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WEAK P-POINTS IN ČECH-STONE COMPACTIFICATIONS

BY

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ABSTRACT. Let X be a nonpseudocompact space which is either nowhere ccc or nowhere of weight $\leq 2^{\omega}$. Then $\beta X - X$ contains a point x which is a weak P-point of βX, i.e. if $F \subset \beta X - \{x\}$ is countable, then $x \notin \overline{F}$. In addition, under MA, if X is any nonpseudocompact space, then $\beta X - X$ contains a point x such that whenever $F \subset \beta X - \{x\}$ is countable and nowhere dense, then $x \notin \overline{F}$.

0. Introduction. *All spaces are completely regular and* X^* *denotes* $\beta X - X$.

Frolík's [**F**] proof that the Čech-Stone remainder of a nonpseudocompact space is not homogeneous is elegant and ingenious, but does not give points which are topologically distinct by an obvious reason. When Kunen [$\mathbf{K_1}$] proved that there are Rudin-Keisler incomparable points in $\beta\omega$, Frolík's ideas were used by Comfort [**C**] and van Douwen [$\mathbf{vD_2}$] to show that, respectively, no infinite compact space in which countable discrete subspaces are C^*-embedded is homogeneous and that βX is not homogeneous for any nonpseudocompact space X. These results showed that certain spaces are not homogeneous but not "why" they are not homogeneous. This suggests an obvious question which has been considered by several authors during the last years.

The first promising partial answers to this question were obtained by van Douwen [$\mathbf{vD_3}$], who showed that each nonpseudocompact space of countable π-weight has a remote point (independently, this was also shown by Chae and Smith [**CS**]), and that remote points can be used to show that certain Čech-Stone remainders are not homogeneous. Unfortunately, it was soon clear, by examples in van Douwen and van Mill [$\mathbf{vDvM_1}$], that this line of attack did not solve the entire problem, since many spaces do not have remote points. Earlier, van Douwen [$\mathbf{vD_1}$] had introduced far points and ω-far points and showed that these points exist in certain Čech-Stone remainders and that they also could be used to show that a restrictive class of Čech-Stone remainders is not homogeneous. Each remote point is a far point and each far point is an ω-far point when we restrict our attention to spaces without isolated points. When it was shown that not every nonpseudocompact space has a remote point, van Douwen's [$\mathbf{vD_1}$] question whether every nonpseudocompact space without isolated points has an ω-far point again became interesting. The examples in [$\mathbf{vDvM_1}$] did not clarify this question since they all have far points. From [$\mathbf{vDvM_2}$] it

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became clear that a counterexample to this question promised to be very difficult and this motivated the author to try to prove a theorem instead of finding a counterexample.

In the meanwhile, Kunen [$\mathbf{K_2}$] proved the important and highly nontrivial result that weak P-points in ω^* exist. This result provides a very satisfactory solution to the problem of the nonhomogeneity of X^*. It is easily seen that whenever X is a nonpseudocompact space, Kunen's theorem implies that there exists a point $x \in X^*$ such that whenever $F \subset X^* - \{x\}$ is countable and has compact closure in X^*, then $x \notin \bar{F}$. Consequently, X^* contains points which very much look like weak P-points, but obviously, in general, need not be weak P-points. This suggests the question how close points in X^* can be to weak P-points, and the surprising answer to this question is "very close". Kunen's theorem gives no elegant solution to the problem of the nonhomogeneity of βX. In addition, one would like to find a point $x \in X^*$ that shows that both X^* and βX are nonhomogeneous at the same time. The aim of this paper is to construct such points.

0.1. THEOREM. *Let X be any nonpseudocompact space. Then*

(a) *if X is either nowhere ccc or nowhere of weight $\leq 2^\omega$, then X^* contains a point which is a weak P-point of βX, and*

(b) (*MA*) *X^* contains a point x such that whenever $F \subset \beta X - \{x\}$ is countable and nowhere dense, then $x \notin \bar{F}$.*

A ccc space of weight $\leq 2^\omega$ is usually considered to be a nice space since it is "small" for several reasons. In the theory of Čech-Stone compactification, ccc spaces of weight $\leq 2^\omega$ are extensively studied since in the presence of the Continuum Hypothesis all kinds of nice points can quite easily be constructed, i.e. P-points, remote points, etc. Large spaces were not considered since the small spaces turned out to be difficult to handle without the aid of the Continuum Hypothesis and why should one increase the difficulty by removing hypotheses which seemed essential when deriving results with the Continuum Hypothesis. Theorem 0.1(a) shows that this argumentation contains a severe mistake. One can use the fact that spaces are large to construct "nice" points. The first to observe this was Dow [**D**] (maybe Kunen noticed this earlier than Dow when he remarked in [$\mathbf{K_2}$] that his proof that ω^* contains weak P-points is more complicated than his proof that ω_1^* contains weak P-points). Apparently, the small spaces are complicated and not the large ones (for our specific purposes of course).

Although my proof of Theorem 0.1(b) unfortunately uses Martin's axiom, the result strikes me as a fundamental theorem in Čech-Stone compactifications. No hypotheses on X, beyond nonpseudocompactness of course, are assumed, yet one gets very "special" points in βX. In addition, it solves van Douwen's question stated above.

Our proof of Theorem 0.1 is not easy and unfortunately is rather lengthy. We heavily rely on results and techniques of Bell [**B**], Dow [**D**], Dow and van Mill [**DvM**], Kunen [$\mathbf{K_2}$] and van Mill [$\mathbf{vM_1}$]. Since only one of the above papers has been published as yet, in this paper we will give the complete proof of Theorem 0.1. Therefore, our paper has the character of a survey paper as well as a research paper since we present also a considerable amount of new material. We deliberately have chosen this form of presentation since we hope that this will enlarge the readability of our paper.

1. Preliminary remarks. This paper consists of ten sections, each of which can be read rather independently. At the beginning of each section we state the main result and at the end of each section we give notes.

Let X be a space. A point $x \in X$ is called a *weak P-point* provided that $x \notin \overline{F}$ for any countable $F \subset X - \{x\}$. A space is ccc if each family of pairwise disjoint open subsets is countable. If $\mathcal{P}$ is a topological property, a space is called *nowhere* $\mathcal{P}$ provided that no nonempty open set has $\mathcal{P}$.

If $U \subset X$, then $\mathrm{Ex}(U) = \beta X - \mathrm{cl}_{\beta X}(X - U)$.

Notes for §1. Weak P-points were introduced by Kunen [$\mathbf{K_2}$], after Shelah (see [**M** or **W**]) showed that P-points need not exist in ω^*.

2. Extending nice filters to OK-points. Let X be the topological sum of countably many compact spaces, say X_n $(n < \omega)$. A closed filter $\mathcal{F}$ on X is called *nice* provided that $|\{n < \omega: F \cap X_n = \varnothing\}| < \omega$ for all $F \in \mathcal{F}$, and $\bigcap \mathcal{F} = \varnothing$. In this section we show that whenever $\mathcal{F}$ is a nice filter on X and if X has weight at most 2^ω, then there exists a weak P-point $x \in X^*$ (i.e. a weak P-point of X^* and not necessarily of βX) such that $x \in \bigcap_{F \in \mathcal{F}} \mathrm{cl}_{\beta X} F$.

In §4, we will use this result to show that the same result is true without the weight restriction on X.

A closed subset $A \subset X$ is called a *P-set* provided that the intersection of countably many neighborhoods of A is again a neighborhood of A. We begin with a simple result.

2.1. LEMMA. *Let X be a locally compact and σ-compact space and let A be a closed subspace of X. Then $\mathrm{cl}_{\beta X} A \cap X^*$ is a P-set of X^*.*

PROOF. Let F be an F_σ of X^* disjoint from $A^* = \mathrm{cl}_{\beta X} A \cap X^*$. Assume that $F = \bigcup_{n<\omega} F_n$, where each F_n is closed in X^*. For each $n < \omega$ take a neighborhood U_n of A in X such that

(a) $U_{n+1}^- \subset U_n$;

(b) $\mathrm{Ex}(U_n) \cap F_n = \varnothing$.

