THE REDUCED MEASURE ALGEBRA AND A $K_1$-SPACE WHICH IS NOT $K_0$

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The reduced measure algebra is used to construct, under $CH$, a hereditarily Lindelöf separable $K_1$-space $X$ which is not a $K_0$-space.

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reduced measure algebra $K_0$-space
monotone extension property $K_1$-space

0. Definitions

All topological spaces under discussion are completely regular and $T_1$.

If $X$ is a space, $C^*(X)$ denotes the Banach space of continuous, bounded, real-valued functions on $X$. For a function $f \in C^*(X)$ the sup-norm of $f$ is defined by

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$  

If $A$ is a closed subspace of $X$, then a function $\iota : C^*(A) \to C^*(X)$ satisfying $\iota(f)|A = f$ for each $f \in C^*(A)$ is called an extender. The norm of $\iota$, which is denoted by $\|\iota\|$, is defined by

$$\|\iota\| = \sup\{\|\iota(f)\| : f \in C^*(A), \|f\| = 1\}.$$  

The extender $\iota$ is linear if $\iota(\alpha f + \beta g) = \alpha \iota(f) + \beta \iota(g)$ for all $f, g \in C^*(A)$ and $\alpha, \beta \in \mathbb{R}$; $\iota$ is said to be monotone if $\iota(f) \leq \iota(g)$ provided that $f \leq g$.

A space $X$ is said to have property $D^*_c$, where $c \in \mathbb{R}$, if for every nonempty closed subspace $A$ of $X$ there is a linear extender $\iota : C^*(A) \to C^*(X)$ with norm not exceeding $c$. Similarly, $X$ has the monotone extension property if for every closed subspace $A \subset X$ there is a monotone extender $\iota : C^*(A) \to C^*(X)$. For more information on these concepts see [3, 4, 7, 13, 14].

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A subspace \( A \subseteq X \) is said to be \( K_n \)-
embedded in \( X \) \((n \geq 0)\) provided there is a
function \( \kappa : \tau(A) \rightarrow \tau(X) \) (for each space \( Z \), the
topology of \( Z \) is denoted by \( \tau(Z) \)) such that

(a) \( \kappa(U) \cap A = U \) for all \( U \in \tau(A) \);

(b) if \( n = 0 \), then \( \kappa(\emptyset) = \emptyset \) and \( \kappa(U) \cap \kappa(V) = \kappa(U \cap V) \) for all \( U, V \in \tau(A) \); if
\( n > 0 \), then \( \kappa(U_0) \cap \cdots \cap \kappa(U_n) = \emptyset \) whenever \( U_i \cap U_j = \emptyset \) for \( 0 \leq i < j \leq n \) and
\( U_0, \ldots, U_n \in \tau(A) \).

A space is a \( K_n \)-space if each subspace is \( K_n \)-
embedded. For more information on these concepts see [3, 4, 18, 19].

1. Introduction

The Dugundji Extension Theorem, [9], has been improved in recent years so
that certain versions of it now also work for certain classes of non-metrizable but
mathematically important objects such as CW-complexes [2, 1] and generalized
ordered spaces [13].

One of the most important observations in Dugundji extension theory is that
spaces which satisfy a certain version of the Dugundji Extension Theorem allow
functions on subspaces which simultaneously extend open sets in a nice way. To
be more precise, a space with property \( D^*_\varepsilon \) is \( K_n \)-space where \( n \) is the smallest
integer greater than \( \frac{1}{2}(\varepsilon - 1) \). This observation of van Douwen [4] was used to
construct a first countable, hereditarily Lindelöf, separable space \( H_\infty \) containing a
closed subspace \( A \) having no continuous linear extender from \( C^*(A) \) to \( C^*(H_\infty) \).

Van Douwen’s Example is the topological sum of spaces \( H_n \) \((n \in \mathbb{N})\), where, for
each \( n \), \( H_n \) is a hereditarily Lindelöf, separable \( K_{n+1} \)-space which is not a \( K_n \)-space.
This example left open the question whether every \( K_1 \)-space is a \( K_0 \)-space and the
aim of this paper is to answer this question, [4, p. 301].

