General Topology and its Relations
to Modern Analysis and Algebra V
Proc. Fifth Prague Topol. Symp. 1981
J. Novak (ed.)

481 – 485

TYPES OF WEAK P-POINTS IN $\beta\omega\text{-}\omega$

J. van Mill

Abstract: There are $2^{\mathfrak{c}}$ types of $\mathfrak{c}$-OK points in ω^*. We also construct $2^{\mathfrak{c}}$ types of weak P-points in ω^* which are not ω_1-OK points.

AMS Subject classification: 54D35.

0. *Introduction.*

All spaces are Tychonoff, $\mathfrak{c}$ *denotes* 2^{ω}, *and* $X^* = \beta X - X$.

Let $\kappa \geq \omega$ be a cardinal. A point x of a space X is called κ-OK provided that for each sequence $\{U_n : n < \omega\}$ of neighborhoods of x, there is a sequence $\{V_\alpha : \alpha < \kappa\}$ of neighborhoods of x such that for all $n \geq 1$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \kappa$,

$$\cap_{1 \leq i \leq n} V_{\alpha_i} \subset U_n.$$

Observe that the property of being κ-OK gets stronger as κ gets bigger.

It is easily seen that if x is ω_1-OK, then x is not a limit point of any subset $A \subset X-\{x\}$ which satisfies the countable chain condition, [K]. In particular, if $x \in X$ is ω_1-OK, then x is a *weak P-point* of X, i.e. if $F \subset X-\{x\}$ is countable, then $x \notin \bar{F}$. Weak P-points and κ-OK points were introduced by Kunen [K], who showed that $\mathfrak{c}$-OK points in ω^* exist. Subsequently, the author showed in [vM$_1$] that there is also a weak P-point $x \in \omega^*$ which is not ω_1-OK, since x is a limit point of some subset $A \subset \omega^*-\{x\}$ which satisfies the countable chain condition. Consequently, there are at least two types of weak P-points in ω^*. The aim of this note is to show that there are $2^{\mathfrak{c}}$ types of $\mathfrak{c}$-OK points in ω^* and also that there are $2^{\mathfrak{c}}$ types of weak P-points which are not ω_1-OK. We use a nonhomogeneity trick due to Comfort and Negrepontis [CN, 16.18], which was inspired by ideas of Frolík [F].

1. *Types of* $\mathfrak{c}$-OK *points.*

Let X be a space. If $x \in X$, define

$$\tau(x,X) = \{y \in X: \exists \text{ autohomeomorphism } h: X \to X \text{ with } h(x) = y\}.$$

In addition, let $F_\sigma(X) = \{U \subset X: U \text{ is a nonempty open } F_\sigma\}$. Define

$$C(X) = \{\langle C_n : n<\omega\rangle : C_n \in F_\sigma(X) \text{ for all } n, \text{ and if } n\neq m \text{ then } C_n \cap C_m = \emptyset\}.$$

If $x \in X$, put

$$T(x,X) = \{p \in \omega^*: \exists \langle C_n : n<\omega\rangle \in C(X) \text{ such that } x \in \cap_{P\in p}(\cup_{n\in P} C_n)^-\}.$$

Observe that if $x \in \tau(y,X)$ then $T(x,X) = T(y,X)$.

Since $|C(\omega^*)| = \mathfrak{c}$, and since disjoint open F_σ-subsets of ω^* have disjoint closures, it is easily seen that $|T(x,\omega^*)| \le \mathfrak{c}$ for all $x \in \omega^*$ (see [CN, 16.18]).

Let X be the topological sum of countably many compact spaces, say X_n $(n < \omega)$. A closed filter F on X is called *nice* provided that

(1) if $F \in F$ then $|\{n < \omega: F \cap X_n = \emptyset\}| < \omega$,
(2) $\cap F = \emptyset$.

Whenever we write $X = \Sigma_{n<\omega} X_n$ then, for convenience, we assume that the X_n's are pairwise disjoint.

In [vM_3, 4.5.1], I showed that there is a finite-to-one surjection $\pi: \omega \to \omega$ such that for all $x \in \omega^*$ there is a point $y \in \beta\pi^{-1}(\{x\})$ which is a $\mathfrak{c}$-OK point of ω^*. Observe that if $\bar{\pi} = \beta\pi \restriction \omega^*$, then $\bar{\pi}$ is open and maps ω^* onto ω^*. For later use, let us formulate a generalization of this result, which can be proved by an easy modification of [vM_3, 4.5.1] and [vM_1, 2.4].

1.1. THEOREM: *Let* $X = \Sigma_{n<\omega} X_n$, *where each* X_n *is compact and of weight at most* $\mathfrak{c}$ *and suppose that* F *is a nice filter on* X. *There is a finite-to-one surjection* $\pi: \omega \to \omega$ *such that if* $f: X \to \omega$ *is defined by* $f(x) = n$ *iff* $x \in X_{\pi^{-1}(n)}$, *then for all* $p \in \omega^*$ *there is a point* $x \in \beta f^{-1}(\{p\}) \cap \cap_{F\in F} \text{cl}_{\beta X} F$ *which is a* $\mathfrak{c}$-OK *point of* X^*.