Since X is σ-compact, so is A. So we may write $A = \bigcup_{n<\omega} A_n$, where the A_n's are compact. For each $n < \omega$ let V_n be an open subset of X such that $A_n \subset V_n \subset U_n$ while, in addition, V_n^- is compact. Let $V = \bigcup_{n<\omega} V_n$. Then $\mathrm{Ex}(V)$ is a neighborhood of A^* which misses F. □

Let X be a normal space. A point $p \in X^*$ is called κ-OK provided that for each sequence $\{U_n: n < \omega\}$ of neighborhoods of p in X^* there are closed sets $A_\alpha \subset X$ $(\alpha < \kappa)$ such that $p \in \bigcap_{\alpha<\kappa} \mathrm{cl}_{\beta X} A_\alpha$ while, moreover, for each $n \geqslant 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$,

$$\bigcap_{1 \leqslant i \leqslant n} \mathrm{cl}_{\beta X} A_{\alpha_i} \cap X^* \subset U_n.$$

Observe that the property of κ-OK gets stronger as κ gets bigger.

2.2. LEMMA. *Let X be a locally compact and σ-compact space and let $p \in X^*$ be ω_1-OK. Then p is a weak P-point of X^*.*

PROOF. Let $F \subset X^* - \{p\}$ be countable. List F as $\{x_n: 1 \leq n < \omega\}$. Take closed $A_\alpha \subset X$ $(\alpha < \omega_1)$ such that $p \in \bigcap_{\alpha<\omega_1} \mathrm{cl}_{\beta X} A_\alpha$ while, moreover, for each $n \geq 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \omega_1$,

$$(1) \qquad \bigcap_{1\leq i\leq n} \mathrm{cl}_{\beta X} A_{\alpha_i} \cap X^* \subset X^* - \{x_n\}.$$

For convenience, put $A_\alpha^* = \mathrm{cl}_{\beta X} A_\alpha \cap X^*$. If $A_\alpha^* \cap F \neq \varnothing$ for every $\alpha < \omega_1$, then there is an uncountable $E \subset \omega_1$ and $1 \leq n < \omega$ such that $x_n \in \bigcap_{\alpha \in E} A_\alpha^*$, which contradicts (1). Hence $A_{\alpha_0}^* \cap F = \varnothing$ for certain $\alpha_0 < \omega_1$. Since, by Lemma 2.1, $A_{\alpha_0}^*$ is a P-set of X^*, it follows that $A_{\alpha_0}^* \cap \bar{F} = \varnothing$, i.e. $p \notin \bar{F}$. $\square$

Whenever X is a set and κ is a cardinal we define (as usual)

$[X]^\kappa = \{A \subset X: |A| = \kappa\}$,

$[X]^{\leq\kappa} = \{A \subset X: |A| \leq \kappa\}$, and

$[X]^{<\kappa} = \{A \subset X: |A| < \kappa\}$, respectively.

2.3. DEFINITION. Let $\mathcal{F}$ be a closed filter on X and assume that no $F \in \mathcal{F}$ is compact.

If $1 \leq n < \omega$, an indexed family $\{A_i: i \in I\}$ of closed subsets of X is *precisely n-linked* w.r.t. $\mathcal{F}$ if for all $\sigma \in [I]^n$ and $F \in \mathcal{F}$, $\bigcap_{i\in\sigma} A_i \cap F$ is not compact, but for all $\sigma \in [I]^{n+1}$, $\bigcap_{i\in\sigma} A_i$ is compact.

An indexed family $\{A_{in}: i \in I, 1 \leq n < \omega\}$ is a *linked system* w.r.t. $\mathcal{F}$ if for each n, $\{A_{in}: i \in I\}$ is precisely n-linked w.r.t. $\mathcal{F}$, and for each n and i, $A_{in} \subset A_{i,n+1}$.

An indexed family $\{A_{in}^j: i \in I, 1 \leq n < \omega, j \in J\}$ is an *I by J independent linked family* w.r.t. $\mathcal{F}$ if for each $j \in J$, $\{A_{in}^j: i \in I, 1 \leq n < \omega\}$ is a linked system w.r.t. $\mathcal{F}$, and $\bigcap_{j\in\tau}(\bigcap_{i\in\sigma_j} A_{in_j}^j) \cap F$ is not compact, whenever $\tau \in [J]^{<\omega}$, and for each $j \in \tau$, $1 \leq n_j < \omega$ and $\sigma_j \in [I]^{n_j}$ and $F \in \mathcal{F}$.

The filter of cofinite subsets of ω is denoted by $\mathcal{CF}$.

2.4. LEMMA. *There is a 2^ω by 2^ω independent linked family of subsets of ω w.r.t. $\mathcal{CF}$.*

PROOF. Let $S = \{\langle k, f\rangle: k \in \omega \,\&\, f \in \mathcal{PP}(k)^{\mathcal{P}(k)}\}$. The required family (defined on the countable set S) will be of the form

$$\{A_{Xn}^Y: X \in \mathcal{P}(\omega), 1 \leq n < \omega, Y \in \mathcal{P}(\omega)\},$$

where

$$A_{Xn}^Y = \{\langle k, f\rangle \in S: |f(Y \cap k)| \leq n \,\&\, X \cap k \in f(Y \cap k)\}. \quad \square$$

We now come to the main result in this section.

2.5. THEOREM. *Let X be the sum of countably many compact nonempty spaces of weight at most 2^ω, say X_n $(n < \omega)$ and let $\mathcal{F}$ be a nice filter on X. Then there is a 2^ω-OK point $p \in \bigcap_{F\in\mathcal{F}} \mathrm{cl}_{\beta X} F \cap X^*$.*

PROOF. Without loss of generality, assume that $X_n \cap X_m = \varnothing$ for all distinct n, $m < \omega$. Let $\{Z_\mu: \mu < 2^\omega \,\&\, \mu \text{ is even}\}$ enumerate all nonempty closed G_δ's of X (there are clearly only 2^ω closed G_δ's). In addition, let $\{\langle C_{\mu n}: n < \omega\rangle: \mu < 2^\omega \,\&\, \mu \text{ is odd}\}$

enumerate all sequences of nonempty closed G_δ's satisfying $C_{\mu,n+1} \subset \operatorname{int} C_{\mu n} - \bigcup_{i \leq n} X_i$. Furthermore, assume that each sequence is listed cofinally often. Finally, let $\{A^\beta_{\alpha n}: \alpha < 2^\omega, 1 \leq n < \omega, \beta < 2^\omega\}$ be an independent linked family of subsets of ω with respect to $\mathcal{C}\mathfrak{F}$.

By induction on μ we construct $\mathfrak{F}_\mu$ and K_μ so that:

(1) $\mathfrak{F}_\mu$ is a closed filter on X, $K_\mu \subset 2^\omega$, and $\{\bigcup\{X_i: i \in A^\beta_{\alpha n}\}: \alpha < 2^\omega, 1 \leq n < \omega, \beta \in K_\mu\}$ is an independent linked family w.r.t. $\mathfrak{F}_\mu$;

(2) $K_0 = 2^\omega$ and $\mathfrak{F}_0 = \mathfrak{F}$;

(3) $\nu < \mu$ implies $\mathfrak{F}_\nu \subset \mathfrak{F}_\mu$ and $K_\nu \supset K_\mu$;

(4) if μ is a limit ordinal, $\mathfrak{F}_\mu = \bigcup_{\nu<\mu} \mathfrak{F}_\nu$ and $K_\mu = \bigcap_{\nu<\mu} K_\nu$;

(5) for each μ, $K_\mu - K_{\mu+1}$ is finite;

(6) if μ is even, either $Z_\mu \in \mathfrak{F}_\mu$ or some $F \in \mathfrak{F}_\mu$ misses Z_μ;

(7) if μ is odd and each $C_{\mu n} \in \mathfrak{F}_\mu$, then there are $D_{\mu\alpha} \in \mathfrak{F}_{\mu+1}$ for $\alpha < 2^\omega$ such that for all $n \geq 1$ and all $\alpha_1 < \alpha_2 < \cdots < \alpha_n < 2^\omega$ the set $(D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) - C_{\mu n}$ has compact closure in X.

Notice that since $\mathfrak{F}$ is a nice filter, the collection

$$\left\{\bigcup \left\{X_i: i \in A^\beta_{\alpha n}\right\}: \alpha < 2^\omega, 1 \leq n < \omega, \beta < 2^\omega\right\}$$

is indeed an independent linked family w.r.t. $\mathfrak{F}$. Put $E^\beta_{\alpha n} = \bigcup\{X_i: i \in A^\beta_{\alpha n}\}$ for all $\alpha < 2^\omega$, $1 \leq n < \omega$ and $\beta < 2^\omega$.

