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Representing Countable Groups by Homeomorphism Groups in Hilbert Space

Jan van Mill

Subfaculteit Wiskunde, Vrije Universiteit, De Boelelaan 1081, Amsterdam, The Netherlands

1. Introduction

All topological spaces under discussion are separable metric and if X is a space then $\mathrm{Aut}\,h(X)$ *denotes the autohomeomorphism group of X.*

De Groot and Wille [8] showed that each countable group is isomorphic to the autohomeomorphism group of some 1-dimensional Peano continuum. Subsequently, de Groot [7] proved that each group is isomorphic to the autohomeomorphism group of some metrizable space.

The aim of this paper is to prove Theorem 1.1 below, which shows that there is a very peculiar partition of Hilbert space ℓ_2 such that each countable group is isomorphic to the autohomeomorphism group of a unique element of the partition.

1.1. Theorem. *There is a partition $\{X_i : i \in I\}$ of Hilbert space ℓ_2 such that for each $i \in I$,*

(1) *X_i is connected, locally connected and dense,*

(2) *each autohomeomorphism of X_i extends to an autohomeomorphism of ℓ_2,*

(3) $\mathrm{Aut}\,h(X_i)$ *is countable,*

(4) *if $H \subset \mathrm{Aut}\,h(X_i)$ is a subgroup, then there is a countable dense set $D \subset X_i$ such that H is isomorphic to* $\mathrm{Aut}\,h(X_i \backslash D)$*; moreover,*

(5) *for each countable group G there is precisely one index $i \in I$ such that G and* $\mathrm{Aut}\,h(X_i)$ *are isomorphic.*

In the proof of this Theorem we use an idea in van Mill [10] and results from infinite-dimensional topology.

2. Preliminaries

A cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. $\mathfrak{c}$ denotes $2^{\aleph_0}$.

The following classical results will be important in our construction.

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2.1. Lemma. (a) (Lavrentieff [9]): *Let X and Y be topologically complete. If $A \subset X$ and $B \subset Y$ and if $h: A \to B$ is a homeomorphism, then there are G_δ-subsets $A' \subset X$ and $B' \subset Y$ such that $A \subset A'$ and $B \subset B'$ while moreover h can be extended to a homeomorphism $h': A' \to B'$.*

(b) (Sierpiński [11]): *If X is a continuum and if $\mathscr{A}$ is a partition of X in countably many closed sets, then at most one element of $\mathscr{A}$ is nonempty.*

The domain and range of a function f will be denoted by dom(f) and range(f), respectively.

As usual, Hilbert space is denoted by ℓ_2, and the Hilbert cube $\prod_{1}^{\infty} [-1, 1]_i$ by Q.

A closed subset A of a space X is called a *Z-set* provided that for each $\varepsilon > 0$ and for each map $f: Q \to X$, there is a map $g: Q \to X \backslash A$ such that $d(f, g) < \varepsilon$. A $\sigma - Z$-set is a countable union of Z-sets. Observe that a Z-set is nowhere dense. AR is an abbreviation for Absolute Retract.

A subset X of a space Y is called *homogeneously embedded* in Y provided that each $h \in \mathrm{Aut}\,h(X)$ can be extended to an $\tilde{h} \in \mathrm{Aut}\,h(Y)$.

Let A be a countable set. For each $a \in A$, the a^{th} coordinate of a point $x \in [-1, 1]^A$ is denoted by x_a.

If $H \subset \mathrm{Aut}\,h(\ell_2)$ and if $x \in \ell_2$, then $V(x, H) = \{h(x): h \in H\}$.

3. Special subgroups of Aut$h(\ell_2)$

It is well-known that each countably infinite group G is isomorphic to a subgroup of Aut$h(Q)$. Simply represent Q by $[-1, 1]^G$ and define a function $f: G \to \mathrm{Aut}\,h([-1, 1]^G)$ by

$$f(g)(x)_h = x_{hg}.$$

It is easy to see that $f(G)$ is a subgroup of Aut$h([-1, 1]^G)$ and that $f: G \to f(G)$ is an isomorphism.

3.1. Definition. Let X be a space. A subgroup G of Aut$h(X)$ is called *special*, provided that for all distinct $g, h \in G$ and for each $x \in X$ we have that $g(x) \neq h(x)$.

