Topology and its Applications 13 (1982) 123–132
North-Holland Publishing Company

THE REDUCED MEASURE ALGEBRA AND A K_1-SPACE WHICH IS NOT K_0

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Received 26 February 1980
Revised 29 May 1981

The reduced measure algebra is used to construct, under *CH*, a hereditarily Lindelöf separable K_1-space X which is not a K_0-space.

AMS Subj. Class.: 54C20, 54C99

reduced measure algebra — K_0-space
monotone extension property — K_1-space

0. Definitions

All topological spaces under discussion are completely regular and T_1.

If X is a space, $C^*(X)$ denotes the Banach space of continuous, bounded, real-valued functions on X. For a function $f \in C^*(X)$ the sup-norm of f is defined by

$$\|f\| = \sup\{|f(x)|: x \in X\}.$$

If A is a closed subspace of X, then a function $\iota: C^*(A) \to C^*(X)$ satisfying $\iota(f)|A = f$ for each $f \in C^*(A)$ is called an *extender*. The norm of ι, which is denoted by $\|\iota\|$, is defined by

$$\|\iota\| = \sup\{\|\iota(f)\|: f \in C^*(A), \|f\| = 1\}.$$

The extender ι is linear if $\iota(\alpha f + \beta g) = \alpha\iota(f) + \beta\iota(g)$ for all $f, g \in C^*(A)$ and $\alpha, \beta \in \mathbb{R}$; ι is said to be monotone if $\iota(f) \leq \iota(g)$ provided that $f \leq g$.

A space X is said to have property D_c^*, where $c \in \mathbb{R}$, if for every nonempty closed subspace A of X there is a linear extender $\iota: C^*(A) \to C^*(X)$ with norm not exceeding c. Similarly, X has the *monotone extension property* if for every closed subspace $A \subset X$ there is a monotone extender $\iota: C^*(A) \to C^*(X)$. For more information on these concepts see [3, 4, 7, 13, 14].

* Partially supported by NATO grant 1927.

0166-8641/82/0000–0000/$02.75

A subspace $A \subset X$ is said to be *K_n-embedded* in X $(n \geq 0)$ provided there is a function $\kappa : \tau(A) \to \tau(X)$ (for each space Z, the topology of Z is denoted by $\tau(Z)$) such that

(a) $\kappa(U) \cap A = U$ for all $U \in \tau(A)$;

(b) if $n = 0$, then $\kappa(\emptyset) = \emptyset$ and $\kappa(U) \cap \kappa(V) = \kappa(U \cap V)$ for all $U, V \in \tau(A)$; if $n > 0$, then $\kappa(U_0) \cap \cdots \cap \kappa(U_n) = \emptyset$ whenever $U_i \cap U_j = \emptyset$ for $0 \leq i < j \leq n$ and $U_0, \ldots, U_n \in \tau(A)$.

A space is a *K_n-space* if each subspace is K_n-embedded. For more information on these concepts see [3, 4, 18, 19].

1. Introduction

The Dugundji Extension Theorem, [9], has been improved in recent years so that certain versions of it now also work for certain classes of non-metrizable but mathematically important objects such as CW-complexes [2, 1] and generalized ordered spaces [13].

One of the most important observations in Dugundji extension theory is that spaces which satisfy a certain version of the Dugundji Extension Theorem allow functions on subspaces which simultaneously extend open sets in a nice way. To be more precise, a space with property D_c^* is K_n-space where n is the smallest integer greater than $\frac{1}{2}(c-1)$. This observation of van Douwen [4] was used to construct a first countable, hereditarily Lindelöf, separable space H_∞ containing a closed subspace A having no continuous linear extender from $C^*(A)$ to $C^*(H_\infty)$.

Van Douwen's Example is the topological sum of spaces H_n $(n \in \mathbb{N})$, where, for each n, H_n is a hereditarily Lindelöf, separable K_{n+1}-space which is not a K_n-space. This example left open the question whether every K_1-space is a K_0-space and the aim of this paper is to answer this question, [4, p. 301].

