Homogeneous subsets of the real line which do not admit the structure of a topological group

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ABSTRACT

We show that if X is one of the real line R or the irrationals P then X can be decomposed in two dense homeomorphic and (topologically) homogeneous parts which do not admit the structure of a topological group. We also show that the space of the irrationals can be decomposed in two dense homeomorphic topological groups.

1. INTRODUCTION

In [2], J. Menu showed that the real line R can be decomposed in two homeomorphic subsets which are topologically homogeneous[1]. Menu's construction is extremely complicated which motivated the author to find an easier proof of this interesting result. This was done in [4]. As remarked in [4], none of the decompositions constructed there has the property that one of the elements of the decomposition is a subgroup of R. This suggests two questions, namely, 1) *can the real line be decomposed in two homeomorphic topological groups?*, and 2) *can the real line be decomposed in two homogeneous homeomorphic parts which do not admit the structure of a topological group?* In this paper we will answer question 2 in the affirmative but we leave question 1 unanswered. As a partial answer to question 1 we will show that the space of irrationals P can be decomposed in two dense homeomorphic topological groups. In addition, P

[1] A space X is called *homogeneous* provided that for all $x, y \in X$ there is an autohomeomorphism h of X such that $h(x) = y$.

can also be decomposed in two dense homeomorphic homogeneous parts which do not admit the structure of a topological group. The method of proof used in this paper combines ideas in van Mill [4] and van Mill [5].

2. PRELIMINARIES

All topological spaces under discussion are separable metric.

A cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. $\mathfrak{c}$ denotes $2^{\aleph_0}$.

The real line will be denoted by R. P and Q denote the space of irrationals and the space of rationals, respectively. If $A, B \subset R$ then define, as usual,

$$A + B = \{a + b\colon a \in A,\ b \in B\}.$$

Similarly, define $A - B, xA, x + A$, etc.

The following results will be important in our construction.

2.1. LEMMA: (a) (Lavrentieff [1]): *Let X and Y be topologically complete. If $A \subset X$ and $B \subset Y$ and if $h\colon A \to B$ is a homeomorphism, then there are G_δ-subsets $A' \subset X$ and $B' \subset Y$ such that $A \subset A'$ and $B \subset B'$ while moreover h can be extended to a homeomorphism $h'\colon A' \to B'$,*

(b) (van Mill [4]): *Let $A \subset R$ be such that $A + Q = A$. Then A is homogeneous.*

The domain and range of a function f will be denoted by dom(f) and range(f), respectively. Observe that the collection

$$\mathscr{F} = \{f\colon \mathrm{dom}(f) \text{ and } \mathrm{range}(f) \text{ are } G_\delta\text{-subsets of } R \text{ and}$$
$$f\colon \mathrm{dom}(f) \to \mathrm{range}(f) \text{ is a homeomorphism}\}$$

has cardinality $\mathfrak{c}$.

2. THE CONSTRUCTION

Let $Z = \{0, \pm 1, \pm 2, \ldots)$ and $Z' = Z \setminus \{0\}$. If $x \in R$ define

(1) $$V(x) = x + Q + \pi Z.$$

Observe that $V(x)$ is countable. Let $\mathscr{F}$ be as in section 2. As in [5] define

(2) $$\mathscr{G} = \{f \in \mathscr{F}\colon |\{x \in \mathrm{dom}(f)\colon f(x) \notin V(x)\}| = \mathfrak{c}\}.$$

Clearly, $|\mathscr{G}| = \mathfrak{c}$ and therefore we may list $\mathscr{G}$ as $\{f_\alpha\colon \alpha < \mathfrak{c}\}$. By transfinite induction we will construct for each $\alpha < \mathfrak{c}$ a point $x_\alpha \in \mathrm{dom}(f_\alpha)$ such that

(3) $$(V(x_\alpha) \cup V(f_\alpha(x_\alpha))) \cap (\textstyle\bigcup_{\beta<\alpha} V(x_\beta) \cup V(f_\beta(x_\beta)) = \emptyset,$$

and

(4) $$f_\alpha(x_\alpha) \notin V(x_\alpha).$$

This construction is a triviality. Suppose that the points x_β for $\beta<\alpha$ have been defined. Put

$$A=\{x\in \mathrm{dom}(f_\alpha)\colon f_\alpha(x)\notin V(x)\}.$$

By assumption, $|A|=\mathfrak{c}$. Define

$$S=\bigcup_{\beta<\alpha}V(x_\beta)\cup V(f_\beta(x_\beta)).$$

Observe that $|S|<\mathfrak{c}$ since $V(x)$ is countable for each $x\in R$ and $|\alpha|<\mathfrak{c}$.