Observe that the autohomeomorphism group of a space with the fixed point property has only one special subgroup, namely the subgroup consisting of the identity only. Consequently, the autohomeomorphism group of the Hilbert cube has no interesting special subgroups. We will show that for Aut$h(\ell_2)$ the situation is completely different.

Let G be a countably infinite group and let $f: G \to \mathrm{Aut}\,h([-1, 1]^G)$ be as above. For convenience, put $\tilde{Q} = [-1, 1]^G$.

3.1. Lemma. *Let $g, h \in G$ be distinct. The set $A(g, h) = \{x \in \tilde{Q}: f(g)(x) = f(h)(x)\}$ is a Z-set in $\tilde{Q}$.*

Proof. It is clear that $A(g, h)$ is closed. Let $\varepsilon > 0$ and enumerate G by $\{g_n: n \in \mathbb{N}\}$ in such a way that $n \neq m$ implies that $g_n \neq g_m$. Find a sufficiently large $n \in \mathbb{N}$ such that the map $r:\tilde{Q}\to\tilde{Q}$ defined by

$$r(x)_{g_i}=\begin{cases} x_{g_i} & \text{if } i\leqq n,\\ \dfrac{1}{i} & \text{if } i>n,\end{cases}$$

moves the points less than ε. We claim that range$(r)\cap A(g,h)=\emptyset$. Since G is countably infinite and since the sets

$$A=\{g_1g^{-1},\ldots,g_ng^{-1}\},$$

and

$$B=\{g_1h^{-1},\ldots,g_nh^{-1}\}$$

are both finite, there is an $s\in G\backslash(A\cup B)$. Now take any point $t\in$range(r). We claim that $f(g)(t)_s\neq f(h)(t)_s$, which will prove that $t\notin A(g,h)$. First observe that

$$f(g)(t)_s=t_{sg} \quad\text{and}\quad f(h)(t)_s=t_{sh}.$$

Since by construction, $\{sg,sh\}\cap(g_1,\ldots,g_n\}=\emptyset$, by the definition of r and by the fact that $sg\neq sh$, we conclude that $t_{sg}\neq t_{sh}$, which is as required. We therefore may conclude that $A(g,h)$ is a Z-set. □

For all distinct $g,h\in G$, let $A(g,h)$ be defined as in the previous Lemma and put

$$A=\bigcup\{A(g,h):g,h\in G \text{ and } g\neq h\}.$$

By Lemma 3.1, A is a $\sigma-Z$-set.

3.2. Lemma. *For all $g\in G$ the restriction $f(g)|\tilde{Q}\backslash A\in \mathrm{Aut}\,h(\tilde{Q}\backslash A)$.*

Proof. Take $x\in\tilde{Q}\backslash A$. We first claim that $y=f(g)(x)\in\tilde{Q}\backslash A$. Indeed, suppose that this is not true. Then we can find distinct $h,k\in G$ such that $f(h)(y)=f(k)(y)$. Consequently,

$$f(hg)(x)=f(kg)(x),$$

and since $hg\neq kg$ this proves that $x\in A$, contradiction. Consequently, $f(g)|\tilde{Q}\backslash A$ is an embedding.

By the above observation, it is also true that $z=f(g^{-1})(x)\in\tilde{Q}\backslash A$. Since

$$f(g)(z)=f(gg^{-1})(x)=x,$$

we conclude that $f(g)|\tilde{Q}\backslash A$ is surjective.

These two observations show that $f(g)|\tilde{Q}\backslash A\in \mathrm{Aut}\,h(\tilde{Q}\backslash A)$. □

3.3.3 Corollary. *G is isomorphic to a special subgroup of* $\mathrm{Aut}\,h(\tilde{Q}\backslash A)$.

Proof. Define $F:G\to\mathrm{Aut}\,h(\tilde{Q}\backslash A)$ by $F(g)=f(g)|\tilde{Q}\backslash A$. It is clear that $F(G)$ is a special subgroup of $\mathrm{Aut}\,h(\tilde{Q}\backslash A)$ and that $F:G\to F(G)$ is an isomorphism (use that $\tilde{Q}\backslash A$ is dense in $\tilde{Q}$). □

We now come to the main result in this section.