We will construct, assuming the Continuum Hypothesis, a hereditarily Lindelöf, separable K_1-space which is not a K_0-space. Our example is inspired by an example in van Mill [19] where we constructed a first countable compact space Z containing a closed subspace A which is K_1-embedded but not K_0-embedded (Z is not a K_1-space since Z is separable and contains an uncountable discrete subspace). Our example is also interesting for another reason. In [3, 3.1] it was shown that any space with the monotone extension property is a K_1-space. Our example has the monotone extension property but is not K_0. That answers another question of van Douwen.

2. Certain subspaces of extremally disconnected compacta

A space is *extremally disconnected* if the closure of any open set is again open. As usual, we call a space without isolated points a *Luzin* (*nodec*) space, if each nowhere dense set is countable (closed). There are, under CH, spaces which are both Luzin and nodec, see [21, 20].

The following results are of independent interest.

2.1. Theorem(CH). *Let X be an extremally disconnected, dense in itself compactum of weight 2^ω. Then X contains a dense nodec subspace. If X moreover satisfies the countable chain condition, then X contains a dense subspace which is both nodec and Luzin.*

Proof. Let $\mathscr{C}$ be the Boolean algebra of clopen subsets of X and, by CH, list $\mathscr{C}-\{\emptyset\}$ as $\{C_\alpha: \alpha<\omega_1\}$. By induction we will construct, for each $\alpha<\omega_1$ a point $x_\alpha \in X$ and a nowhere dense closed set $Z_\alpha \subset X$ such that

(a) $x_\alpha \in Z_\alpha \subset C_\alpha$,

(b) if $\beta<\alpha$ and if $x_\beta \neq x_\alpha$, then $Z_\beta \cap Z_\alpha = \emptyset$, $x_\beta \notin Z_\alpha$ and $x_\alpha \notin Z_\beta$,

(c) if $D \subset X - Z_\alpha$ is nowhere dense, then $x_\alpha \notin \bar{D}$.

Suppose that we have constructed the x_β's and the Z_β's for all $\beta<\alpha<\omega_1$. If there is a $\gamma<\alpha$ such that $x_\gamma \in C_\alpha$ then define $x_\alpha = x_\gamma$ and $Z_\alpha = Z_\gamma$. If not, take $x \in C_\alpha$ so that

$$x \notin \bigcup_{\beta<\alpha} Z_\beta.$$

Let $Z \subset C_\alpha$ be a closed G_δ subset of X missing $\bigcup_{\beta<\alpha} Z_\beta \cup \{x_\beta: \beta<\alpha\}$ but containing x. Since X is extremally disconnected and since the cellularity of X is non-measurable, x is not a P-point, i.e. there is a closed nowhere dense G_δ set S containing x, [10, 12H].

Define $S' = S \cap Z$. Then S' is also nowhere dense, so $Y = X - S'$ is a locally compact, σ-compact, non-compact, dense subspace of X. Since dense subspaces of X are C^*-embedded, $\beta Y = X$. Since Y is not pseudocompact and βY has weight 2^ω, by Kunen, van Mill & Mills [16, 1.3] there is a point $x' \in \beta Y - Y = S'$ such that $x' \notin \bar{D}$ for any nowhere dense subspace $D \subset Y$. Define $x_\alpha = x'$ and $Z_\alpha = S'$.

Now put $P = \{x_\alpha: \alpha<\omega_1\}$. Clearly P is dense and we claim that P is nodec. Let $D \subset P$ be nowhere dense and suppose that D is not closed. Take $x \in (P \cap \bar{D}) - D$. Choose $\alpha<\omega_1$ such that $x = x_\alpha$. By (b), $Z_\alpha \cap (P - \{x_\alpha\}) = \emptyset$ and therefore $Z_\alpha \cap D = \emptyset$. Since $D \subset Z_\alpha$ is nowhere dense, by (c), $x \notin \bar{D}$. Contradiction.