We will construct, assuming the Continuum Hypothesis, a hereditarily Lindelöf,
separable \( K_1 \)-space which is not a \( K_0 \)-space. Our example is inspired by an example
in van Mill [19] where we constructed a first countable compact space \( Z \) containing a
closed subspace \( A \) which is \( K_1 \)-embedded but not \( K_0 \)-embedded \((Z \) is not a
\( K_1 \)-space since \( Z \) is separable and contains an uncountable discrete subspace). Our
example is also interesting for another reason. In [3, 3.1] it was shown that any
space with the monotone extension property is a \( K_1 \)-space. Our example has the
monotone extension property but is not \( K_0 \). That answers another question of van
Douwen.

2. Certain subspaces of extremally disconnected compacta

A space is extremally disconnected if the closure of any open set is again open.
As usual, we call a space without isolated points a Luzin (nodec) space, if each
nowhere dense set is countable (closed). There are, under CH, spaces which are both Luzin and nodec, see [21, 20].

The following results are of independent interest.

2.1. Theorem (CH). Let $X$ be an extremally disconnected, dense in itself compactum of weight $2^\omega$. Then $X$ contains a dense nodec subspace. If $X$ moreover satisfies the countable chain condition, then $X$ contains a dense subspace which is both nodec and Luzin.

**Proof.** Let $\mathcal{C}$ be the Boolean algebra of clopen subsets of $X$ and, by CH, list $\mathcal{C}-\{\emptyset\}$ as $\{C_\alpha : \alpha < \omega_1\}$. By induction we will construct, for each $\alpha < \omega_1$, a point $x_\alpha \in X$ and a nowhere dense closed set $Z_\alpha \subset X$ such that

(a) $x_\alpha \in Z_\alpha \subset C_\alpha$,

(b) if $\beta < \alpha$ and if $x_\beta \neq x_\alpha$, then $Z_\beta \cap Z_\alpha = \emptyset$, $x_\beta \notin Z_\alpha$ and $x_\alpha \notin Z_\beta$,

(c) if $D \subset X - Z_\alpha$ is nowhere dense, then $x_\alpha \notin D$.

Suppose that we have constructed the $x_\beta$'s and the $Z_\beta$'s for all $\beta < \alpha < \omega_1$. If there is a $\gamma < \alpha$ such that $x_\gamma \in C_\alpha$, then define $x_\alpha = x_\gamma$ and $Z_\alpha = Z_\gamma$. If not, take $x \in C_\alpha$ so that

$$x \notin \bigcup_{\beta < \alpha} Z_\beta.$$ 

Let $Z \subset C_\alpha$ be a closed $G_\delta$ subset of $X$ missing $\bigcup_{\beta < \alpha} Z_\beta \cup \{x_\beta : \beta < \alpha\}$ but containing $x$. Since $X$ is extremally disconnected and since the cellularity of $X$ is non-measurable, $x$ is not a $P$-point, i.e. there is a closed nowhere dense $G_\delta$ set $S$ containing $x$, [10, 12H].

Define $S' = S \cap Z$. Then $S'$ is also nowhere dense, so $Y = X - S'$ is a locally compact, $\sigma$-compact, dense subspace of $X$. Since dense subspaces of $X$ are $C^\kappa$-embedded, $\beta Y = X$. Since $Y$ is not pseudocompact and $\beta Y$ has weight $2^\omega$, by Kunen, van Mill & Mills [16, 1.3] there is a point $x' \in Y - Y = S'$ such that $x' \notin D$ for any nowhere dense subspace $D \subset Y$. Define $x_\alpha = x'$ and $Z_\alpha = S'$.

Now put $P = \{x_\alpha : \alpha < \omega_1\}$. Clearly $P$ is dense and we claim that $P$ is nodec. Let $D \subset P$ be nowhere dense and suppose that $D$ is not closed. Take $x \in (P \cap \overline{D}) - D$. Choose $\alpha < \omega_1$ such that $x = x_\alpha$. By (b), $Z_\alpha \cap (P - \{x_\alpha\}) = \emptyset$ and therefore $Z_\alpha \cap D = \emptyset$. Since $D \subset Z_\alpha$ is nowhere dense, by (c), $x \notin D$. Contradiction.