3.4. Theorem. *Each countable group is isomorphic to a special subgroup of* $\mathrm{Aut}\,h(\ell_2)$.

Proof. Let G be a countably infinite group and let A be as above. Then G is isomorphic to a special subgroup of $\mathrm{Aut}\,h(\tilde{Q}\backslash A)$ (Corollary 3.3). It is well-known, and easy to prove, that the complement of any $\sigma-Z$-set in Q is a topologically complete AR. For a proof, see for instance Anderson et al. [3]. By a deep result of Toruńczyk [12], if X is any topologically complete AR, then $X\times\ell_2\approx\ell_2$. Consequently, $(\tilde{Q}\backslash A)\times\ell_2\approx\ell_2$. The desired result now immediately follows from the trivial observation that if $G\subset\mathrm{Aut}\,h(S)$ is a special subgroup, and T is any space, then $\{g\times\mathrm{id}:g\in G\}\subset\mathrm{Aut}\,h(S\times T)$ is a special subgroup which is isomorphic to G.

Next, if G is finite, observe that G is isomorphic to a subgroup of $G\times\mathbb{Z}$. Therefore, this case follows from the countably infinite case. □

4. A Technical Lemma

The aim of this section is to prove a technical Lemma which is of crucial importance in the proof of Theorem 1.1, which we will give in Sect. 5.

4.1. Lemma. *Let H be a countable group and let $A\subset\ell_2$ be of cardinality less than $\mathfrak{c}$. If $x\in\ell_2\backslash A$ then there exists a countable special subgroup $G\subset\mathrm{Aut}\,h(\ell_2)$ such that:*

(1) *H and G are isomorphic,*

(2) *$\{g(x):g\in G\}\cap A=\emptyset$.*

Proof. By a result of Anderson [2], the spaces ℓ_2 and $\mathbb{R}^\infty$ are homeomorphic. It therefore suffices to prove the Lemma for $\mathbb{R}^\infty$ instead of ℓ_2. It is clear that there is a topologically complete space X containing a point t and a family $\mathscr{A}$ of $\mathfrak{c}$ closed copies of $\mathbb{R}^\infty$ such that for all distinct $A, B\in\mathscr{A}$ we have that $A\cap B=\{t\}$. Since X is topologically complete, it embeds in $\mathbb{R}^\infty$ as a closed subset and it is clear that we can reembed any closed subset in $\mathbb{R}^\infty$ such that it projects onto a point in infinitely many coordinates. An appeal to the homogeneity of $\mathbb{R}^\infty$ and using Anderson [1] or Bessaga and Pelczyński [4, 6.1], yields the following: there is a family $\mathscr{B}$ of $\mathfrak{c}$ Z-set copies of $\mathbb{R}^\infty$ in $\mathbb{R}^\infty$ such that for all distinct $B, E\in\mathscr{B}$ we have that $B\cap E=\{x\}$.

Since $|\mathscr{B}|=\mathfrak{c}$ and since $|A|<\mathfrak{c}$, we can find a $B\in\mathscr{B}$ such that $B\cap A=\emptyset$. By Theorem 3.4 we can find a countable special subgroup $G'\subset\mathrm{Aut}\,h(B)$ such that G' and H are isomorphic. Since B is a Z-set in $\mathbb{R}^\infty$, by the Homeomorphism Extension Theorem of Anderson [1], see also [4, 6.2], there is a homeomorphism of pairs $(B,\mathbb{R}^\infty)\approx(B\times\{(0,0,\ldots)\}, B\times\mathbb{R}^\infty)$. As in the proof of Theorem 3.4, we may therefore conclude that G' is isomorphic to a countable special subgroup $G\subset\mathrm{Aut}\,h(\mathbb{R}^\infty)$ with the property that for all $g\in G$ we have that $g|B\in G'$. It is clear that G is as required. □

5. Proof of Theorem 1.1

In this section we will give the proof of Theorem 1.1.

Let $\{G_\alpha:\alpha<\mathfrak{c}\}$ be a family of countable groups such that

(1) if H is any countable group, then there is an index $\alpha<\mathfrak{c}$ such that H and G_α are isomorphic.

(2) if $\alpha<\beta<\mathfrak{c}$, then G_α and G_β are not isomorphic.