If X is ccc, then the x_α's must be chosen more carefully in order for $P = \{x_\alpha: \alpha<\omega_1\}$ to be Luzin. First observe the well-known fact that there is a family $\mathscr{A}$ of 2^ω nowhere dense subsets of X so that each nowhere dense subset of X is contained in some element of $\mathscr{A}$. Indeed, since X is ccc each nowhere dense subset of X is contained in a nowhere dense G_δ, and, since there are only $(2^\omega)^\omega = 2^\omega$ G_δ's in X, we can simply let $\mathscr{A}$ be the family of all nowhere dense G_δ's of X. To make P Luzin we must simply add in the induction hypotheses that $x_\alpha \notin \bigcup_{\beta \leq \alpha} A_\beta$ (let $\{A_\alpha: \alpha<\omega_1\}$ enumerate $\mathscr{A}$). The rest is routine. □

2.2. Remark. For related ideas see [8, 20].

A space is called *retractable* if each nonempty closed subspace is a retract.

The following Lemma generalizes [3, 3.3].

2.3. Lemma. *A Lindelöf nodec space is retractable.*

Proof. Let X be a Lindelöf nodec space. First observe that each nowhere dense subset of X is discrete and hence, because X is Lindelöf, countable. Therefore X is Luzin, which easily implies that X is zero-dimensional, [15].

Observe that it clearly suffices that each nowhere dense closed subspace of X is a retract. So let $D \subset X$ be closed and discrete. Since X is strongly zero-dimensional there is a disjoint clopen cover $\{U_d : d \in D\}$ of X with $d \in U_d$. Then, as in [3, 3.3], define $r: X \to D$ by $r(x) = d$ iff $x \in U_d$. □

2.4. Corollary. *A Lindelöf nodec space is* K_0.

Proof. By [4, 2.1] it suffices to prove that closed subspaces are K_0-embedded. But that immediately follows from Lemma 2.3. □

2.5. Question. Is the statement "Each dense in itself extremally disconnected compactum of weight 2^ω contains a dense nodec subspace" equivalent to CH?

3. The reduced measure algebra

Let I denote the closed unit interval $[0, 1]$, let $\mathcal{M}$ be the Boolean algebra of measurable subsets of I, and let $\mathcal{N}$ be the ideal of null-sets. The quotient algebra $\mathcal{M}/\mathcal{N}$ is called the reduced measure algebra. Let M denote its Stone space. Notice that $\mathcal{M}/\mathcal{N}$ is complete and has cardinality 2^ω, so that M is an extremally disconnected compactum of weight 2^ω.

Let λ denote Lebesgue measure on I, and for $A \in \mathcal{M}$ let $[A]$ denote the $\mathcal{N}$-equivalence class of A.

The following lemma is well-known. The proof is included for completeness sake.

3.1. Lemma. (a) *The family $\mathcal{C}$ of nonempty clopen (≡closed and open) subsets of M can be written as $\mathcal{C} = \bigcup_{n<\omega} \mathcal{C}_n$, where, for each $n < \omega$,*

(α) $\bigcap \mathcal{C}_n = \emptyset$, *and*

(β) *any two members of $\mathcal{C}_n$ meet.*

(b) *M is not separable.*

Proof. For (a), let us first prove a corresponding statement for $\mathcal{M} - \mathcal{N}$. Let $\mathcal{B}$ be a countable (open) basis for I which is closed under finite unions. For each $B \in \mathcal{B}$ define

$$\mathcal{L}(B) = \{A \in \mathcal{M} : \lambda(A \cap B) > \tfrac{1}{2}\lambda(B)\}.$$

It is clear that any two members of $\mathscr{L}(B)$ meet in a set of positive measure. Also, $\mathscr{L}(B)$ contains three elements which have empty intersection.