Take a point $x \in \omega^*$ which is not a P-point and let $\{F_n : n < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^* such that

$$x \in (\bigcup_{n<\omega} F_n)^- - \bigcup_{n<\omega} F_n.$$

Let $\pi\colon \omega \to \omega$ be as above. By transfinite induction, for every $\xi < 2^{\mathfrak{c}}$ we will construct a point $q_\xi \in \omega^*$, a point $x_\xi \in (\bigcup_{n<\omega} F_n)^- - \bigcup_{n<\omega} F_n$, and a $\mathfrak{c}$-OK point $y_\xi \in \bar{\pi}^{-1}(\{x_\xi\})$ such that

$$q_\xi \in T(y_\xi,\omega^*) - \bigcup_{\eta<\xi} T(y_\eta,\omega^*)$$

Suppose that this has been done for all $\eta < \xi < 2^{\mathfrak{c}}$. Pick a point $q \in \omega^* - \bigcup_{\eta<\xi} T(y_\eta,\omega^*)$, and define $q_\xi = q$. In addition, take a point

$$x \in \bigcap_{Q\in q} (\bigcup_{n\in Q} F_n)^-$$

arbitrarily, and define $x_\xi = x$. Let y_ξ be an arbitrarily chosen $\mathfrak{c}$-OK point from $\bar{\pi}^{-1}(\{x_\xi\})$. Since $\bar{\pi}$ is open, $q_\xi \in T(y_\xi,\omega^*)$.

If $\eta < \xi < 2^{\mathfrak{c}}$ then, by construction, $q_\xi \in T(y_\xi,\omega^*) - T(y_\eta,\omega^*)$. We therefore can conclude that $y_\xi \notin \tau(y_\eta,\omega^*)$.

2. *Types of weak P-points.*

It should be clear what we mean by a κ-OK set.

2.1. LEMMA: *Let $X \subset \omega^*$ be a closed ω_1-OK set which satisfies the countable chain condition. If $x \in X$, then $\tau(x,\omega^*) \cap X \subset \tau(x,X)$.*

PROOF: Take $y \in \tau(x,\omega^*) \cap X$ and let $h\colon \omega^* \to \omega^*$ be an autohomeomorphism of ω^* such that $h(x) = y$. Since $h(X)$ is ω_1-OK and since X-$h(X)$ satisfies the countable chain condition, we conclude that $\overline{X-h(X)} \cap h(X) = \emptyset$. This implies that $X \cap h(X)$ is clopen in X. By the same argument, $X \cap h^{-1}(X)$ is clopen in X. Consequently, x and y have homeomorphic clopen neighborhoods in X. Since X is zero-dimensional, this easily implies that $y \in \tau(x,X)$. □

By a result of Bell [B], there is a compact, nowhere separable space Y which satisfies the countable chain condition, and which is a continuous image of ω^*. Applying [vM_1, 2.4], yields the existence of a compact ccc nowhere separable space X such that $\beta(\omega \times X)$ can be embedded in ω^* as a $\mathfrak{c}$-OK set.

It is easy to construct a nice filter $\mathcal{F}$ on $\omega \times X$ such that for any countable subset $D \subset \omega \times X$ there is an element $F \in \mathcal{F}$ such that $\bar{D} \cap F = \emptyset$, [vM$_1$, 3.5]. Let π and f be as in Theorem 1.1 and for every $p \in \omega^*$ choose a point $y(p) \in \beta f^{-1}(\{p\}) \cap \bigcap_{F \in \mathcal{F}} \mathrm{cl}_{\beta(\omega \times X)} F$ which is a $\mathfrak{c}$-OK point of $(\omega \times X)^*$. Routine arguments show that the collection $\{y(p): p \in \omega^*\}$ consists of weak P-points of $\beta(\omega \times X)$. Using the same technique as in section 1, it can be shown that there is a subset $\{p_\xi: \xi < 2^{\mathfrak{c}}\} \subset \omega^*$ such that for all $\eta < \xi < 2^{\mathfrak{c}}$ we have that

$$p_\xi \in T(y(p_\xi),\beta(\omega \times X)) - \bigcup_{\eta<\xi} T(y(p_\eta),\beta(\omega \times X))$$

(observe that f is open). Consequently, if $\eta < \xi < 2^{\mathfrak{c}}$, then

$$y(p_\xi) \notin \tau(y(p_\eta),\beta(\omega \times X)).$$

As remarked above, we may assume that $\beta(\omega \times X)$ is a $\mathfrak{c}$-OK set in ω^*. It is clear that any point of $\beta(\omega \times X)$ which is a weak P-point of $\beta(\omega \times X)$ is also a weak P-point of ω^*. Therefore, the collection $\{y(p_\xi): \xi < 2^{\mathfrak{c}}\}$ consists of weak P-points of ω^*. It is also clear that no $y(p_\xi)$ is ω_1-OK. By Lemma 2.1 we may conclude that if $\eta < \xi < 2^{\mathfrak{c}}$, then $y(p_\xi) \notin \tau(y(p_\eta),\omega^*)$.

2.2. *Remark:* Using the same technique as in this note, it can be shown that of all the "special" points constructed in [vM$_1$] and [vM$_2$], there are at least $2^{\mathfrak{c}}$ different types.

References:

[B] M.G. Bell, *Compact* ccc *non-separable spaces of small weight*, to appear in Top. Proc.

[CN] W.W. Comfort & S. Negrepontis, *The theory of ultrafilters*, Grundlehren math. Wiss., Bd. 211,Springer-Verlag, Berlin (1974).

[F] Z. Frolík, *Sums of ultrafilters*, Bull. Am. Math. Soc. 73(1967)87-91.

[K] K. Kunen, *Weak* P-*points in* N^*, Proc. Bolyái János Society Colloquium on Topology, Budapest (1978) 741-749.

[vM$_1$] J. van Mill, *Sixteen topological types in* $\beta\omega$-ω, to appear in Top. Appl.

[vM$_2$] J. van Mill, *Weak* P-*points in Čech-Stone compactifications*, to appear in Trans. Am. Math. Soc.

[vM$_3$] J. van Mill, *An introduction to* $\beta\omega$-ω, to appear.

Subfaculteit Wiskunde
Vrije Universiteit
De Boelelaan 1081
1081 HV Amsterdam
The Netherlands