Let us assume for a moment that this construction can be carried out and put $\mathcal{G} = \bigcup_{\mu<2^\omega} \mathfrak{F}_\mu$. By (6) $\mathcal{G}$ is a closed ultrafilter, hence $\bigcap_{G\in\mathcal{G}} \operatorname{cl}_{\beta X} G \cap X^*$ consists of precisely one point, say p. by (2), $p \in \bigcap_{F\in\mathfrak{F}} \operatorname{cl}_{\beta X} F$ and by (7), p is 2^ω-OK.

Fix $\mu < 2^\omega$ and assume that the $\mathfrak{F}_\nu$, K_ν have been constructed for $\nu \leq \mu$. We will construct $\mathfrak{F}_{\mu+1}$ and $K_{\mu+1}$.

If μ is even, let $\mathfrak{T}$ be the closed filter generated by $\mathfrak{F}_\mu \cup \{Z_\mu\}$. If $\mathfrak{T}$ has no compact elements and if

$$\left\{E^\beta_{\alpha n}: \alpha < 2^\omega, 1 \leq n < \omega, \beta \in K_\mu\right\}$$

is independent w.r.t. $\mathfrak{T}$ we set $\mathfrak{F}_{\mu+1} = \mathfrak{T}$ and $K_{\mu+1} = K_\mu$. If not, then we can find $E \in \mathfrak{F}_\mu$ such that $Z_\mu \cap E \cap \bigcap_{\beta\in\tau}(\bigcap_{\alpha\in\sigma_\beta} E^\beta_{\alpha n_\beta})$ is compact for some $\tau \in [K_\mu]^{<\omega}$, $n_\beta \in \omega$, and $\sigma_\beta \in [2^\omega]^{n_\beta}$. Then let $K_{\mu+1} = K_\mu - \tau$, and $\mathfrak{F}_{\mu+1}$ be the closed filter generated by $\mathfrak{F}_\mu$ and $\bigcap_{\beta\in\tau}(\bigcap_{\alpha\in\sigma_\beta} E^\beta_{\alpha n_\beta})$. Clearly $\mathfrak{F}_{\mu+1}$ and $K_{\mu+1}$ are as required.

If μ is odd and some $C_{\mu n}$ is not in $\mathfrak{F}_\mu$, put $\mathfrak{F}_{\mu+1} = \mathfrak{F}_\mu$ and $K_{\mu+1} = K_\mu$. In case $C_{\mu n} \in \mathfrak{F}_\mu$ for each $n < \omega$, then fix $\beta \in K_\mu$ and let $K_{\mu+1} = K_\mu - \{\beta\}$. Let $\mathfrak{F}_{\mu+1}$ be the closed filter generated by $\mathfrak{F}_\mu$ and the collection $\{D_{\mu\alpha}: \alpha < 2^\omega\}$, where

$$D_{\mu\alpha} = \bigcup_{1\leq n<\omega} E^\beta_{\alpha n} \cap C_{\mu n}.$$

First observe that $D_{\mu\alpha}$ is closed in X since $C_{\mu n} \subset \bigcup_{i \geq n+1} X_i$ for all $n < \omega$. To verify condition (7), let $\alpha_1 < \alpha_2 < \cdots < \alpha_n < 2^\omega$ and put

$$Y = \left(D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}\right) - C_{\mu n}.$$

If $n = 1$, then clearly $Y = \varnothing$, since $D_{\mu\alpha_1} \subset C_{\mu 1}$. Therefore, assume that $n > 1$.

Claim. $Y \subset E^{\beta}_{\alpha_1,n-1} \cap \cdots \cap E^{\beta}_{\alpha_n,n-1}$.

Take $x \in D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}$ and assume that $x \in \bigcap_{1 \leqslant i \leqslant n} E^{\beta}_{\alpha_i k_i} \cap C_{\mu k_i}$, where $k_{i_0} \geqslant n$ for some $1 \leqslant i_0 \leqslant n$. Since $C_{\mu k_{i_0}} \subset C_{\mu n}$ it follows that $x \notin Y$.

Therefore, if $x \in Y$ then there exist $k_i \leqslant n - 1$ $(1 \leqslant i$

$$x \in \bigcap_{1 \leqslant i \leqslant n} E^{\beta}_{\alpha_i k_i} \cap C_{\mu k_i} \subset \bigcap_{1 \leqslant i \leqslant n} E^{\beta}_{\alpha_i, n-1},$$

since $E^{\beta}_{\alpha_i k_i} \subset E^{\beta}_{\alpha_i, n-1}$ for each $k_i \leqslant n - 1$.

This implies that Y has compact closure since these $E^{\beta}_{\alpha,n-1}$ are precisely $(n-1)$-linked.

Finally, to verify condition (1), observe that $D_{\mu\alpha} \supset C_{\mu n} \cap E^{\beta}_{\alpha n}$ for each n. □

2.6. COROLLARY. *Let X be the sum of countably many nonempty compact spaces of weight at most 2^{ω}, say X_n $(n < \omega)$ and let $\mathcal{F}$ be a nice filter on X. Then there is a point $p \in \bigcap_{F \in \mathcal{F}} \mathrm{cl}_{\beta X} F \cap X^*$ which is a weak P-point of X^*.*

PROOF. Apply Lemma 2.2 and Theorem 2.5. □

2.7. REMARKS. Corollary 2.6 is weaker than Theorem 2.5 since in ω^* there exist weak P-points which are not 2^{ω}-*OK* (see [**vM$_2$**]). As remarked in the beginning of this section, in Corollary 2.6 the condition that the spaces X_n have all weight at most 2^{ω} is superfluous (see §4). However, in the proof of the general result we need the special case Corollary 2.6.

2.8. REMARK. Obviously, being a weak P-point is a nicer topological property than being a 2^{ω}-*OK* point. However, 2^{ω}-*OK* points are more interesting than weak P-points since they seem to have more applications (see e.g. [**vDvM$_3$**]).

Notes for §2. The concept of a κ-*OK* point is due to Kunen [**K$_2$**]. If $x \in X$, then x is called κ-*OK* provided that for each sequence of neighborhoods $\{U_n : n < \omega\}$ of x in X there is a sequence of neighborhoods $\{V_{\alpha} : \alpha < \kappa\}$ of x such that for each $n \geqslant 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$, $\bigcap_{1 \leqslant i \leqslant n} V_{\alpha_i} \subset U_n$. For technical reasons we have slightly changed the definition of a κ-*OK* point in the special case of Čech-Stone remainders.

The technique of proof used in this section is due to Kunen [**K$_2$**] who proved Theorem 2.5 for the special case $X = \omega$ and $\mathcal{F} = \mathcal{C}\mathcal{F}$. Theorem 2.5 as stated here is due to the author [**vM$_1$**], but the proof is almost the same as the proof of Kunen's result. The reason I became interested in Theorem 2.5 is that very nontrivial nice filters exist (see e.g. [**CS, D, vD$_3$, vM$_1$, vM$_2$**]), and that therefore Theorem 2.5 proves the existence of points which are "special" in X^* as well as in βX. For a generalization of Theorem 2.5 see [**vM$_2$**].

Lemma 2.1 is due to van Mill and Mills [**vMM**].

Lemma 2.2 is in fact due to Kunen [**K$_2$**]. For a different proof of this lemma see [**vM$_1$**].

Independent linked families were first defined by Kunen [**K$_2$**].

Lemma 2.4 is due to Kunen [**K$_2$**] who proved it, as he notes in [**K$_2$**], via a tree of trees. The proof presented here, as well as in [**K$_2$**], is due to P. Simon.

3. A ccc nowhere separable remainder of ω. In this section we show that there is a compactification $\gamma\omega$ of ω with $\gamma\omega - \omega$ ccc and nowhere separable. This result we need in §4 to generalize Theorem 2.5.

We start with a simple lemma.