We claim that $\mathscr{M}-\mathscr{N}=\bigcup_{B\in\mathscr{B}}\mathscr{L}(B)$. Indeed, take $A\in\mathscr{M}-\mathscr{N}$ and construct a compact $K\subset A$ with $\lambda(K)>0$. There is a $B\in\mathscr{B}$ with $B\supset K$ and $\lambda(K)>\frac{1}{2}\lambda(B)$. This shows that $K\in\mathscr{L}(B)$. Hence $A\in\mathscr{L}(B)$.

From these observations (a) is immediately clear.

For (b), let $\langle p_n\rangle_n$ be any sequence in M. Since p_n is an ultrafilter in the Boolean algebra $\mathscr{M}/\mathscr{N}$ we can find $P_n\in\mathscr{M}-\mathscr{N}$ with $[P_n]\in p_n$ and $\lambda(P_n)<2^{-2-n}$. Then $\{x\in M: [I-\bigcup_n P_n]\in x\}$ is an open set in M which is nonempty and misses each P_n. □

A family of sets is called *linked* (*centered*) if any two (any finite number) of its members meet. Call a family of sets *σ-linked* (*σ-centered*) if it is the union of countably many linked (centered) subfamilies. A compact space the topology of which is σ-centered is clearly separable [5], so that we can reformulate Lemma 2.1 by saying that $\tau(M)$ is σ-linked but not σ-centered (this is not entirely true; in Lemma 2.1 we proved that $\tau(M)$ is the union of countably many linked subfamilies which all have empty intersection and it is precisely this fact which makes our construction work).

By Lemma 2.1(a) M satisfies the countable chain condition, so that by Theorem 2.1 M has a dense subspace P which is both Luzin and nodec, see also [21]. Since P is dense in M, $\tau(P)$ is σ-linked but not σ-centered. We have constructed the following example.

3.2. Example (CH). *There is a space P which is both Luzin and nodec and which moreover has the following properties*:

(a) *the family of nonempty clopen subsets of P is not σ-centered,*

(b) *the family of nonempty clopen subsets of P is the union of countably many linked subfamilies having all empty intersection.*

4. The example

Let P be the space of Example 3.2. The family of nonempty clopen subsets of P will be denoted by $\mathscr{C}$. By 3.2(b), we can write $\mathscr{C}$ as $\bigcup_{n<\omega}\mathscr{C}_n$ where each $\mathscr{C}_n$ is linked while moreover $\bigcap\mathscr{C}_n=\emptyset$.

The following construction is inspired by de Groot's [11] notion of a *superextension.*

By Zorn's Lemma extend each $\mathscr{C}_n$ to a maximal linked system $\mathscr{L}_n\subset\mathscr{C}$, i.e. a linked system in $\mathscr{C}$ not properly contained in any other linked system in $\mathscr{C}$. For each $C\in\mathscr{C}$ define

$$C^+=C\cup\{n<\omega: C\in\mathscr{L}_n\}.$$

Notice that if $C\in\mathscr{C}$, then either $C\in\mathscr{L}_n$ or $P-C\in\mathscr{L}_n$, and also that $\bigcap\mathscr{L}_n=\emptyset$ since $\bigcap\mathscr{C}_n=\emptyset$.

Fact 1. *If $F \subset \omega$ is finite and if $x \in P$, then there is a $C \in \mathscr{C}$ containing x such that $C^+ \cap F = \emptyset$.*

Take $n \in F$ arbitrarily. Since $\bigcap \mathscr{L}_n = \emptyset$, there is an $L_n \in \mathscr{L}_n$ not containing x. Put $C = \bigcap_{n \in F} (P - L_n)$. Then C is as required.

Fact 2. *Let $F, G \subset \omega$ be finite, let $x \in P$, and, for each $n \in F$ let C_n be a clopen neighborhood of x in P. Then there is a clopen neighborhood C of x such that $C^+ \subset \bigcap_{n \in F} C_n^+ - G$.*

Put $E = \bigcap_{n \in F} C_n$ and let D be a clopen neighborhood of x such that $D^+ \cap G = \emptyset$ (Fact 1).