If $X$ is ccc, then the $x_\alpha$'s must be chosen more carefully in order for $P = \{x_\alpha : \alpha < \omega_1\}$ to be Luzin. First observe the well-known fact that there is a family $\mathcal{A}$ of $2^\omega$ nowhere dense subsets of $X$ so that each nowhere dense subset of $X$ is contained in some element of $\mathcal{A}$. Indeed, since $X$ is ccc each nowhere dense subset of $X$ is contained in a nowhere dense $G_\delta$ and, since there are only $(2^\omega)^\omega = 2^\omega$ $G_\delta$'s in $X$, we can simply let $\mathcal{A}$ be the family of all nowhere dense $G_\delta$'s of $X$. To make $P$ Luzin we must simply add in the induction hypotheses that $x_\alpha \notin \bigcup_{\beta < \alpha} A_\beta$ (let $\{A_\alpha : \alpha < \omega_1\}$ enumerate $\mathcal{A}$). The rest is routine. □

2.2. Remark. For related ideas see [8, 20].
A space is called retractable if each nonempty closed subspace is a retract. The following Lemma generalizes [3, 3.3].

2.3. Lemma. A Lindelöf nodec space is retractable.

Proof. Let $X$ be a Lindelöf nodec space. First observe that each nowhere dense subset of $X$ is discrete and hence, because $X$ is Lindelöf, countable. Therefore $X$ is Luzin, which easily implies that $X$ is zero-dimensional, [15].

Observe that it clearly suffices that each nowhere dense closed subspace of $X$ is a retract. So let $D \subseteq X$ be closed and discrete. Since $X$ is strongly zero-dimensional there is a disjoint clopen cover $\{U_d: d \in D\}$ of $X$ with $d \in U_d$. Then, as in [3, 3.3], define $r: X \to D$ by $r(x) = d$ iff $x \in U_d$. □

2.4. Corollary. A Lindelöf nodec space is $K_0$.

Proof. By [4, 2.1] it suffices to prove that closed subspaces are $K_0$-embedded. But that immediately follows from Lemma 2.3. □

2.5. Question. Is the statement “Each dense in itself extremally disconnected compactum of weight $2^\omega$ contains a dense nodec subspace” equivalent to CH?

3. The reduced measure algebra

Let $I$ denote the closed unit interval $[0, 1]$, let $\mathcal{M}$ be the Boolean algebra of measurable subsets of $I$, and let $\mathcal{N}$ be the ideal of null-sets. The quotient algebra $\mathcal{M}/\mathcal{N}$ is called the reduced measure algebra. Let $M$ denote its Stone space. Notice that $\mathcal{M}/\mathcal{N}$ is complete and has cardinality $2^\omega$, so that $M$ is an extremally disconnected compactum of weight $2^\omega$.

Let $\lambda$ denote Lebesgue measure on $I$, and for $A \in \mathcal{M}$ let $[A]$ denote the $\mathcal{N}$-equivalence class of $A$.

The following lemma is well-known. The proof is included for completeness sake.

3.1. Lemma. (a) The family $\mathcal{C}$ of nonempty clopen (=closed and open) subsets of $M$ can be written as $\mathcal{C} = \bigcup_{n<\omega} \mathcal{C}_n$, where, for each $n<\omega$, 
\[ (\alpha) \ \bigcap_{n<\omega} \mathcal{C}_n = \emptyset, \text{ and} \]
\[ (\beta) \text{ any two members of } \mathcal{C}_n \text{ meet.} \]
(b) $M$ is not separable.

Proof. For (a), let us first prove a corresponding statement for $\mathcal{M}/\mathcal{N}$. Let $\mathcal{B}$ be a countable (open) basis for $I$ which is closed under finite unions. For each $B \in \mathcal{B}$ define 
\[ \mathcal{L}(B) = \{ A \in \mathcal{M}: \lambda(A \cap B) > \frac{1}{2} \lambda(B)\}. \]
It is clear that any two members of $\mathcal{L}(B)$ meet in a set of positive measure. Also, $\mathcal{L}(B)$ contains three elements which have empty intersection.

