\bar{D}\cap\bar{A}$ is clopen in $\bar{A}$. It is easily seen that each x_n is a 2^ω-OK point of ω^*. Hence $D\cap\bar{A}=\emptyset$. We conclude that $A\cap(\bar{D}-D)$ is ccc and that if $x\in\bar{A}$, then $x\in(A\cap(\bar{D}-D))^-$. But this is impossible since x is a 2^ω-OK point of $\bar{D}-D$. Therefore, if we put $x_{\{1\}}=x$, then $x_{\{1\}}$ is as required.

3.3 *Construction of $x_{\{2\}}$*

Since the Cantor set C clearly is a continuous image of ω^*. the projective cover E of C embeds as a 2^ω-OK set in ω^*. So assume that $E\subset\omega^*$ and that E is 2^ω-OK. Let E_n $(n<\omega)$ be a sequence of pairwise disjoint nonempty clopen subspaces of

[4] The projective cover EX of a space X is the unique extremally disconnected (=closure of an open set is open) space which admits an irreducible perfect map onto X.

E whose union is dense in E, while moreover $E_n \approx E$ for all $n<\omega$. Since E is extremally disconnected, $\beta(\bigcup_{n<\omega} E_n)=E$. By Theorems 1.3 and 2.4 there is a point $x \in F=E-\bigcup_{n<\omega} E_n$ such that x is a 2^ω-OK point of F while moreover $x \notin \bar{D}$ for any nowhere dense subset $D \subset \bigcup_{n<\omega} E_n$.

Let $G \subset \omega^*-\{x\}$ be either countable discrete, or countable π-homogeneous of π-weight ω_1, or ccc nowhere separable. Assume that $x \in \bar{G}$. Since E is a 2^ω-OK set, w.l.o.g, $G \subset E$. Since $G \cap \bigcup_{n<\omega} E_n$ is nowhere dense in $\bigcup_{n<\omega} E_n$, w.l.o.g, $G \subset F$. Since x is a 2^ω-OK point of F this is a contradiction. Since x is clearly a point of A_2, we can put $x_{\{2\}}=x$.

Notice that x is a limit point of a countable set but not of any countable discrete set. That such a point exists answers a question of K. Kunen.

3.4. *Construction of* $x_{\{3\}}$

Let $\mathscr{F}$ be a nice filter on $\omega \times 2^{\omega_1}$ which in addition is remote and let E be the projective cover of 2^{ω_1}. There is clearly an irreducible perfect map $f: \omega \times E \to \omega \times 2^{\omega_1}$. If $D \subset \omega \times E$ is nowhere dense, then $f(D)$ is nowhere dense, hence some $F \in \mathscr{F}$ misses $f(D)$. Therefore, if $\mathscr{G}$ is the closed filter on $\omega \times E$ generated by $\{f^{-1}(F): F \in \mathscr{F}\}$, then $\mathscr{G}$ is both nice and remote. By Theorem 2.4 E embeds as a 2^ω-OK set in ω^*. So assume that $E \subset \omega^*$ and that E is 2^ω-OK. Let E_n $(n<\omega)$ be a sequence of pairwise disjoint nonempty clopen subsets of E such that $E_n \approx E$ for all $n<\omega$ and $\bigcup_{n<\omega} E_n$ is dense in E. By Theorem 2.4 and the above remark there is a point $x \in F=E-\bigcup_{n<\omega} E_n$ such that x is a 2^ω-OK point of F while moreover $x \notin \bar{D}$ for any nowhere dense $D \subset \bigcup_{n<\omega} E_n$. By using the same technique as in section 3.3 it can easily be seen that $x_{\{3\}}=x$ is as required.

3.5. *Construction of* $x_{\{4\}}$

By [2], there is a continuous image X of ω^* which is ccc and not separable. We even may assume that X is nowhere separable, [13, 5.1]. Let Y be the one point compactification of $\omega \times X$. It is clear that Y is a continuous image of ω^*. By Theorem 2.4 the projective cover of Y embeds in ω^* as a 2^ω-OK set. This space is obviously homeomorhpic to $\beta(\omega \times E)$, where E is the projective cover of X. Clearly E is ccc and nowhere separable. Let $\pi: \omega \times E \to E$ be the projection. For each countable subset $A \subset \omega \times E$ let $\{U_n(A): n<\omega\}$ be a maximal (faithfully indexed) pairwise disjoint collection of nonempty clopen subsets of E none of which intersects $\overline{\pi(A)}$. Since E is nowhere separable, $\bigcup_{n<\omega} U_n(A)$ is dense in E. Put

$$L(A)=\bigcup_{n<\omega}\left(\{n\} \times \bigcup_{i \leq n} U_i(A)\right).$$

Then $L(A) \cap \bar{A}=\emptyset$ and the closed filter generated by $\{L(A): A \subset \omega \times E\}$ is nice. This construction is implicit in [13, 5.2]. By Theorem 2.4 there is a point $x \in \beta(\omega \times E)$ which is a 2^ω-OK point of $(\omega \times E)^*$ while moreover $x \notin \bar{A}$ for any countable $A \subset \omega \times E$.