Then $C = E \cap D$ is as required.

The underlying set of X is $P \cup \omega$. The topology of X is generated by the collection

$$\{C^+ : C \in \mathscr{C}\} \cup \{\{n\} : n \in \omega\}.$$

Notice that this implies that the points of ω are isolated and that a basic neighborhood of $x \in P \subset X$ has the form C^+, where $x \in C \in \mathscr{C}$. Since $C^+ \cap P = C$ for all $C \in \mathscr{C}$, the inclusion $P \hookrightarrow X$ is an embedding.

Fact 3. *X is a zero-dimensional Hausdorff space.*

Take $C \in \mathscr{C}$. Then $C^+ \cap (P - C)^+ = \emptyset$ and $C^+ \cup (P - C)^+ = X$. This implies that C^+ is clopen. The rest is clear.

Fact 4. *X is Lindelöf and ω is dense in X. In particular, X is separable.*

Since P is Lindelöf and ω is countable, the Lindelöfness of X is trivial. We will now show that $P \subset \omega^-$. Take $x \in P$ and let U be any neighborhood of x in X. Take $C \in \mathscr{C}$ so that $x \in C \subset C^+ \subset U$. Let $C \in \mathscr{C}_n$. Then $C \in \mathscr{L}_n$, or equivalently, $n \in C^+$. That shows that $U \cap \omega \neq \emptyset$.

Fact 5. *X is not a K_0-space.*

We claim that there is no K_0-function $\kappa : \tau(P) \to \tau(X)$. For, to the contrary, assume there is a K_0-function $\kappa : \tau(P) \to \tau(X)$. since $X - P$ is countable this would imply that $\tau(P)$ is σ-centered, a contradiction.

If $U \subset P$ is open, then define

$$U^+ = U \cup \{n < \omega : \exists C \in \mathscr{L}_n (C \subset U)\}.$$

Notice that U^+ is open and that if U is clopen the set U^+ defined here equals the set U^+ defined above.

Fact 6. *X is a K_1-space.*

By [4, 2.1] it suffices to prove that closed subspaces of X allow K_1-functions. Therefore, let $A \subset X$ be closed. Since, by Corollary 2.5, P is a K_0-space, there is a K_0-function $\rho: \tau(A \cap P) \to \tau(P)$ (in fact, a K_1-function would suffice). Define $\kappa: \tau(A) \to \tau(X)$ by

$$\kappa(U) = U \cup ((\rho(U \cap P))^+ - A).$$

It is clear that $\kappa(U) \cap A = U$. Let us observe that $\kappa(U)$ is open. Since $X - P$ consists of isolated points of X we only need to check that $\kappa(U)$ is a neighborhood of any point of $\kappa(U) \cap P$. So take $x \in \kappa(U) \cap P$. If $x \notin A$ take a clopen $C \subset P$ so that $C \subset \rho(U \cap P)$ while moreover $x \in C \subset C^+ \subset X - A$. Then $C^+ \subset \kappa(U)$, and consequently $\kappa(U)$ is a neighborhood of x. If $x \in A$, take a clopen $F \subset P$ so that $x \in F \subset \rho(U \cap P)$ while moreover $F^+ \cap A \subset U$. Then $F^+ \subset \kappa(U)$ so that in this case $\kappa(U)$ is also a neighborhood of x. We conclude that $\kappa(U)$ is open.

If $U \cap V = \emptyset$, then $\kappa(U) \cap \kappa(V) = \emptyset$ since $\rho(U \cap P) \cap \rho(V \cap P) = \emptyset$ (which implies that $(\rho(U \cap P))^+ \cap (\rho(V \cap P))^+ = \emptyset$).

Therefore κ is a K_1-function.

5. The monotone extension property

We will now prove that X has the monotone extension property. From this it also follows that X is a K_1-space, [3, 3.1].

Let P and X be as in Section 4. In the following Lemma we will use a technique essentially due to J. Jensen (see [22, II.4.5]).