We claim that $\mathcal{M} - \mathcal{N} = \bigcup_{B \in \mathcal{L}(B)}$. Indeed, take $A \in \mathcal{M} - \mathcal{N}$ and construct a compact $K \subset A$ with $\lambda(K) > 0$. There is a $B \in \mathcal{B}$ with $B \supseteq K$ and $\lambda(K) > \frac{1}{2}\lambda(B)$. This shows that $K \in \mathcal{L}(B)$. Hence $A \in \mathcal{L}(B)$.

From these observations (a) is immediately clear.

For (b), let $(p_n)_n$ be any sequence in $M$. Since $p_n$ is an ultrafilter in the Boolean algebra $\mathcal{M}/\mathcal{N}$ we can find $P_n \in \mathcal{M} - \mathcal{N}$ with $[P_n] \subseteq p_n$ and $\lambda(P_n) < 2^{-2^{-n}}$. Then $\{x \in M : [I - \bigcup_n P_n] \subseteq x\}$ is an open set in $M$ which is nonempty and misses each $P_n$. □

A family of sets is called linked (centered) if any two (any finite number) of its members meet. Call a family of sets $\sigma$-linked ($\sigma$-centered) if it is the union of countably many linked (centered) subfamilies. A compact space the topology of which is $\sigma$-centered is clearly separable [5], so that we can reformulate Lemma 2.1 by saying that $\tau(M)$ is $\sigma$-linked but not $\sigma$-centered (this is not entirely true; in Lemma 2.1 we proved that $\tau(M)$ is the union of countably many linked subfamilies which all have empty intersection and it is precisely this fact which makes our construction work).

By Lemma 2.1(a) $M$ satisfies the countable chain condition, so that by Theorem 2.1 $M$ has a dense subspace $P$ which is both Luzin and nodec, see also [21]. Since $P$ is dense in $M$, $\tau(P)$ is $\sigma$-linked but not $\sigma$-centered. We have constructed the following example.

3.2. Example (CH). There is a space $P$ which is both Luzin and nodec and which moreover has the following properties:

(a) the family of nonempty clopen subsets of $P$ is not $\sigma$-centered,

(b) the family of nonempty clopen subsets of $P$ is the union of countably many linked subfamilies having all empty intersection.

4. The example

Let $P$ be the space of Example 3.2. The family of nonempty clopen subsets of $P$ will be denoted by $\mathcal{C}$. By 3.2(b), we can write $\mathcal{C}$ as $\bigcup_{n<\omega} \mathcal{C}_n$ where each $\mathcal{C}_n$ is linked while moreover $\bigcap \mathcal{C}_n = \emptyset$.


By Zorn's Lemma extend each $\mathcal{C}_n$ to a maximal linked system $\mathcal{L}_n \subset \mathcal{C}$, i.e. a linked system in $\mathcal{C}$ not properly contained in any other linked system in $\mathcal{C}$. For each $C \in \mathcal{C}$ define

$$C^+ = C \cup \{n < \omega : C \in \mathcal{L}_n\}.$$ 

Notice that if $C \in \mathcal{C}$, then either $C \in \mathcal{L}_n$ or $P - C \in \mathcal{L}_n$, and also that $\bigcap \mathcal{L}_n = \emptyset$ since $\bigcap \mathcal{C}_n = \emptyset$. 


Fact 1. If \( F \subseteq \omega \) is finite and if \( x \in P \), then there is a \( C \in \mathcal{C} \) containing \( x \) such that \( C^+ \cap F = \emptyset \).

Take \( n \in F \) arbitrarily. Since \( \bigcap \mathcal{L}_n = \emptyset \), there is an \( L_n \in \mathcal{L}_n \) not containing \( x \). Put \( C = \bigcap_{n \in F} (P - L_n) \). Then \( C \) is as required.

Fact 2. Let \( F, G \subseteq \omega \) be finite, let \( x \in P \), and, for each \( n \in F \) let \( C_n \) be a clopen neighborhood of \( x \) in \( P \). Then there is a clopen neighborhood \( C \) of \( x \) such that \( C^+ \subseteq \bigcap_{n \in F} C_n^+ - G \).

Put \( E = \bigcap_{n \in F} C_n \) and let \( D \) be a clopen neighborhood of \( x \) such that \( D^+ \cap G = \emptyset \) (Fact 1).

Then \( C = E \cap D \) is as required.