3.1. LEMMA. *If there is a compactification $\gamma\omega$ of ω with $\gamma\omega - \omega$ ccc and not separable, then there is a compactification $b\omega$ of ω with $b\omega - \omega$ ccc and nowhere separable.*

PROOF. Let $\mathfrak{U}$ be a maximal family of pairwise disjoint separable open subsets of $\gamma\omega - \omega$. Since $|\mathfrak{U}| \leq \omega$, $\bigcup\mathfrak{U}$ is not dense. Let $b\omega$ be the space we get from $\gamma\omega$ by collapsing $(\bigcup\mathfrak{U})^- \cup \{x\}$, where $x \in \gamma\omega - \omega - (\bigcup\mathfrak{U})^-$, to a point. □

If f is a function, then dom f denotes the domain of f.

Let $2 \leq n < \omega$. A family of sets is called *n-linked* provided that each subfamily of cardinality at most n has nonempty intersection. Call a family of sets *$\sigma - n$-linked* provided that it is the union of countably many n-linked subfamilies. Observe that a space with a $\sigma - 2$-linked base in ccc. It therefore suffices to prove the following.

3.2. EXAMPLE. *There is a compactification $\gamma\omega$ of ω such that $\gamma\omega - \omega$ is not separable although $\gamma\omega - \omega$ has a $\sigma - 2$-linked base.*

PROOF. Let $P = \{f \in \omega^\omega \colon 0 \leq f(n) \leq n+1 \text{ for each } n \in \omega\}$ and $N = \{f \restriction n \colon f \in P \text{ and } n \in \omega\}$. Define $T = \{\pi \in N^\omega \colon \operatorname{dom} \pi(n) = n+1 \text{ for each } n \in \omega\}$. For each $s \in N$, let $C_s = \{t \in N \colon s \subset t\}$ and for each $\pi \in T$ put

$$C_\pi = \bigcup_{n \in \omega} C_{\pi(n)}.$$

Observe that $N - C_\pi$ is infinite for each π. Let $\mathfrak{B}$ denote the smallest Boolean subalgebra of $\mathfrak{P}(N)$ containing $\mathcal{C} = \{C_\pi \colon \pi \in T\} \cup \{N - C_\pi \colon \pi \in T\}$. Notice that $\{\{s\} \colon s \in N\} \cup \{C_s \colon s \in N\} \subset \mathfrak{B}$. Let $\gamma\omega$ denote the Stone space of $\mathfrak{B}$. It is clear that $\gamma\omega$ is a compactification of the countable discrete space $\{\{B \in \mathfrak{B} \colon s \in B\} \colon s \in N\}$ which we identify with ω. Put $X = \gamma\omega - \omega$.

Claim 1. X is not separable.

Let $\{p_n \colon n \in \omega\}$ be countably many free ultrafilters of $\mathfrak{B}$. For each $n \in \omega$, there exists $\pi(n)$ with $\operatorname{dom} \pi(n) = n+1$ such that $C_{\pi(n)} \in P_n$. This is so, since $N = \{s \in N \colon \operatorname{dom} s \leq n\} \cup \bigcup\{C_s \colon \operatorname{dom} s = n+1\}$ for each $n \in \omega$. Consequently, $\{p \in X \colon N - C_\pi \in p\}$ is a nonempty open set of X disjoint from $\{p_n \colon n \in \omega\}$.

Claim 2. X has a $\sigma - 2$-linked base.

It suffices to show that $\{B \in \mathfrak{B} \colon |B| = \omega\} = \bigcup_{n \in \omega} \mathfrak{B}_n$ such that for each n every two members of $\mathfrak{B}_n$ have infinite intersection. To this end, for each $j \in \omega$ and for each $s \in N$ with $2j - 1 \leq \operatorname{dom} s$, define

$$\mathfrak{B}(j, s) = \Big\{B \in \mathfrak{B} \colon \exists K \in [T]^{<\omega} \text{ and } L \in [T]^j \text{ with } s \in \bigcap_{\pi \in K} C_\pi \cap \bigcap_{\pi \in L} N - C_\pi \in [B]^\omega\Big\}.$$

Since for each $B \in \mathfrak{B}$ with $|B| = \omega$, there exists a set D which is a finite intersection of elements of $\mathcal{C}$, with $D \in [B]^\omega$ and since any infinite subset of N contains elements of arbitrarily large domain, it follows that

$$\{B \in \mathfrak{B} \colon |B| = \omega\} = \bigcup \{\mathfrak{B}(j, s) \colon j \in \omega, s \in N, \text{ and } 2j - 1 \leq \operatorname{dom} s\}.$$

Fix an index j and $s \in N$ with $2j - 1 \leqslant \operatorname{dom} s$. If $\{B_0, B_1\} \subset \mathfrak{B}(j, s)$, then there exist $K_i \in [T]^{<\omega}$ and $L_i \in [T]^j$ such that for each $i = 0, 1$,

$$s \in D_i = \bigcap_{\pi \in K_i} C_\pi \cap \bigcap_{\pi \in L_i} N - C_\pi \in [B_i]^\omega.$$

We now define, by induction on $\operatorname{dom} s \leqslant n$, an $h \in P$ such that $\{h \restriction n\colon \operatorname{dom} s \leqslant n\} \subset D_0 \cap D_1$.

Stage $\operatorname{dom} s$. Let $h \restriction \operatorname{dom} s = s$. Then $h \restriction \operatorname{dom} s \in D_0 \cap D_1$. Assume we have defined $h \restriction n$ for some $\operatorname{dom} s \leqslant n$ such that $h \restriction n \in D_0 \cap D_1$.

Stage $n + 1$. Define $h \restriction n + 1$ to be some sequence in N of domain $n + 1$ that extends $h \restriction n$ and such that $h \restriction n + 1 \notin \{\pi(n)\colon \pi \in L_0 \cup L_1\}$. This is possible because there are $n + 2$ sequences in N of domain $n + 1$ that extend $h \restriction n$ and $|L_0 \cup L_1| \leqslant 2j < \operatorname{dom} s + 2 \leqslant n + 2$. Then $h \restriction n + 1 \in D_0 \cap D_1$. $\square$

Notes for §3. The question whether a ccc nonseparable growth of ω exists was asked in [**vM$_1$**] and such a growth was constructed by Bell [**B**]. The compactification $\gamma\omega$ of ω constructed in this section is precisely the same as in [**B**]. We have also used Bell's write-up of the example. The existence of a ccc nonseparable growth of ω is an important ingredient in the proof of Theorem 0.1.

Lemma 3.1 was first shown in van Mill [**vM$_1$**] and the proof is due to the referee of [**vM$_1$**].

4. The extension theorem. Let X be the topological sum of countably many compact spaces, say X_n $(n < \omega)$, and let $\mathfrak{F}$ be a nice filter on X (see §2). We will show that there is a weak P-point $x \in X^*$ such that $x \in \bigcap_{F \in \mathfrak{F}} \operatorname{cl}_{\beta X} F$. This generalizes results in §2.

For each $n < \omega$, let X_n be a space. The disjoint topological sum of the spaces X_n will be denoted by $\Sigma_{n<\omega} X_n$. Whenever we write $\Sigma_{n<\omega} X_n$, for convenience, we assume that the spaces X_n are disjoint.

We start with a simple but important lemma.

4.1. LEMMA. *Let Y be a ccc nowhere separable space and let, for each $n < \omega$, X_n be a compact space which can be mapped onto Y, say be g_n. In addition, for each $n < \omega$, let $Z_n \subset X_n$ be closed such that $g_n(Z_n) = Y$. Then there is a nice filter $\mathfrak{F}$ on $Z = \Sigma_{n<\omega} Z_n$ such that for each countable $D \subset X = \Sigma_{n<\omega} X_n$ some $F \in \mathfrak{F}$ misses the closure (in X) of D.*

PROOF. For each $n < \omega$, let $f_n = g_n \restriction Z_n$. In addition, for any countable $D \subset X$, let $\{U_n(D)\colon n < \omega\}$ be a maximal disjoint collection of nonempty regular closed subsets of Y none of which intersects $(\bigcup_{n<\omega} g_n(D \cap X_n))^-$. Define

$$F(D) = \bigcup_{n<\omega} f_n^{-1}\Big(\bigcup_{i \leqslant n} U_i(D)\Big).$$

Observe that $F(D) \cap \bar{D} = \varnothing$.