Let $\{t_\alpha:\alpha<\mathfrak{c}\}$ be an enumeration of ℓ_2 and let $\{f_\alpha:\alpha<\mathfrak{c}\}$ enumerate the family

$\mathscr{F}=\{f:\mathrm{dom}(f)$ and $\mathrm{range}(f)$ are G_δ-subsets of ℓ_2 and

$f:\mathrm{dom}(f)\to\mathrm{range}(f)$ is a homeomorphism$\}$

such that every $f\in\mathscr{F}$ is listed $\mathfrak{c}$ times (it is very important, as we will see later, that $\mathscr{F}$ is listed in this way).

For each $\alpha<\mathfrak{c}$ we will construct a countable special subgroup $H_\alpha\subset\mathrm{Aut}\,h(\ell_2)$ and for each $\xi\leqq\alpha$ a point $x^\alpha_\xi\in\ell_2$ such that

(3) H_α and G_α are isomorphic,

(4) if κ, $\mu\leqq\alpha$, $\xi\leqq\kappa$ and $\eta\leqq\mu$ and if either $\kappa\neq\mu$ or $\xi\neq\eta$, then $V(x^\kappa_\xi,H_\xi)\cap V(x^\mu_\eta,H_\eta)=\emptyset$.

(5) if $\xi<\alpha$ then $V(x^\alpha_\xi,H_\xi)\cap\{f_\gamma(x^\gamma_\xi):\xi\leqq\gamma<\alpha\,\&\,x^\gamma_\xi\in\mathrm{dom}(f_\gamma)\}=\emptyset$,

(6) if $t_\alpha\notin\bigcup_{\beta<\alpha}\bigcup_{\xi\leqq\beta}V(x^\beta_\xi,H_\xi)$, then $x^\alpha_\alpha=t_\alpha$,

(7) for each $\xi<\alpha$ such that $|\{x\in\mathrm{dom}(f_\alpha):f_\alpha(x)\notin V(x,H_\xi)\}|=\mathfrak{c}$ we have that $x^\alpha_\xi\in\mathrm{dom}(f_\alpha)$ and $f_\alpha(x^\alpha_\xi)\notin\bigcup_{\xi\leqq\gamma\leqq\alpha}V(x^\gamma_\xi,H_\xi)$.

Suppose that we have defined H_β and for each $\xi\leqq\beta$ the points x^β_ξ for all $\beta<\alpha$.

If $t_\alpha\notin Z=\bigcup_{\beta<\alpha}\bigcup_{\xi\leqq\beta}V(x^\beta_\xi,H_\xi)$, then let $x^\alpha_\alpha=t_\alpha$ and by Lemma 4.1 find a countable special subgroup $H\subset\mathrm{Aut}\,h(\ell_2)$ such that H and G_α are isomorphic, while moreover $V(t_\alpha,H)\cap\bigcup_{\beta<\alpha}\bigcup_{\xi\leqq\beta}V(x^\beta_\xi,H_\xi)=\emptyset$.

Define $H_\alpha=H$.

If $t_\alpha\in Z$, then let x^α_α be any point of $\ell_2\setminus Z$ and again apply Lemma 4.1 to find a countable special subgroup $H\subset\mathrm{Aut}\,h(\ell_2)$ such that H and G_α are isomorphic, while moreover $V(x^\alpha_\alpha,H)\cap Z=\emptyset$. Again define $H_\alpha=H$.

This defines x^α_α and H_α and we will now proceed to construct the points x^α_ξ for all $\xi<\alpha$. Suppose that the points x^α_ξ have been defined for all $\xi<\gamma<\alpha$ such that their choice does not contradict one of the statements (3)–(7) [formally we have to formulate again appropriate induction hypotheses, but since it follows directly from (3)–(7) what these hypotheses should be, we will not bother to state them explicitly]. We will construct x^α_γ. Define

$$S=\bigcup_{\beta<\alpha}\bigcup_{\xi\leqq\beta}V(x^\beta_\xi,H_\xi)\cup\bigcup_{\xi<\gamma}V(x^\alpha_\xi,H_\xi)\cup V(x^\alpha_\alpha,H_\alpha).$$

Observe that $|S|\leqq|\alpha|\cdot\aleph_0<\mathfrak{c}$. This implies that the set

$$T=\bigcup_{\beta\leqq\alpha}f_\beta(\mathrm{dom}(f_\beta)\cap S)$$

does also have cardinality at most $|\alpha|\cdot\aleph_0<\mathfrak{c}$.