2.1. LEMMA: *If $B\subset R$ has cardinality $\mathfrak{c}$, then* $|\{x\in B\colon V(x)\cap S=\emptyset\}|=\mathfrak{c}$.

PROOF: Suppose that $B_0=\{x\in B\colon V(x)\cap S\neq\emptyset\}$ has cardinality $\mathfrak{c}$. We will derive a contradiction. For all $x\in B_0$ choose a point $q_x\in Q$ and a point $n_x\in Z$ such that $x+q_x+\pi n_x\in S$. Since $\mathfrak{c}$ has uncountable cofinality, i.e. $\mathfrak{c}$ is not the sum of countably many smaller cardinals, there is a subset $B_1\subset B_0$ of cardinality $\mathfrak{c}$ such that for all $x,y\in B_1$ we have that $q_x=q_y$ and $n_x=n_y$. Define a function $f\colon B_1\to S$ by $f(x)=x+q_x+\pi n_x$. Then f is clearly one to one and this contradicts the fact that S has cardinality less than $\mathfrak{c}$. □

This Lemma implies that we can find a subset $A_0\subset A$ of cardinality $\mathfrak{c}$ such that for all $x\in A_0$ it is true that $V(x)\cap S=\emptyset$. Since f_α is one to one, the set $f_\alpha(A_0)$ has also cardinality $\mathfrak{c}$. Therefore, we can find a subset $A_1\subset A_0$ of cardinality $\mathfrak{c}$ such that for every $x\in A_1$ we have that $V(f_\alpha(x))\cap S=\emptyset$. Take any $x\in A_1$ and define $x_\alpha=x$. It is clear that this choice of x_α is as required.

Define

$$X=\bigcup_{\alpha<\mathfrak{c}}(x_\alpha+Q)\cup\bigcup_{\alpha<\mathfrak{c}}(f_\alpha(x_\alpha)-\pi+Q).$$

In addition, as in [4] define

$$\mathscr{A}=\{A\subset R\colon X\subset A,\ A+Q=A \text{ and } A\cap(A+\pi Z')=\emptyset\}.$$

2.2. LEMMA: $X\in\mathscr{A}$.

PROOF: Suppose that for certain $\alpha<\mathfrak{c}$, $q\in Q$ and $n\in Z'$ we have that $x_\alpha+q+n\pi\in X$. Since $x_\alpha+q+n\pi\in V(x_\alpha)$, by (3), $x_\alpha+q+n\pi\in(x_\alpha+Q)\cup\cup(f_\alpha(x_\alpha)-\pi+Q)$. Since $n\neq 0$, it is clear that $x_\alpha+q+n\pi\notin(x_\alpha+Q)$ and consequently, we can find a point $s\in Q$ such that

$$x_\alpha+q+n\pi=f_\alpha(x_\alpha)-\pi+s,$$

which obviously contradicts (4).

Suppose now that for certain $\alpha<\mathfrak{c}$, $q\in Q$ and $n\in Z'$ we have that $f_\alpha(x_\alpha)-\pi++q+n\pi\in X$. Since $f_\alpha(x_\alpha)+q+(n-1)\pi\in V(f_\alpha(x_\alpha))$, by (3), $f_\alpha(x_\alpha)+q++(n-1)\pi\in(x_\alpha+Q)\cup(f_\alpha(x_\alpha)-\pi+Q)$. Suppose first that for certain $s\in Q$ we have that

$$f_\alpha(x_\alpha)+q+(n-1)\pi=f_\alpha(x_\alpha)-\pi+s.$$

This obviously leads to a contradiction since $n \neq 0$ and $\pi \notin Q$. Therefore there exists a point $t \in Q$ such that

$$f_\alpha(x_\alpha) + q + (n-1)\pi = x_\alpha + t,$$

which again contradicts (4). □

It is clear that any chain $\mathscr{K} \subset \mathscr{A}$ has the property that $\cup \mathscr{K} \in \mathscr{A}$, so that by the Kuratowski-Zorn Lemma we can find an $A_0 \in \mathscr{A}$ such that whenever $B \in \mathscr{A}$ and $A_0 \subset B$ then $A_0 = B$.

Define $A_n = \{x \in R\colon \exists a \in A_0 \text{ such that } x = a + n\pi\}$ $(n \in Z)$.

2.3. LEMMA: *$\{A_n\colon n \in Z\}$ is a partition of R.*

PROOF: See van Mill [4], section 3. □

As in [4], define $A = \bigcup\{A_n\colon n \in Z \text{ and } n \text{ is even}\}$ and $B = \bigcup\{A_n\colon n \in Z$ and n is odd$\}$. Observe that $A + Q = A$, so that by Lemma 2.1(b), A is homogeneous. Also, A is homeomorphic to B since the map $h(r) = r + \pi$, sends A onto B. So the only remaining thing to prove is that A does not admit the structure of a topological group. This will be done in section 3.