Claim. The closed filter on Z generated by $\{F(D)\colon D \in [X]^\omega\}$ is nice. Take $D_1, \ldots, D_k \in [X]^\omega$. Since $\bigcup_{n<\omega} \operatorname{int} U_n(D_i)$ is dense for each $1 \leqslant i \leqslant k$,

$$\bigcap_{1 \leqslant i \leqslant k} \bigcup_{n<\omega} \operatorname{int} U_n(D_i)$$

is also dense. We can therefore find $n_1, n_2, \ldots, n_k < \omega$ so that $\bigcap_{1 \leqslant i \leqslant k} U_{n_i}(D_i) \neq \varnothing$. Let $n = \max\{n_1, n_2, \ldots, n_k\}$ and take $l \geqslant n$ arbitrarily. Then

$$\bigcap_{1 \leqslant i \leqslant k} F(D_i) \cap Z_l \supset f_l^{-1}\Big(\bigcap_{1 \leqslant i \leqslant k} U_{n_i}(D_i)\Big) \neq \varnothing .$$

This proves our claim. □

An *F-space* is a space in which cozero-sets are C^*-embedded. It is easily seen that a normal space is an F-space iff any two disjoint open F_σ's have disjoint closures.

4.2. Lemma. *Let X be a locally compact and σ-compact space. Then each F_σ $F \subset X^*$ is C^*-embedded in X^*. Consequently, X^* is an F-space.*

Proof. Let $F \subset X^*$ be any F_σ and let f: $F \to I$ be continuous. Since F is closed in $X \cup F$ and since $X \cup F$ is normal, being σ-compact, f extends to a map $\bar{f}$: $X \cup F \to I$. Since $\beta(X \cup F) = \beta X$ [**GJ**], $\bar{f}$ extends to a map $\bar{\bar{f}}$: $\beta X \to I$. □

Let f: $X \to Y$ be a continuous surjection. The map f is called *irreducible* provided that $f(A) \neq Y$ for any proper closed subset $A \subset X$.

For each space X let $RO(X)$ be the Boolean algebra of regular open subsets of X. It is clear that $|RO(X)| \leqslant w(X)^{c(X)}$, where $w(X)$ and $c(X)$ denote the weight and cellularity of X. If f: $X \to Y$ is a closed irreducible surjection then $f^{\#}$: $RO(X) \to RO(Y)$ defined by $f^{\#}(U) = Y - f(X - U)$ clearly is a Boolean isomorphism; hence $|RO(X)| = |RO(Y)| \leqslant w(Y)^{c(Y)}$. This observation will be used in the proof of the main result in this section.

4.3. Theorem. *Let X be the topological sum of countably many nonempty compact spaces, say X_n $(n < \omega)$ and let $\mathfrak{F}$ be a nice filter on X. Then there is a weak P-point $x \in X^*$ such that $x \in \bigcap_{F \in \mathfrak{F}} \mathrm{cl}_{\beta X} F$.*

Proof. Let $\{E_n: n < \omega\}$ be a partition of ω in countably many infinite sets. For each $n < \omega$, let

$$\mathfrak{F}_n = \Big\{F \cap \bigcup \{X_i: i \in E_n\}: F \in \mathfrak{F}\Big\}$$

and notice that $\mathfrak{F}_n$ is a nice filter on $\bigcup\{X_i: i \in E_n\}$. Let

$$F(n) = \bigcap_{F \in \mathfrak{F}_n} \mathrm{cl}_{\beta X} F \cap X^*.$$

Notice that $F(n) \cap F(m) = \varnothing$ whenever $n \neq m$ and that, by Lemma 4.2, $\bigcup_{n<\omega} F(n)$ is C^*-embedded in X^*.

Define f: $X \to \omega$ by $f(x) = n$ iff $x \in X_n$ and, for each $n < \omega$, let $f_n = f \restriction \bigcup_{i \in E_n} X_i$. In addition, let βf and βf_n $(n < \omega)$ be the Stone extensions of f and f_n $(n < \omega)$.

For each $n < \omega$, put

$$S(n) = \mathrm{cl}_{\beta X}\Big(\bigcup \{X_i: i \in E_n\}\Big) \cap X^*.$$

Observe that $F(n) \subset S(n)$ and that $\beta f_n(S(n)) = E_n^* \approx \omega^*$. Since $\mathfrak{F}_n$ is a nice filter we also have that $\beta f_n(F(n)) = E_n^*$.

Let Y be a ccc nowhere separable remainder of ω, see §3, and for each $n < \omega$ let g_n map E_n^* onto Y and let h_n be the composition of $\beta f_n \restriction S_n$ and g_n. Notice that

$h_n(F(n)) = Y$. For each $n < \omega$, let $Y(n) \subset F(n)$ be closed such that $h_n \restriction Y(n)$: $Y(n) \to Y$ is an irreducible surjection. Then $|RO(Y(n))| = |RO(Y)| \leq w(Y)^{c(Y)} = (2^\omega)^\omega = 2^\omega$. We conclude that $Y(n)$ has weight 2^ω.

By Lemma 4.1 there is a nice filter $\mathcal{G}$ on $\Sigma_{n<\omega} Y(n)$ such that whenever $D \subset S = \bigcup_{n<\omega} S(n)$ is countable, then $\mathrm{cl}_S D \cap G = \varnothing$ for some $G \in \mathcal{G}$. By Theorem 2.5 we can "extend" $\mathcal{G}$ to a 2^ω-*OK* point p of $(\bigcup_{n<\omega} Y(n))^*$. Since, by Lemma 4.2, $\bigcup_{n<\omega} Y(n)$ is C^*-embedded in X^*, p is a point of X^*. We claim that p is a weak P-point of X^* and that $p \in \bigcap_{F \in \mathcal{F}} \mathrm{cl}_{\beta X} F$.

Since $\bigcup_{n<\omega} F(n) \subset \bigcap_{F \in \mathcal{F}} \mathrm{cl}_{\beta X} F$, and since $p \in (\bigcup_{n<\omega} F(n))^-$, the second claim is trivial.

Let $H \subset X^* - \{p\}$ be countable. Put

$$H_0 = H - \Big(\bigcup_{n<\omega} S(n)\Big)^-.$$

Since $\bigcup_{n<\omega} S(n)$ is an open F_σ of X^* and since, by Lemma 4.2, X^* is an F-space, it is clear that $\overline{H}_0 \cap (\bigcup_{n<\omega} S(n))^- = \varnothing$; we conclude that $p \notin \overline{H}_0$.

Let

$$H_1 = H \cap \bigcup_{n<\omega} S(n).$$

By construction, some $G \in \mathcal{G}$ misses the closure, in $S = \bigcup_{n<\omega} S_n$, of H_1. Since S is normal, being σ-compact, this implies that $\overline{G} \cap \overline{H}_1 = \varnothing$, consequently, $p \notin \overline{H}$.

Let

$$H_2 = (H \cap (\overline{S} - S)) - \Big(\bigcup_{n<\omega} Y(n)\Big)^-.$$

Since, by Lemma 2.1, $(\bigcup_{n<\omega} Y(n))^- \cap (\overline{S} - S)$ is a P-set of $\overline{S} - S$, we conclude that $\overline{H}_2 \cap (\bigcup_{n<\omega} Y(n))^- = \varnothing$, consequently, $p \notin \overline{H}_2$.

Finally, let

$$H_3 = H \cap \left(\Big(\bigcup_{n<\omega} Y(n)\Big)^- - \Big(\bigcup_{n<\omega} Y(n)\Big)\right).$$

Since p is a 2^ω-*OK* point of $(\bigcup_{n<\omega} Y(n))^*$, by Lemma 2.2, $p \notin \overline{H}_3$.

We conclude that $p \notin \overline{H}$. □

Notes for §4. Lemma 4.1 is implicit in van Mill [**vM**$_1$, 5.2].

Lemma 4.2 is due to Gillman and Henriksen [**GH**]. The easy proof presented here is due to Negrepontis [**N**].

Theorem 4.3 is new. The proof of Theorem 4.3 is implicit in van Mill [**vM**$_1$, 5.2].

5. The nowhere ccc case. Let X be a nonpseudocompact space which is nowhere ccc. We show that there is a point $x \in X^*$ which is a weak P-point of βX.

Let X be a nonpseudocompact nowhere ccc space. We aim to apply Theorem 4.3, so we will construct a nice filter on a certain closed subspace of X which "avoids" all separable subspaces of X. Since a nowhere ccc space can have "many" separable subspaces there is no hope to do this by an induction avoiding one separable subspace at each stage of the induction. We therefore use a different technique.