If $A \subset \mathbb{R}$ let $h(A)$ denote the closed convex hull of A in $\mathbb{R}$.

5.1. Lemma. *There is an extender $\Phi: C^*(P) \to C^*(X)$ so that*
(a) *$\|\Phi(f)\| = \|f\|$ for all $f \in C^*(P)$, and*
(b) *if $f \leq g$, then $\Phi(f) \leq \Phi(g)$.*

Proof. Let $f \in C^*(P)$. Define $\Phi(f): X \to \mathbb{R}$ by

$$\Phi(f)(x) = f(x) \qquad (x \in P),$$

and

$$\{\Phi(f)(n)\} = \bigcap \{h(f(L)) \colon L \in \mathscr{L}_n\} \quad (n \in \omega).$$

Clearly $\Phi(f) | P = f$. We claim that $\Phi(f)$ defined in this way is as required.

Claim 1. $\Phi(f)$ is well-defined.

First observe that the fact that $f \in C^*(P)$ and the fact that $\mathscr{L}_n$ is a linked system imply that $\bigcap \{h(f(L)) \colon L \in \mathscr{L}_n\} \neq \emptyset$ for all $n \in \omega$. Suppose that for certain $n \in \omega$ $\bigcap \{h(f(L)) \colon L \in \mathscr{L}_n\}$ contains two distinct points, say a and b. Without loss of generality $a<b$. Take a clopen set $E\subset P$ such that

$$f^{-1}(-\infty, \tfrac{3}{4}a+\tfrac{1}{4}b]\subset E\subset f^{-1}(-\infty, \tfrac{1}{4}a+\tfrac{3}{4}b),$$

Since $\mathscr{L}_n$ is a maximal linked system, either $E\in\mathscr{L}_n$ or $P-E\in\mathscr{L}_n$. If $E\in\mathscr{L}_n$, then $b\in h(f(E))\subset(-\infty, \tfrac{1}{4}a+\tfrac{3}{4}b]$, which is impossible. If $P-E\in\mathscr{L}_n$, then

$$f^{-1}[\tfrac{1}{4}a+\tfrac{3}{4}b, \infty)\subset P-E\subset f^{-1}(\tfrac{3}{4}a+\tfrac{1}{4}b, \infty),$$

so the same contradiction can be derived.

Claim 2. $\Phi(f)$ is continuous.

The reader can easily check that

$$\Phi(f)^{-1}(-\infty, s]=\bigcap\{C^+\colon C\in\mathscr{C} \text{ and } \exists\varepsilon>0\colon f^{-1}(-\infty, s+\varepsilon]\subset C\},$$
$$\Phi(f)^{-1}[s, \infty)=\bigcap\{C^+\colon C\in\mathscr{C} \text{ and } \exists\varepsilon>0\colon f^{-1}[s-\varepsilon, \infty)\subset C\}$$

for all $s\in\mathbb{R}$.

Claim 3. $\|\Phi(f)\|=\|f\|$ for all $f\in C^*(P)$, in particular, $\Phi(f)\in C^*(X)$.

This follows immediately from the definition of $\Phi(f)$.

Claim 4. If $f\leqslant g$, then $\Phi(f)\leqslant\Phi(g)$.

This requires proof. Suppose that $f\leqslant g$ but $\Phi(f)\not\leqslant\Phi(g)$ for certain $f, g\in C^*(P)$. since $\Phi(f)|P=f$ and $\Phi(g)|P=g$, we can find $n\in\omega$ such that

$$\Phi(g)(n)<\Phi(f)(n).$$

Since $f, g\in C^*(P)$ we can find $M, N\in\mathscr{L}_n$ such that

$$h(f(M))\cap h(g(N))=\emptyset$$

(argument: if $h(f(M))\cap h(g(N))\neq\emptyset$ for all $M, N\in\mathscr{L}_n$ then

$$\bigcap\{h(f(M))\colon M\in\mathscr{L}_n\}\cap\bigcap\{h(g(N))\colon N\in\mathscr{L}_n\}\cap h(f(P)\cup g(P))\neq\emptyset,$$

i.e. $\Phi(g)(n)=\Phi(f)(n)$, which is impossible).