5.1. Lemma. *Let $A,B\subset\ell_2$ such that $|A|=\mathfrak{c}$ and $|B|<\mathfrak{c}$. Then $|\{x\in A:V(x,H_\gamma)\cap B=\emptyset\}|=\mathfrak{c}$.*

Proof. Suppose that $|\{x\in A:V(x,H_\gamma)\cap B\neq\emptyset\}|=\mathfrak{c}$. We will derive a contradiction. Since $\mathfrak{c}$ has uncountable cofinality, i.e. $\mathfrak{c}$ is not the sum of countably many smaller cardinals, there is a set $A_0\subset A$ of cardinality $\mathfrak{c}$ and an $h\in H_\gamma$ such that $h(A_0)\subset B$. Since h is one to one and $|B|<\mathfrak{c}$, this is impossible. Therefore

$|\{x\in A : V(x,H_\gamma)\cap B \neq \emptyset\}| < \mathfrak{c}$, from which it follows that $|\{x\in A : V(x,H_\gamma)\cap B = \emptyset\}| = \mathfrak{c}$. □

Now if $|\{x\in \mathrm{dom}(f_\alpha) : f_\alpha(x)\notin V(x,H_\gamma)\}| < \mathfrak{c}$, then let x_γ^α be any point of $\ell_2\setminus(S\cup T)$ such that

$$V(x_\gamma^\alpha, H_\gamma)\cap(S\cup T) = \emptyset.$$

Such a choice for x_γ^α is possible by Lemma 5.1. Therefore, suppose that

$$A = \{x\in \mathrm{dom}(f_\alpha) : f_\alpha(x)\notin V(x,H_\gamma)\}$$

has cardinality $\mathfrak{c}$. By Lemma 5.1, $A_0 = \{x\in A : V(x,H_\gamma)\cap(S\cup T) = \emptyset\}$ has cardinality $\mathfrak{c}$. Since f_α is one to one, there is also a set $A_1 \subset A_0$ of cardinality $\mathfrak{c}$ such that

$$f_\alpha(A_1)\cap(S\cup T) = \emptyset.$$

Let $x_\gamma^\alpha \in A_1$ be arbitrarily chosen. It is clear that this choice of x_γ^α is as required.

This completes the transfinite induction.

For every $\alpha < \mathfrak{c}$ put

$$X_\alpha = \bigcup_{\gamma\geqq\alpha} V(x_\alpha^\gamma, H_\alpha).$$

5.2. Lemma. *If* $\alpha<\beta<\mathfrak{c}$ *then* $X_\alpha\cap X_\beta = \emptyset$ *and* $\bigcup_{\alpha<\mathfrak{c}} X_\alpha = \ell_2$.

Proof. Let $\alpha<\beta$ and suppose that we can find $\kappa\geqq\alpha$ and $\mu\geqq\beta$ such that

$$V(x_\alpha^\kappa, H_\alpha)\cap V(x_\beta^\mu, H_\beta) \neq \emptyset. \tag{$*$}$$

Let $\theta = \max\{\beta,\kappa,\mu\}$. Then $\theta<\mathfrak{c}$ and therefore (∗) contradicts (4).

That $\bigcup_{\alpha<\mathfrak{c}} X_\alpha = \ell_2$ immediately follows from (6). □

It follows that $\{X_\alpha : \alpha<\mathfrak{c}\}$ is a partition of ℓ_2 and we claim that this is the required partition.

5.3. Lemma. *For each* $a<\mathfrak{c}$ *and for each Cantor set* $K\subset\ell_2$ *we have that* $X_\alpha\cap K\neq\emptyset$.

Proof. Let $L\subset\ell_2$ be a Cantor set not intersecting

$$\bigcup_{h\in H_\alpha} h(K)$$

and let $f : K\to L$ be any homeomorphism. Choose an index $\gamma>\alpha$ such that $f = f_\gamma$. By (7) it now follows that $x_\alpha^\gamma \in \mathrm{dom}(f_\gamma)\cap X_\alpha$. □

5.4. Corollary. *Each* X_α *is connected, locally connected and dense in* ℓ_2.

Proof. Since no countable set separates some nonempty open subset of ℓ_2, this immediately follows from Lemma 5.3. □

5.5. Lemma. *Let S and T be* G_δ*-subsets of* ℓ_2 *both containing* X_α. *If* $f : S\to T$ *is a homeomorphism such that* $f(X_\alpha) = X_\alpha$, *then* $|\{x\in S : f(x)\notin V(x,H_\alpha)\}| < \mathfrak{c}$.