The underlying set of \( X \) is \( P \cup \omega \). The topology of \( X \) is generated by the collection \( \{C^+: C \in \mathcal{C}\} \cup \{\{n\}: n \in \omega\} \).

Notice that this implies that the points of \( \omega \) are isolated and that a basic neighborhood of \( x \in P \subseteq X \) has the form \( C^+ \), where \( x \in C \in \mathcal{C} \). Since \( C^+ \cap P = C \) for all \( C \in \mathcal{C} \), the inclusion \( P \rightarrow X \) is an embedding.

Fact 3. \( X \) is a zero-dimensional Hausdorff space.

Take \( C \in \mathcal{C} \). Then \( C^+ \cap (P-C)^+ = \emptyset \) and \( C^+ \cup (P-C)^+ = X \). This implies that \( C^+ \) is clopen. The rest is clear.

Fact 4. \( X \) is Lindelöf and \( \omega \) is dense in \( X \). In particular, \( X \) is separable.

Since \( P \) is Lindelöf and \( \omega \) is countable, the Lindelöfness of \( X \) is trivial. We will now show that \( P \subseteq \omega^\circ \). Take \( x \in P \) and let \( U \) be any neighborhood of \( x \) in \( X \). Take \( C \in \mathcal{C} \) so that \( x \in C \subseteq C^+ \subseteq U \). Let \( C \in \mathcal{C}_n \). Then \( C \in \mathcal{L}_n \), or equivalently, \( n \in C^+ \).

That shows that \( U \cap \omega \neq \emptyset \).

Fact 5. \( X \) is not a \( K_0 \)-space.

We claim that there is no \( K_0 \)-function \( \kappa: \tau(P) \rightarrow \tau(X) \). For, to the contrary, assume there is a \( K_0 \)-function \( \kappa: \tau(P) \rightarrow \tau(X) \). since \( X - P \) is countable this would imply that \( \tau(P) \) is \( \sigma \)-centered, a contradiction.

If \( U \subseteq P \) is open, then define
\[
U^+ = U \cup \{n < \omega: \exists C \in \mathcal{L}_n (C \subseteq U)\}.
\]
Notice that \( U^+ \) is open and that if \( U \) is clopen the set \( U^+ \) defined here equals the set \( U^+ \) defined above.
Fact 6. $X$ is a $K_1$-space.

By [4, 2.1] it suffices to prove that closed subspaces of $X$ allow $K_1$-functions. Therefore, let $A \subseteq X$ be closed. Since, by Corollary 2.5, $P$ is a $K_0$-space, there is a $K_0$-function $\rho: \tau(A \cap P) \to \tau(P)$ (in fact, a $K_1$-function would suffice). Define $\kappa: \tau(A) \to \tau(X)$ by

$$\kappa(U) = U \cup ((\rho(U \cap P))^+ - A).$$

It is clear that $\kappa(U) \cap A = U$. Let us observe that $\kappa(U)$ is open. Since $X - P$ consists of isolated points of $X$ we only need to check that $\kappa(U)$ is a neighborhood of any point of $\kappa(U) \cap P$. So take $x \in \kappa(U) \cap P$. If $x \notin A$ take a clopen $C \subseteq P$ so that $C \subseteq \rho(U \cap P)$ while moreover $x \in C \subseteq C^+ \subseteq X - A$. Then $C^+ \subseteq \kappa(U)$, and consequently $\kappa(U)$ is a neighborhood of $x$. If $x \in A$, take a clopen $F \subseteq P$ so that $x \in F \subseteq \rho(U \cap P)$ while moreover $F^+ \cap A \subseteq U$. Then $F^+ \subseteq \kappa(U)$ so that in this case $\kappa(U)$ is also a neighborhood of $x$. We conclude that $\kappa(U)$ is open.

If $U \cap V = \emptyset$, then $\kappa(U) \cap \kappa(V) = \emptyset$ since $\rho(U \cap P) \cap \rho(V \cap P) = \emptyset$ (which implies that $(\rho(U \cap P))^+ \cap (\rho(V \cap P))^+ = \emptyset$).

Therefore $\kappa$ is a $K_1$-function.

5. The monotone extension property

We will now prove that $X$ has the monotone extension property. From this it also follows that $X$ is a $K_1$-space, [3, 3.1].