5.1. LEMMA. *Let* $X = \sum_{n<\omega} X_n$, *where each* X_n *is compact and nowhere ccc. In addition, for each* $n < \omega$, *let* D_n *be a closed nowhere dense subset of* X_n. *Then there exists a nice filter* $\mathcal{F}$ *on* X *such that*:

(1) *there is an* $F \in \mathcal{F}$ *with* $F \cap D_n = \varnothing$ *for all* $n < \omega$,

(2) *if* $E \subset X$ *is ccc then there is an* $F \in \mathcal{F}$ *with* $\overline{E} \cap F = \varnothing$.

PROOF. For each finite subset $F \subset \omega_1$ (possibly empty) and each $n < \omega$ we will define an open set $C_F^n \subset X_n$ and a nonempty regular closed set $B_F^n \subset C_F^n$ such that:

(1) $C_{F\cup\{\alpha\}}^n \subset B_F^n$ for all $\max F < \alpha < \omega_1$;

(2) $C_{F\cup\{\alpha\}}^n \cap C_{F\cup\{\beta\}}^n = \varnothing$ if $\max F < \alpha < \beta < \omega_1$;

(3) $C_F^n \cap D_n = \varnothing$.

We will induct on the cardinality of F. Let $C_\varnothing^n = X_n - D_n$ and let $B_\varnothing^n \subset C_\varnothing^n$ be any nonempty regular closed set.

Suppose that we have defined the C_F^n and B_F^n for all $F \subset \omega_1$ of cardinality i. Let $\{C_{F\cup\{\alpha\}}^n: \max F < \alpha < \omega_1\}$ be a "faithfully indexed" collection of pairwise disjoint nonempty open subsets of B_F^n. In addition, let $B_{F\cup\{\alpha\}}^n$ be any nonempty regular closed subset of $C_{F\cup\{\alpha\}}^n$. This completes the induction.

FACT 1. $C_F^n \cap C_G^n \neq \varnothing \to (F \subset G) \vee (G \subset F)$.

We induct on the cardinality of $|F| + |G|$. If $|F| + |G| = 1$, then there is nothing to prove. Suppose that we have proved Fact 1 for all finite sets F, $G \subset \omega_1$ satisfying $|F| + |G| \leq i - 1$. Now take finite sets S, $T \subset \omega_1$ so that $|S| + |T| \leq i$. Define $S^1 = S - \{\max S\}$. By (1) we have that $C_S^n \subset C_{S^1}^n$ and consequently $C_{S^1}^n \cap C_T^n \neq \varnothing$. By induction hypothesis, $S^1 \subset T$ or $T \subset S^1$. If $T \subset S^1$ then we are done, so we may assume that $S^1 \subset T$. Define $T^1 = T - \{\max T\}$. By precisely the same argumentation we may conclude that $T^1 \subset S$. Then clearly

$$(S \cap T) \cup \{\max S\} = S \quad \text{and} \quad (S \cap T) \cup \{\max T\} = T.$$

If $\max S \in T$ or $\max T \in S$ then there is nothing to prove. So assume that this is not true. Then, by (2) we have that $C_S^n \cap C_T^n = \varnothing$, which is a contradiction.

Let $f\colon \omega_1 \to \omega_1 \times \omega_1$ be one-to-one and onto. For each $\langle \alpha, \beta \rangle \in \omega_1 \times \omega_1$ and $n < \omega$ define

$$U_\beta^\alpha(n) = \bigcup \left\{C_{F\cup\{f^{-1}(\langle\alpha,\beta\rangle)\}}^n: \max F < f^{-1}(\langle \alpha, \beta \rangle) \text{ and } f(F) \cap (\{\alpha\} \times \omega_1) = \varnothing.\right.$$

Notice that $U_\beta^\alpha(n)$ is open.

FACT 2. $U_\beta^\alpha(n) \cap U_\gamma^\alpha(n) = \varnothing$ whenever $\beta \neq \gamma$.

Assume that this is not true. Without loss of generality, assume that $f^{-1}(\langle \alpha, \beta \rangle) < f^{-1}(\langle \alpha, \gamma \rangle)$. There are finite sets F_0, $F_1 \subset \omega_1$ so that:

(a) $C_{F_0\cup\{f^{-1}(\langle\alpha,\beta\rangle)\}}^n \cap C_{F_1\cup\{f^{-1}(\langle\alpha,\gamma\rangle)\}}^n \neq \varnothing$;

(b) $\max F_0 < f^{-1}(\langle \alpha, \beta \rangle)$ and $f(F_0) \cap (\{\alpha\} \times \omega_1) = \varnothing$;

(c) $\max F_1 < f^{-1}(\langle \alpha, \gamma \rangle)$ and $f(F_1) \cap (\{\alpha\} \times \omega_1) = \varnothing$.

Since $f^{-1}(\langle \alpha, \gamma \rangle) \notin F_0 \cup \{f^{-1}(\langle \alpha, \beta \rangle)\}$, by Fact 1, $F_0 \cup \{f^{-1}(\langle \alpha, \beta \rangle)\} \subset F_1 \cup \{f^{-1}(\langle \alpha, \gamma \rangle)\}$. Therefore $f^{-1}(\langle \alpha, \beta \rangle) \in F_1$, since $f^{-1}(\langle \alpha, \beta \rangle) \neq f^{-1}(\langle \alpha, \gamma \rangle)$. However, this contradicts (c).

For each $i < \omega$ and $\beta < \omega_1$ let $\{F_k(\langle i, \beta \rangle): k < \omega\}$ enumerate all finite subsets $F \subset \omega_1$ with $\max F < f^{-1}(\langle i, \beta \rangle)$ and $f(F) \cap (\{i\} \times \omega_1) = \varnothing$.

If $\beta < \omega_1$ and $i < \omega$ define

$$G_\beta^i = \bigcup_{l<\omega} \bigcup_{m\leqslant l} B^{i+l}_{F_m(\langle i,\beta\rangle)\cup\{f^{-1}(\langle i,\beta\rangle)\}} \quad \text{and} \quad G_\beta = \bigcup_{i<\omega} G_\beta^i.$$

Observe that G_β is a closed subset of X which misses $\bigcup_{n<\omega} D_n$ for each $\beta < \omega_1$.

FACT 3. If $E \subset X$ is ccc then $G_\beta \cap \bar{E} = \varnothing$ for some $\beta < \omega_1$.

By Fact 2 we can find $\beta < \omega_1$ such that $U^i_\beta(n) \cap E = \varnothing$ for each $n, i < \omega$ (in fact this is true for all but countably many $\beta < \omega_1$). Since $G_\beta \subset \bigcup_{i<\omega} \bigcup_{n<\omega} U^i_\beta(n)$, and since $\bigcup_{i<\omega} \bigcup_{n<\omega} U^i_\beta(n)$ is open, $G_\beta \cap \bar{E} = \varnothing$.

FACT 4. The closed filter generated by $\{G_\beta : \beta < \omega_1\}$ is nice.

Take $\beta_1, \beta_2, \ldots, \beta_n \in \omega_1$ arbitrarily and put $\gamma_i = f^{-1}(\langle i, \beta_i\rangle)$ for all $1 \leqslant i \leqslant n$. Without loss of generality assume that $\gamma_1 < \gamma_2 < \cdots < \gamma_n$. For all $1 \leqslant i \leqslant n-1$ let $k_i < \omega$ be such that $\{\gamma_1, \ldots, \gamma_i\} = F_{k_i}(\langle i+1, \beta_{i+1}\rangle)$. In addition, let $k_0 < \omega$ be such that $\varnothing = F_{k_0}(\langle 1, \beta_1\rangle)$. Define $l = n + \max\{k_i : 0 \leqslant i \leqslant n-1\}$. We claim that

$$(*) \qquad \bigcup_{m\geqslant l} B^m_{\{\gamma_1,\gamma_2,\ldots,\gamma_n\}} \subset \bigcap_{1\leqslant i\leqslant n} G_{\beta_i}.$$

Take $m \geqslant l$ arbitrarily. Since

$$B^m_{\{\gamma_1,\ldots,\gamma_n\}} = B^m_{F_{k_{n-1}}(\langle n,\beta_n\rangle)\cup\{f^{-1}(\langle n,\beta_n\rangle)\}}$$

and since $m - n \geqslant k_{n-1}$, it follows that $B^m_{\{\gamma_1,\gamma_2,\ldots,\gamma_n\}} \subset G^n_{\beta_n} \subset G_{\beta_n}$. Since $B^m_{\{\gamma_1,\gamma_2,\ldots,\gamma_n\}} \subset B^m_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}}$, by (1), by the same argument,

$$B^m_{\{\gamma_1,\gamma_2,\ldots,\gamma_n\}} \subset B^m_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}} \subset G^{n-1}_{\beta_{n-1}} \subset G_{\beta_{n-1}}.$$

Continuing this process inductively one can now easily prove (*). □

5.2. REMARK. Notice that in the proof of the above lemma we found a filterbase of cardinality ω_1 which avoids all ccc subsets of X. This is truly remarkable. For example, if one wants to avoid all nowhere dense subsets of $\omega \times [0, 1]$, then, under MA + ¬CH, one needs 2^ω closed subsets. This justifies our claim in the Introduction that the "small" spaces are more complicated than the "large" ones.