Proof. Suppose not, and choose an index $\gamma>\alpha$ such that $f = f_\gamma$. Then, by (7), at stage γ we have chosen a point $x_\alpha^\gamma\in\mathrm{dom}(f_\gamma)\cap X_\alpha$ such that

$$f_\gamma(x_\alpha^\gamma)\notin \bigcup_{\alpha\leqq\xi\leqq\gamma} V(x_\alpha^\xi, H_\alpha).$$

If $\kappa > \gamma$ then, by (5).

$$f_\gamma(x_\alpha^\gamma) \notin V(x_\alpha^\kappa, H_\alpha).$$

We therefore conclude that $f_\gamma(x_\alpha^\gamma) \notin X_\alpha$. But this contradicts our assumption that $f(X_\alpha) = X_\alpha$. □

5.6. Corollary. *Let S and T be G_δ-subsets of ℓ_2 both containing X_α. If $f: S \to T$ is a homeomorphism such that $f(X_\alpha) = X_\alpha$, then*

$$|\{x \in S : f(x) \notin V(x, H_\alpha)\}| \leqq \aleph_0.$$

Proof. For each $h \in H_\alpha$, put

$$A_h = \{x \in S : h(x) = f(x)\}.$$

Observe that each A_h is a closed subset of S and that, by Lemma 5.5,

$$B = S \Big\backslash \bigcup_{h \in H} A_h$$

has cardinality less than $\mathfrak{c}$. Since B is a G_δ-subset of S, it has to be topologically complete. Consequently, B is at most countably infinite, [6, 4.5.5(b)]. □

5.7. Lemma. *Let S and T be G_δ-subsets of ℓ_2 both containing X_α. If $f: S \to T$ is a homeomorphism such that $f(X_\alpha) = X_\alpha$, then there is an $h \in H_\alpha$ such that $f(x) = h(x)$ for all $x \in S$.*

Proof. As in the proof of Corollary 5.6, for every $h \in H_\alpha$, put

$$A_h = \{x \in S : h(x) = f(x)\}.$$

By Corollary 5.6, the set

$$B = S \Big\backslash \bigcup_{h \in H_\alpha} A_h$$

is at most countable. Since $\ell_2 \backslash S$ is a countable union of topologically complete spaces which all have to contain a Cantor set or have to be countable, from Lemma 5.3, we may conclude that $\ell_2 \backslash S$ is countable. Consequently,

$$E = \ell_2 \Big\backslash \bigcup_{h \in H_\alpha} A_h$$

is countable. Now suppose that there are at least two distinct $h, g \in H_\alpha$ such that $A_h \neq \emptyset$ and $A_g \neq \emptyset$. Take a point $x \in A_h$ and a point $y \in A_g$ and let J be an arc in ℓ_2 connecting x and y but missing E (it is clear that such arcs exist, since E is countable). Therefore, J is covered by the family $\{A_h : h \in H_\alpha\}$ which is pairwise disjoint, since H_α is special. Since J misses E each set $A_k \cap J$, where $k \in H_\alpha$, is closed in J. An appeal to Sierpiński's Lemma, 2.1(b), now yields the desired contradiction. Since each A_h is closed in S, we are done. □

5.8. Remark. The fact that the groups H_α are special is essential in the proof of Lemma 5.7. For details see van Mill [10].

5.9. Theorem. *Let $f: X_\alpha \to X_\alpha$ be a homeomorphism. Then there is an $h \in H_\alpha$ such that $f = h|X_\alpha$. As a consequence, X_α is homogeneously embedded in ℓ_2. In addition, if $h \in H_\alpha$, then $h|X_\alpha \in \mathrm{Aut}\,h(X_\alpha)$. Consequently, $\mathrm{Aut}\,h(X_\alpha)$ is isomorphic to H_α.*