3. A IS NOT A TOPOLOGICAL GROUP

Let A be as in section 2. We will show here that A does not admit the structure of a topological group. We use ideas from [5].

3.1. LEMMA: *If $h\colon A \to A$ is any homeomorphism, then $|\{x \in A\colon h(x) \notin V(x)\}| < \mathfrak{c}$.*

PROOF: Suppose not. By Lemma 2.1(a) we can find G_δ's S and T containing A such that h can be extended to a homeomorphism $h'\colon S \to T$. Then $h' \in \mathscr{G}$, say $h' = f_\alpha$. Since, by construction, $f_\alpha(x_\alpha) - \pi \in A_0 \subset A$, we find that $f_\alpha(x_\alpha) = (f_\alpha(x_\alpha) - \pi) + \pi \in A_1 \subset R \setminus A = B$. However, since f_α extends h, we have that $f_\alpha(x_\alpha) = h(x_\alpha) \in A$, which is a contradiction. □.

3.2. LEMMA: *Let $U \subset A$ be open and nonempty and let $\mathscr{D}$ be a family of countably many nowhere dense subsets of A. Then $|U \setminus \cup \mathscr{D}| = \mathfrak{c}$.*

PROOF: Let $^-$ denote the closure operator in R and let $U' \subset R$ be open such that $U' \cap A = U$. Put $\mathscr{E} = \{\bar{D}\colon D \in \mathscr{D}\}$. Since A is dense in R, each element of $\mathscr{E}$ is a nowhere dense subset of R. Consequently, $U' \setminus \cup \mathscr{E}$ contains a Cantor set, say K. Choose a Cantor set L in R such that $L \cap (K + Q + \pi Z) = \emptyset$. Let $\mathscr{F}$ denote a family of $\mathfrak{c}$ pairwise disjoint Cantor sets in K. For each $F \in \mathscr{F}$ let $h_F\colon F \to L$ be a homeomorphism. Then $h_F \in \mathscr{G}$ and therefore, by construction, $A \cap \operatorname{dom}(h_F) \neq \emptyset$. We conclude that $|U \setminus \cup \mathscr{D}| = \mathfrak{c}$. □

These Lemma's imply the following

3.3. THEOREM: *There is a countable dense set $E\subset A$ such that for each homeomorphism $h\colon A\to A$ we have that $E\cap h(E)\neq\emptyset$.*

PROOF: Let $D\subset A$ be any countable dense set and put $E=\bigcup_{d\in D}V(d)\cap A$ (observe that we have to intersect with A since $V(x)\cap A\neq\emptyset\neq B\cap V(x)$ for all $x\in A$). We claim that E is as required. To this end, let $h\colon A\to A$ be any homeomorphism. For each $q\in Q$ and $n\in Z$ put

$$H_n^q=\{x\in A\colon h(x)=x+q+n\pi\}.$$

Notice that each H_n^q is closed in A and that, by Lemma 3.1,

$$|A\setminus\bigcup_{q\in Q}\bigcup_{n\in Z}H_n^q|<\mathfrak{c}.$$

Consequently, by Lemma 3.2, there is a $q\in Q$ and an $n\in Z$ such that H_n^q is not nowhere dense. Since H_n^q is closed, it must therefore contain a nonempty open set and consequently, it must intersect D. Take $d\in D$ arbitrarily. Then

$$h(d)=d+q+n\pi\in V(d)\cap A\subset E.$$

Therefore, $E\cap h(E)\neq\emptyset$. □

3.4. LEMMA: *Let G denote a topological group such that $|G|=\mathfrak{c}$. If $D\subset G$ is countable, then there is a homeomorphism $h\colon G\to G$ such that $h(D)\cap D=\emptyset$.*

PROOF: We claim that there is an $x\in G$ such that $xD\cap D=\emptyset$. If this is not true, then for all $x\in G$ we can find a point $d_x\in D$ such that $xd_x\in D$. For each $d\in D$ define $G_d=\{x\in G\colon xd_x=d\}$. Since $\mathfrak{c}$ has uncountable cofinality, there is a $d\in D$ such that $|G_d|=\mathfrak{c}$. Again, since $\mathfrak{c}$ has uncountable cofinality there is a subset $H\subset G_d$ such that $|H|=\mathfrak{c}$ while moreover for distinct $x,y\in H$ we have that $d_x=d_y$. Since the equation $xa=b$ has only one solution in G, this obviously is a contradiction. Therefore, we can find $x\in G$ with $xD\cap D=\emptyset$. Now define $h\colon G\to G$ by $h(g)=xg$. Then h is clearly as required. □

3.4. COROLLARY: *A does not admit the structure of a topological group.*

PROOF: Immediate from Theorem 3.3 and Lemma's 3.2 and 3.4. □

3.5. REMARK: Observe that Theorem 3.3 and the proof of Lemma 3.4 in fact show that A does not admit a binary operation "." such that all right translations are homeomorphisms.