Let $P$ and $X$ be as in Section 4. In the following Lemma we will use a technique essentially due to J. Jensen (see [22, II.4.5]).

If $A \subseteq \mathbb{R}$ let $h(A)$ denote the closed convex hull of $A$ in $\mathbb{R}$.

5.1. Lemma. There is an extender $\Phi: C^*(P) \to C^*(X)$ so that

(a) $\|\Phi(f)\| = \|f\|$ for all $f \in C^*(P)$, and

(b) if $f \leq g$, then $\Phi(f) \leq \Phi(g)$.

Proof. Let $f \in C^*(P)$. Define $\Phi(f): X \to \mathbb{R}$ by

$$\Phi(f)(x) = f(x) \quad (x \in P),$$

and

$$\{\Phi(f)(n)\} = \bigcap \{h(f(L)) : L \in \mathcal{L}_n\} \quad (n \in \omega).$$

Clearly $\Phi(f)|P = f$. We claim that $\Phi(f)$ defined in this way is as required.

Claim 1. $\Phi(f)$ is well-defined.

First observe that the fact that $f \in C^*(P)$ and the fact that $\mathcal{L}_n$ is a linked system imply that $\bigcap \{h(f(L)) : L \in \mathcal{L}_n\} \neq \emptyset$ for all $n \in \omega$. Suppose that for certain $n \in \omega$ $\bigcap \{h(f(L)) : L \in \mathcal{L}_n\}$ contains two distinct points, say $a$ and $b$. Without loss of
generality $a < b$. Take a clopen set $E \subset P$ such that
\[ f^{-1}(-\infty, \frac{3}{4}a + \frac{1}{4}b) \subset E \subset f^{-1}(-\infty, \frac{1}{4}a + \frac{3}{4}b), \]
Since $\mathcal{L}_n$ is a maximal linked system, either $E \in \mathcal{L}_n$ or $P - E \in \mathcal{L}_n$. If $E \in \mathcal{L}_n$, then $b \in h(f(E)) \subset (-\infty, \frac{1}{4}a + \frac{3}{4}b]$, which is impossible. If $P - E \in \mathcal{L}_n$, then
\[ f^{-1}[\frac{3}{4}a + \frac{1}{4}b, \infty) \subset P - E \subset f^{-1}(\frac{3}{4}a + \frac{1}{4}b, \infty), \]
so the same contradiction can be derived.

Claim 2. $\Phi(f)$ is continuous.

The reader can easily check that
\[ \Phi(f)^{-1}(-\infty, s) = \bigcap \{ C^+ : C \in \mathcal{C} \text{ and } \exists \varepsilon > 0 : f^{-1}(-\infty, s + \varepsilon) \subset C \}, \]
\[ \Phi(f)^{-1}[s, \infty) = \bigcap \{ C^+ : C \in \mathcal{C} \text{ and } \exists \varepsilon > 0 : f^{-1}[s - \varepsilon, \infty) \subset C \} \]
for all $s \in \mathbb{R}$.

Claim 3. $\|\Phi(f)\| = \|f\|$ for all $f \in C^*(P)$, in particular, $\Phi(f) \in C^*(X)$.

This follows immediately from the definition of $\Phi(f)$.

Claim 4. If $f \leq g$, then $\Phi(f) \leq \Phi(g)$.

This requires proof. Suppose that $f \leq g$ but $\Phi(f) \not\leq \Phi(g)$ for certain $f, g \in C^*(P)$. Since $\Phi(f)|P = f$ and $\Phi(g)|P = g$, we can find $n \in \omega$ such that
\[ \Phi(g)(n) < \Phi(f)(n). \]
Since $f, g \in C^*(P)$ we can find $M, N \in \mathcal{L}_n$ such that
\[ h(f(M)) \cap h(g(N)) = \emptyset \]
(argument: if $h(f(M)) \cap h(g(N)) \neq \emptyset$ for all $M, N \in \mathcal{L}_n$ then
\[ \bigcap \{ h(f(M)) : M \in \mathcal{L}_n \} \cap \bigcap \{ h(g(N)) : N \in \mathcal{L}_n \} \cap h(f(P) \cup g(P)) \neq \emptyset, \]
i.e. $\Phi(g)(n) = \Phi(f)(n)$, which is impossible).