Proof. By Lavrentieff's Lemma, 2.1(a), there are G_δ-subsets S and T in ℓ_2 which both contain X_α such that f can be extended to a homeomorphism $\bar{f}:S\to T$. By Lemma 5.7, there is an $h\in H_\alpha$ such that $h|S=\bar{f}$. Consequently, $h|X_\alpha=f$. All other assertions stated in the Theorem are obvious now, since observe that we have constructed X_α in such a way that $h|X_\alpha\in \mathrm{Aut}h(X_\alpha)$ for all $h\in H_\alpha$ and that, since X_α is dense in ℓ_2, the above result implies that the function $F:H_\alpha\to \mathrm{Aut}h(X_\alpha)$ defined by $F(h)=h|X_\alpha$ is an isomorphism. □

Since, by construction, for each countable group G there is precisely one index $\alpha<\mathfrak{c}$ such that G and H_α are isomorphic, we see that the partition $\{X_\alpha:\alpha<\mathfrak{c}\}$ is as required, except that we still have to prove 1.1(4). In the remaining part of this section we will do that. For a result related to 1.1(4), see [5].

5.10. Theorem. *Let $H\subset H_\alpha$ be a subgroup. Then there is a countable dense set $D\subset X_\alpha$ such that* $\mathrm{Aut}h(X_\alpha\backslash D)$ *is isomorphic to H.*

Proof. Since X_α is dense in the Baire space ℓ_2, by Lemma 5.3, it has to be Baire also. We therefore can find a countable dense set $\{e_n:n\in\mathbb{N}\}$ in X_α such that $n\neq m$ implies that $V(e_n,H_\alpha)\cap V(e_m,H_\alpha)=\emptyset$. Put

$$D=\{h(e_n):h\in H_\alpha\backslash H, n\in\mathbb{N}\}.$$

We claim that D is as required. For convenience, put $Y=X_\alpha\backslash D$.

Fact 1. If $h\in H$ and $x\in Y$, then $h(x)\in Y$.

This need only to be shown for the points $\{e_n:n\in\mathbb{N}\}$. If $h(e_n)\notin Y$ for certain $h\in H$, then there is a $g\in H_\alpha\backslash H$ such that $h(e_n)=g(e_n)$. Since $h\neq g$ and since H_α is special, this is impossible.

Since H is a subgroup of H_α, for each $h\in H$ and $x\in Y$, by Fact 1, also $h^{-1}(x)\in Y$. Therefore, the following is immediate.

Fact 2. If $h\in H$, then $h|Y\in \mathrm{Aut}h(Y)$.

Since Y is dense, by similar arguments as in the proof of Theorem 5.9, it therefore suffices to prove

Fact 3. If $f\in \mathrm{Aut}h(Y)$, then there is an $h\in H$ such that $f=h|Y$.

Choose $f\in\mathscr{F}$ such that $|\{x\in\mathrm{dom}(f):f(x)\notin V(x,H_\alpha)\}|=\mathfrak{c}$. Since f occurs $\mathfrak{c}$ times in the list $\{f_\alpha:\alpha<\mathfrak{c}\}$, by (7), $\mathrm{dom}(f)\cap X_\alpha$ has cardinality $\mathfrak{c}$. Since we have removed only countably many points from X_α, we may also conclude that $\mathrm{dom}(f)\cap Y$ has cardinality $\mathfrak{c}$. Having this in mind, we can prove in precisely the same way as in the X_α case, that there is an $h\in H_\alpha$ such that $f=h|Y$. If $h\notin H$, then $h(e_1)\in D\subset\ell_2\backslash Y$, which shows that h is not an extension of f. This proves that $h\in H$.

This concludes the proof of the Theorem. □

6. Remarks

At several places in this paper, we used essentially the fact that we only considered *countable* groups. However, several arguments that worked for ℓ_2 will also work for Hilbert spaces of larger weight, but for example the use of Sierpiński's Lemma in the proof of Lemma 5.7 does not generalize. Maybe a modification of our technique will work to answer Question 6.1 below, but it is clear that additional new ideas will be needed.

6.1. Question. *Let κ be an infinite cardinal. Is there a Hilbert space (preferably of weight κ) H which admits a partition $\{X_i : i \in I\}$ such that for each group G of cardinality at most κ there is precisely one index $i \in I$ such that G and* $\mathrm{Aut}h(X_i)$ *are isomorphic?*

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Note added in proof. I recently noticed that a result stronger than Theorem 3.4 is known, see Bessaga, C., Pelczyński, A.: On spaces of measurable functions. Studia Math. **44**, 597–615 (1972)