We now come to the main result in this section.

5.3. THEOREM. *Let X be a nowhere ccc nonpseudocompact space. Then X^* contains a point x which is a weak P-point of βX.*

PROOF. Since X is nonpseudocompact, there is a nonempty closed G_δ Z of βX which is comtained in X^*. Put $Y = \beta X - Z$ and, since $\beta Y = \beta X$ [**GJ**, 6.7] and Y is clearly nowhere ccc, we need only prove the theorem for Y. For each $n < \omega$ take a compact nonempty regular closed set $V_n \subset Y$ such that:

(a) if $n \neq m$ then $V_n \cap V_m = \varnothing$, and

(b) for any $E \subset \omega$, $\bigcup_{n\in E} V_n$ is closed in Y.

Let $V = \bigcup_{n<\omega} V_n$. By Lemma 5.1 there is a nice filter $\mathfrak{F}$ on V such that $F \subset \operatorname{int}_Y V$ for all $F \in \mathfrak{F}$ while, moreover, for each ccc subset $D \subset V$ there is some $F \in \mathfrak{F}$ which misses the closure of D (in V). By Theorem 4.3 there is a point $x \in V^*$ such that $x \in \bigcap_{F\in\mathfrak{F}} \operatorname{cl}_{\beta V} F$, while, moreover, x is a weak P-point of V^*.

By normality of Y, the closed set V is C^*-embedded in Y, hence $\operatorname{cl}_{\beta Y} V = \beta V$, consequently, x is a point of Y^*. We claim that x is as required.

Let $H \subset \beta Y - \{x\}$ be countable. Put $H_0 = (H \cap Y) - \bigcup_{n<\omega} \operatorname{int} V_n$. Then, by construction, some $F \in \mathcal{F}$ misses the closure, in Y, of H_0. Hence, by normality of Y,

$$\mathrm{cl}_{\beta Y} F \cap \mathrm{cl}_{\beta Y} H_0 = \varnothing,$$

i.e. $x \notin \mathrm{cl}_{\beta Y} H_0$. Put $H_1 = H \cap \bigcup_{n<\omega} \operatorname{int} V_n$. By construction, some $F \in \mathcal{F}$ misses the closure in V (= the closure in Y) of H_1. In the same way as above we conclude that $x \notin \mathrm{cl}_{\beta Y} H_1$. Consequently $x \notin \mathrm{cl}_{\beta Y}(H \cap Y)$.

Since, by Lemma 2.1, V^* is a P-set of Y^* and since x is a weak P-point of V^* it easily follows that also $x \notin \mathrm{cl}_{\beta Y}(H \cap Y^*)$. □

5.4. REMARK. The weak P-points we get from the proof of Theorem 4.3 are all limit points of some subset which satisfies the countable chain condition. The way these points were constructed shows that one cannot avoid this. Since the filter of Lemma 5.1 avoids all ccc subsets one is led to the following.

5.5. *Question. Let X be a nowhere ccc nonpseudocompact space. Is there a point $x \in X^*$ such that whenever $A \subset \beta X - \{x\}$ is ccc, then $x \notin \bar{A}$?*

I have absolutely no idea how to answer this question.

Notes for §5. The proof of Lemma 5.1 is similar to, but more complicated than, Dow and van Mill [**DvM**, 2.1]. The idea of using matrices of sets as constructed in the proof of Lemma 5.1 goes back to Kunen [**K₂**] and the actual filter constructed from this matrix is similar to, but of more complicated nature than, filters in [**vM₁**, **vM₃**, **DvM**].

6. The nowhere of weight $\leqslant 2^\omega$ case: Part 1. Let X be a compact space of weight greater than 2^ω which satsifies the countable chain condition. We prove that for each $1 \leqslant n < \omega$ there is a family $\{(A^0_{\alpha n}, A^1_{\alpha n})\colon \alpha < (2^\omega)^+\}$ of pairs of disjoint nonempty closed sets in X such that whenever $F \subset (2^\omega)^+$ has cardinality n and $f\colon F \to 2$, then $\bigcap_{\alpha \in F} A^{f(\alpha)}_{\alpha n} \neq \varnothing$.

In this section we prove the result stated above which we will use in §7 to prove the nowhere of weight $\leqslant 2^\omega$ case of our theorem. I have the feeling that the results in this section are of independent interest and since they have nothing to do with Čech-Stone compactifications we have stated them in a separate section.

Let κ denote any infinite cardinal. The statement $(2^\kappa)^+ \to ((2^\kappa)^+, \kappa^+)^2$ means that whenever $|S| = (2^\kappa)^+$ and $[S]^2$ is the union of two sets A and B, then we can either find a set $S_0 \in [S]^{|S|}$ such that $[S_0]^2 \subset A$ or we can find a set $S_1 \in [S]^{\kappa^+}$ such that $[S_1]^2 \subset B$. It is well known that $(2^\kappa)^+ \to ((2^\kappa)^+, \kappa^+)^2$ is true for any infinite cardinal (see e.g. Juhász [**J**, A4.8]). Our results heavily rely on this result.

Until now, our paper is self contained. We have decided not to include a proof of the partition relation $(2^\kappa)^+ \to ((2^\kappa)^+, \kappa^+)$ since it is well known and the proof can be found in any book on combinatorial set theory. In addition, if the reader understands the proof of Lemma 6.2, he or she can easily reconstruct the proof of $(2^\kappa)^+ \to ((2^\kappa)^+, \kappa^+)$.

6.1. DEFINITION. Let X be a space and let $\{(A^0_\alpha, A^1_\alpha)\colon \alpha < \kappa\}$ be a collection of pairs of disjoint nonempty closed subsets of X. If $1 \leqslant n < \omega$ we call this family *n-independent* provided that for each $F \in [\kappa]^n$ and $f\colon F \to 2$ it is true that $\bigcap_{\alpha \in F} A^{f(\alpha)}_\alpha \neq \varnothing$. In addition, we call this family *strongly n-independent* provided that for each $F \in [\kappa]^n$ and $f\colon F \to 2$ we have that $\bigcap_{\alpha \in F} \operatorname{int} A^{f(\alpha)}_\alpha \neq \varnothing$.

Let X be a space. A *π-basis* $\mathfrak{B}$ for X is a collection of nonempty open subsets of X such that every nonempty open subset of X includes a member of $\mathfrak{B}$. The *π-weight*, $\pi(X)$, of X is defined by

$$\pi(X) = \omega \cdot \min\{|\mathfrak{B}|: \mathfrak{B} \text{ is a } \pi\text{-basis of } X\}.$$

Let, as usual, $w(X)$ and $c(X)$ denote the weight and cellularity of X. Trivially, $w(X) \leq \pi(X)^{c(X)}$ for any space X [**J**, 2.3] (recall that all spaces are Tychonoff).

Let X be a space. The Boolean algebra of regular open subsets of X is denoted by *RO(X)*. If $\mathcal{E} \subset RO(X)$, then $[\![\mathcal{E}]\!]$ denotes the smallest Boolean subalgebra of $RO(X)$ which contains $\mathcal{E}$. Notice that $|[\![\mathcal{E}]\!]| \leq \omega \cdot |\mathcal{E}|$.

For the remaining part of this section, let X be a ccc space of weight greater than 2^ω. We aim at proving that for each $n \geq 1$ there is a strongly n-independent family in X of cardinality $(2^\omega)^+$.