Since $\Phi(g)(n) < \Phi(f)(n)$ for all $r \in h(f(M))$ and $s \in h(g(N))$ we have that $s < r$. Now, $\mathcal{L}_n$ is a linked system, so that $M$ and $N$ meet, say $x \in M \cap N$. Since $f(x) \in h(f(M))$ and $g(x) \in h(g(N))$ it follows that $g(x) < f(x)$. But this contradicts the fact that $f \leq g$. \qed

5.2. Remark. From the proof of Lemma 4.1, the definition of $\Phi$, Claim 1 and Claim 2 are known, see [22, II 4.5], since $\Phi(f)$ is already continuous in the weaker superextension topology on $X$. Since we used the explicit construction of $\Phi(f)$ in Claim 3 and Claim 4, for completeness sake we have also included the proofs of Claim 1 and Claim 2.

We now come to the main result in this section.

5.3. Theorem. Let $A$ be any closed subspace of $X$. Then there is an extender $\Phi : C^*(A) \to C^*(X)$ such that
(a) \( \|\Phi(f)\| = \|f\| \) for all \( f \in C^*(A) \), and

(b) if \( f \leq g \), then \( \Phi(f) \leq \Phi(g) \).

In particular, \( X \) has the monotone extension property.

**Proof.** Let \( A \subseteq X \) be closed. Without loss of generality \( A \neq \emptyset \).

Suppose first that \( A \cap P = \emptyset \). Then, since \( X - P \) is countable, \( A \) is a closed discrete subspace of the zero-dimensional Lindelöf space \( X \). This implies that \( A \) is a retract of \( X \). Let \( r \) retract \( X \) onto \( A \). Define \( \Phi: C^*(A) \to C^*(X) \) by

\[
\Phi(f)(x) = f(r(x)).
\]

Then \( \Phi \) is as required (this is well-known of course).

Now suppose that \( A \cap P \neq \emptyset \) and let \( \iota: C^*(P) \to C^*(X) \) be an extender as in Lemma 5.1. By Lemma 2.3, \( P \) is retractable, so let \( r: P \to A \cap P \) be a retraction. Define \( \Phi: C^*(A) \to C^*(X) \) by

\[
\begin{align*}
\Phi(f)(x) &= f(x), \quad (x \in A), \\
\Phi(f)(x) &= \iota((f|A \cap P) \circ r)(x), \quad (x \notin A).
\end{align*}
\]

A straightforward check shows that \( \Phi \) defined in this way is as required. \( \square \)

**5.4. Remark.** The extender \( \Phi \) in Theorem 5.3 is in general not linear.

6. Remarks

The results derived in this paper suggest the following question:

**6.1. Question.** Is there, in ZFC, a first countable zero-dimensional Lindelöf \( K_0 \)-space \( X \) for which the family of all nonempty clopen subsets is the union of countably many linked subfamilies all having empty intersection but is not \( \sigma \)-centered?

Let us indicate why this question is nontrivial and interesting. It is interesting since a positive answer would yield, using the same technique as in Section 3 of this paper, an example of a first countable separable Lindelöf \( K_1 \)-space which is not \( K_0 \). The question is nontrivial, since if such an example exists, it cannot be locally compact and it cannot have a first countable compactification, by [12]. Of course there are first countable Lindelöf spaces having no first countable compactification, but these examples are all difficult. At first glance one would hope that a space asked for in Question 6.1 can be linearly orderable, or, generalized orderable, since the only known (nontrivial) class of \( K_0 \)-spaces not related to metrizable spaces are the generalized orderable spaces, [3, 2.3.1; 17]. However, unfortunately, the example cannot be generalized orderable. For suppose \( X \) is generalized orderable and has all properties listed in Question 6.1. Let \( X^+ \) be the
Dedekind completion of $X$. Then $X^+$ is supercompact (this will not be defined here) and the topology of $X^+$ is $\sigma$-linked. But van Douwen [6] has recently shown that such a space must be separable. Therefore $X^+$ is separable, which in turn implies that $X$ is separable, contradicting the fact that $\tau(X)$ is not $\sigma$-centered.

References