3.6. REMARK: Notice that we found a topological property of A that shows that A is not a topological group (of course, not being a topological group is also a topological property, but we found one which in a sense shows "why" A is not a topological group).

3.7. REMARK: Eric van Douwen has previously constructed a homogeneous zero-dimensional space which does not admit the structure of a topological group (unpublished). It can be shown that his space does not have the property that for some embedding in R it is homeomorphic to its complement.

4. A PARTITION OF P

In this section we will show that P can be decomposed in two dense homeomorphic topological groups.

Let C denote the Cantor cube $\{0,1\}^N$. Then C is a topological group under coordinatewise addition. We will first show that C can be decomposed in two homeomorphic dense topological groups. To find the required decomposition of P is then easy. Simply observe that P is a topological group, being homeomorphic to Z^N, and that $P \approx C \times P$.

Let $\mathscr{P}(N)$ denote the power set of N and let f: $\mathscr{P}(N) \to \{0,1\}^N$ denote the canonical bijection, i.e. $f(A)_i = 0$ iff $i \notin A$. Fix a free ultrafilter $\mathscr{F}$ on N, i.e. $\mathscr{F} \subset \mathscr{P}(N)$ and $\mathscr{F}$ is a maximal with respect to the finite intersection property and $\bigcap \mathscr{F} = \emptyset$. The dual ideal of $\mathscr{F}$ is denoted by $\mathscr{I}$. It is easily seen that $f(\mathscr{I})$ is a subgroup of C which is homeomorphic to its complement, which is equal to $f(\mathscr{F})$, since the map g: $C \to C$ defined by $g(x)_i = x_i + 1$ (mod 2) obviously sends $f(\mathscr{I})$ onto $f(\mathscr{F})$. So it remains to be shown that $f(\mathscr{F})$ is dense in C. To this end, let F and G be two disjoint finite subsets of N. We have to show that there is an element $H \in \mathscr{F}$ which misses F and contains G. It is clear that such an H exists, since $\mathscr{F}$ is a free ultrafilter (observe that, since F is finite, $N \setminus F \in \mathscr{F}$).

Let (A, B) be the decomposition of R constructed in section 2. Put $E = \bigcup_{q \in Q} V(q)$. Since E is countable, $R \setminus E \approx P$. Therefore, $(A \setminus E, B \setminus E)$ is a decomposition of (a homeomorph of) P in homogeneous dense sets which are obviously homeomorphic and which can be seen not to admit the structure of a topological group by precisely the same construction as in section 3.

5. REMARKS

Our use of Lavrentieff's Lemma to kill certain homeomorphisms is not the first construction of this type. For references, see [5].

Let us recall that a zero-dimensional space is called *strongly homogeneous* provided that all nonempty clopen subsets are homeomorphic. Strongly homogeneous zero-dimensional spaces have the pleasant property that any homeomorphism between closed and nowhere dense sets can be extended to a homeomorphism of the whole space, [3]. Until now I have not been able to show that there exists a decomposition of the real line in two homeomorphic strongly homogeneous sets (whether such a decomposition exists, was asked in [4]). The results in this paper suggest the question whether a decomposition of the real line exists in strongly homogeneous homeomorphic sets which do not admit the structure of a topological group.

Evert Wattel has used one of our decompositions in [4] to show that R^n can be decomposed in $n+1$ homeomorphic homogeneous sets, [6]. The decomposition of this paper can be modified so that by Wattel's construction one gets a decomposition of R^n in $n+1$ homogeneous and homeomorphic parts which do not admit the structure of a topological group.

REFERENCES

1. Lavrentieff, M. – Contribution à la théorie des ensembles homéomorphes, Fund. Math. **6,** 149–160 (1924).
2. Menu, J. – A partition of R in two homogeneous and homeomorphic parts, to appear.
3. Mill, J. van – Characterization of some zero-dimensional separable metric spaces, Trans. Amer. Math. Soc. **264,** 205–215 (1981).
4. Mill, J. van – Homogeneous subsets of the real line, to appear in Comp. Math.
5. Mill, J. van – Strong local homogeneity does not imply countable dense homogeneity, to appear in Proc. Amer. Math. Soc.
6. Wattel, E. – A partition of R^n in $n+1$ homogeneous and homeomorphic parts, to appear.