Since $\Phi(g)(n)<\Phi(f)(n)$ for all $r\in h(f(M))$ and $s\in h(g(N))$ we have that $s<r$. Now, $\mathscr{L}_n$ is a linked system, so that M and N meet, say $x\in M\cap N$. Since $f(x)\in h(f(M))$ and $g(x)\in h(g(N))$ it follows that $g(x)<f(x)$. But this contradicts the fact that $f\leqslant g$. □

5.2. Remark. From the proof of Lemma 4.1, the definition of Φ, Claim 1 and Claim 2 are known, see [22, II 4.5], since $\Phi(f)$ is already continuous in the weaker superextension topology on X. Since we used the explicit construction of $\Phi(f)$ in Claim 3 and Claim 4, for completeness sake we have also included the proofs of Claim 1 and Claim 2.

We now come to the main result in this section.

5.3. Theorem. *Let A be any closed subspace of X. Then there is an extender $\Phi\colon C^*(A)\to C^*(X)$ such that*

(a) $\|\Phi(f)\| = \|f\|$ *for all* $f \in C^*(A)$, *and*
(b) *if* $f \leq g$, *then* $\Phi(f) \leq \Phi(g)$.
In particular, X has the monotone extension property.

Proof. Let $A \subset X$ be closed. Without loss of generality $A \neq \emptyset$.

Suppose first that $A \cap P = \emptyset$. Then, since $X - P$ is countable, A is a closed discrete subspace of the zero-dimensional Lindelöf space X. This implies that A is a retract of X. Let r retract X onto A. Define $\Phi: C^*(A) \to C^*(X)$ by

$$\Phi(f)(x) = f(r(x)).$$

Then Φ is as required (this is well-known of course).

Now suppose that $A \cap P \neq \emptyset$ and let $\iota: C^*(P) \to C^*(X)$ be an extender as in Lemma 5.1. By Lemma 2.3, P is retractable, so let $r: P \to A \cap P$ be a retraction. Define $\Phi: C^*(A) \to C^*(X)$ by

$$\begin{cases} \Phi(f)(x) = f(x), & (x \in A), \\ \Phi(f)(x) = \iota((f|(A \cap P)) \circ r)(x), & (x \notin A). \end{cases}$$

A straightforward check shows that Φ defined in this way is as required. □

5.4. Remark. The extender Φ in Theorem 5.3 is in general not linear.

6. Remarks

The results derived in this paper suggest the following question:

6.1. Question. Is there, in ZFC, a first countable zero-dimensional Lindelöf K_0-space X for which the family of all nonempty clopen subsets is the union of countably many linked subfamilies all having empty intersection but is not σ-centered?

Let us indicate why this question is nontrivial and interesting. It is interesting since a positive answer would yield, using the same technique as in Section 3 of this paper, an example of a first countable separable Lindelöf K_1-space which is not K_0. The question is nontrivial, since if such an example exists, it cannot be locally compact and it cannot have a first countable compactification, by [12]. Of course there are first countable Lindelöf spaces having no first countable compactification, but these examples are all difficult. At first glance one would hope that a space asked for in Question 6.1 can be linearly orderable, or, generalized orderable, since the only known (nontrivial) class of K_0-spaces not related to metrizable spaces are the generalized orderable spaces, [3, 2.3.1; 17]. However, unfortunately, the example cannot be generalized orderable. For suppose X is generalized orderable and has all properties listed in Question 6.1. Let X^+ be the

Dedekind completion of X. Then X^+ is supercompact (this will not be defined here) and the topology of X^+ is σ-linked. But van Douwen [6] has recently shown that such a space must be separable. Therefore X^+ is separable, which in turn implies that X is separable, contradicting the fact that $\tau(X)$ is not σ-centered.

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