As remarked above, we may assume that $\beta(\omega\times E)\subset\omega^*$ and that $\beta(\omega\times E)$ is a 2^ω-OK set in ω^*. It is easily seen that if we put $x_{\{4\}}=x$, then $x_{\{4\}}$ is a weak P-point of ω^* which is a limit point of some ccc subset of $\omega^*-\{x_{\{4\}}\}$.

That such weak P-point exists was first shown, under MA, in [8]. This answers a question implicit in [9, § 1].

3.6. *Construction of* $x_{\{1,2\}}$

Let E be the projective cover of the Cantor set C and assume that $\beta(\omega\times E)\subset\omega^*$ is 2^ω-OK. Let $D\subset(\omega\times E)^*$ be countable and discrete such that $\bar{D}$ is a 2^ω-OK set of $(\omega\times E)^*$ while moreover $\bar{D}\cap\bar{A}=\emptyset$ for any nowhere dense $A\subset\omega\times E$. Let $x\in\bar{D}-D$ be a 2^ω-OK point of $\bar{D}-D$ and define $x_{\{1,2\}}=x$.

3.7. *Construction of* $x_{\{1,3\}}$

Replace E in Section 3.6 by the projective cover of 2^{ω_1}.

3.8. *Construction of* $x_{\{1,4\}}$

Replace E in Section 3.6 by the projective cover of Y where Y is a ccc nowhere separable image of ω^* and use the filter $\{L(A)\colon A\subset\omega\times E \text{ is countable}\}$ constructed in Section 3.5.

3.9. *Construction of* $x_{\{2,3\}}$

Let E_0 be the projective cover of the Cantor set and let E_1 be the projective cover of 2^{ω_1}. Assume that $\beta(\omega\times E_0)\subset\omega^*$ is 2^ω-OK. In addition, assume that $\beta(\omega\times E_1)\subset(\omega\times E_0)^*$ is 2^ω-OK in $(\omega\times E_0)^*$ while moreover

$$\beta(\omega\times E_1)\cap\bar{D}=\emptyset$$

for any nowhere dense $D\subset\omega\times E_0$. Finally, let $x\in(\omega\times E_1)^*$ be a 2^ω-OK point of $(\omega\times E_1)^*$ such that $x\notin F$ for any nowhere dense $F\subset\omega\times E_1$. Define $x_{\{2,3\}}=x$.

3.10. *Construction of* $x_{\{2,4\}}$

Replace E_1 in Section 3.9 by the projective cover of Y, where Y is a ccc nowhere separable image of ω^* and use the filter $\{L(A)\colon A\subset\omega\times E_1 \text{ is countable}\}$ constructed in Section 3.5.

3.11 *Construction of* $x_{\{3,4\}}$

Replace E_0 in Section 3.10 by the projective cover of 2^{ω_1}.

3.12 *Construction of* $x_{\{1,2,3\}}$, $x_{\{1,2,4\}}$, $x_{\{1,3,4\}}$ *and* $x_{\{2,3,4\}}$

Use the same technique as in Sections 3.6–3.11.

3.13. *Construction of* $x_{\{1,2,3,4\}}$

This is easy.

4. Remarks

The results in Section 1 suggest the question whether each nonpseudocompact space of π-weight at most ω_1 has a remote point. Since there are spaces of weight ω_2 which do not have remote points [10] and since each space which is nonpseudocompact and which has π-weight ω has a remote point [3, 4], our question is relevant. We don't have any information concerning this question. The only additional fact we know is the rather curious result that the statement "$\omega \times \omega^*$ has a remote point" is both consistent with and independent of the usual axioms of set theory. Under CH, any small nonpseudocompact space has a remote point, [10], hence, under CH, $\omega \times \omega^*$ has a remote point. On the other hand, it is consistent that ω^* can be covered by nowhere dense closed P-sets [1], which implies that $\omega \times \omega^*$ has no remote points [5].

Let x and y be distinct weak P-points in ω^* which are not P-points and let X be the space we get from ω^* by identifying x and y, It is easily seen that X is a compact space which is not an F-space although every countable subspace is C^*-embedded. This example is due to C.F. Mills. That a space with these properties exists was previously proved, by a different technique, by the author and independently, but earlier, by E.K. van Douwen. Finally, let x and y be distinct points in ω^* which are both limits of countable sets but not a limit of any countable discrete set. Let Y be the space we get from ω^* by identifying x and y. It is easily seen that Y is a compact space in which not every countable subspace is C^*-embedded although each countable discrete subspace is C^*-embedded. This example is also due to C.F. Mills.

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