First observe that $\pi(X) > 2^\omega$. For convenience put $\kappa = (2^\omega)^+$. Inductively, let us construct nonempty U_α, $C_\alpha \subset X$ for $\alpha < \kappa$ such that:

(1) U_α is regular open and C_α is regular closed;

(2) $C_\alpha \subset U_\alpha$;

(3) if $\mathcal{E}_\alpha = [\![\{U_\beta\colon \beta < \alpha\} \cup \{\text{int}\, C_\beta\colon \beta < \alpha\}]\!] - \{\varnothing\}$ and if $E \in \mathcal{E}_\alpha$ is chosen arbitrarily, then $E \not\subset U_\alpha$.

Suppose that we have constructed C_β and U_β for each $\beta < \alpha$. Since the family

$$\mathcal{E} = [\![\{U_\beta\colon \beta < \alpha\} \cup \{\text{int}\, C_\beta\colon \beta < \alpha\}]\!] - \{\varnothing\}$$

has cardinality at most $\omega \cdot 2^\omega = 2^\omega$, it is not a π-basis for X. Hence we can find a nonempty $U \in RO(X)$ such that $E \not\subset U$ for each $E \in \mathcal{E}$. Put $U_\alpha = U$ and let $C_\alpha \subset U_\alpha$ be any nonempty regular closed set.

6.2. LEMMA. *There is a set $A \in [\kappa]^\kappa$ such that whenever $\alpha \in A$ and $F \subset A \cap \alpha$ is finite, then $C_\alpha \not\subset (\bigcup_{\beta \in F} U_\beta)^-$.*

PROOF. For each $\nu \leq \omega_1$ put $S_\nu = \{s\colon \nu \to [2^\omega]^{<\omega}\}$. Let $R_0 = \kappa$ and let $A_s \subset R_s$ be a maximal subset such that whenever $\alpha \in A_s$ and $F \subset A_s \cap \alpha$ is finite, then $C_\alpha \not\subset (\bigcup_{\beta \in F} U_\beta)^-$ (for each $s \in \bigcup\{S_\nu\colon \nu \leq \omega_1\}$ for which R_s is defined). While defining the sets R_s and A_s we will assume that each A_s has cardinality at most 2^ω.

Let $\nu \leq \omega_1$ be an ordinal such that R_s has been defined for all s of length $< \nu$. We define R_s for $s \in S_\nu$ as follows:

Case 1. If ν is a limit and $s \in S_\nu$ then $R_s = \bigcap_{\eta < \nu} R_{s \restriction \eta}$.

Case 2. If ν is a successor, and $s \in S_{\nu - 1}$, then we define $R_{[s, F]}$ for all $F \in [2^\omega]^{<\omega}$ at once (by definition, $[s, F]$ is the function which restricted to $\nu - 1$ is equal to s and which has the value F in the point $\nu - 1$). Since, by assumption, $|A_s| \leq 2^\omega$, we may list A_s as $\{p_\xi^s\colon \xi < \beta_s\}$ for some $\beta_s \leq 2^\omega$. Put $\tilde{R}_s = \{x \in R_s\colon x > \alpha$ for each $\alpha \in A_s\}$. For each $x \in \tilde{R}_s$ we can find $p_{\xi_1}^s, p_{\xi_2}^s, \ldots, p_{\xi_n}^s \in A_s$ such that $C_x \subset (\bigcup_{1 \leq i \leq n} U_{p_{\xi_i}^s})^-$ (by maximality of A_s). Define a function $\phi_s\colon \tilde{R}_s \to [2^\omega]^{<\omega}$ in such a way that $C_x \subset (\bigcup\{U_{p_\xi^s}\colon \xi \in \phi_s(x)\})^-$ for each $x \in \tilde{R}_s$. For each $F \in [2^\omega]^{<\omega}$ define $R_{[s,F]}$ by

$$R_{[s,F]} = \{x \in \tilde{R}_s\colon \phi_s(x) = F\}.$$

We claim that for some $s_0 \in S_{\omega_1}$ we have that $R_{s_0} \neq \varnothing$. Notice that

$$\left| \bigcup \{A_s\colon \text{length}\, s < \omega_1\} \right| \leq \sum_{\nu<\omega_1} \sum_{s \in S_\nu} |A_s| \leq \sum_{\nu<\omega_1} (2^\omega)^\nu \cdot 2^\omega = 2^\omega < \kappa;$$

hence we may choose a point $y \in \kappa$ such that $y > \alpha$ for each $\alpha \in \bigcup \{A_s\colon \text{length}\, s < \omega_1\}$. Put $S(y) = \{s \in \bigcup S_\nu\colon y \in R_s\}$ (notice that $S(y) \neq \varnothing$). Let $s_0 \in S(y)$ be such that $s_0 \neq s \restriction \text{dom}\, s_0$ for any $s \in S(y) - \{s_0\}$. We claim that length $s_0 = \omega_1$. If length $s_0 < \omega_1$, then $y \in \tilde{R}_{s_0}$ by definition of y. Consequently, $y \in R_{[s_0, \phi_{s_0}(y)]}$, which implies that $[s_0, \phi_{s_0}(y)] \in S(y)$, contradicting the maximality of s_0.

For each $\xi < \omega_1$ let

$$F_\xi = \left\{ p_\alpha^{s_0 \restriction \xi}\colon \alpha \in s_0(\xi + 1) \right\} \subset A_{s_0 \restriction \xi}.$$

Notice that if $\xi < \eta < \omega_1$ then

(4) $\max F_\xi < \min F_\eta$, and

(5) if $x \in F_\eta$ then $C_x \subset (\bigcup_{y \in F_\xi} U_y)^-$.

For each $\beta < \omega_1$ define

$$V_\beta = \text{int}\left(\bigcup_{\gamma \geq \beta} \bigcup_{x \in F_\gamma} C_x \right)^-.$$

Since X is ccc, and since each V_β is regular open, there is a $\beta < \omega_1$ such that $V_\eta = V_\beta$ for all $\eta \geq \beta$. Since $\bigcup_{\gamma \geq \beta+2} \bigcup_{x \in F_\gamma} C_x \subset (\bigcup_{y \in F_{\beta+1}} U_y)^-$, we conclude that $V_\beta \subset \text{int}(\bigcup_{y \in F_{\beta+1}} U_y)^-$, and consequently $\bigcup_{x \in F_\beta} \text{int}\, C_x \subset \text{int}(\bigcup_{y \in F_{\beta+1}} U_y)^-$. Let $\alpha = \min F_\beta$ and $F_{\beta+1} = \{\gamma_1, \ldots, \gamma_n\}$, where $\gamma_i < \gamma_j$ whenever $i < j$. Since $\alpha < \gamma_1$, by (3), $\text{int}\, C_\alpha \not\subset U_{\gamma_1}$, and since U_{γ_1} is regular open, $W_1 = \text{int}\, C_\alpha - \overline{U}_{\gamma_1} \neq \varnothing$. Since $W_1 \in \mathcal{E}_{\gamma_2}$, again by (3), $W_1 \not\subset U_{\gamma_2}$ and by the same argument, $W_2 = W_1 - \overline{U}_{\gamma_2} \neq \varnothing$. Proceeding in this way we find that $\text{int}\, C_\alpha - (\overline{U}_{\gamma_1} \cup \cdots \cup \overline{U}_{\gamma_n}) \neq \varnothing$, which obviously is a contradiction. □

Let A be as in Lemma 6.2 and for each $\alpha \in A$ put $B_\alpha^0 = C_\alpha$ and $B_\alpha^1 = X - U_\alpha$. Observe that $\{(B_\alpha^0, B_\alpha^1)\colon \alpha \in A\}$ has the property that whenever $\alpha, \beta \in A$ are distinct, $\text{int}\, B_\alpha^i \cap \text{int}\, B_\beta^j \neq \varnothing$, for any $i, j \in 2$ with $i \neq j$. Clearly $|\{B_\alpha^i\colon \alpha \in A\}| = \kappa$ for each $i \in 2$.

By two trivial applications of $\kappa \to (\kappa, \omega_1)^2$ we find that there is a subset $A_0 \in [A]^\kappa$ such that whenever $\alpha, \beta \in A_0$ are distinct and $i \in 2$, $\text{int}\, B_\alpha^i \cap \text{int}\, B_\beta^i \neq \varnothing$. (Put $F = \{\{\alpha, \beta\} \in [A]^2\colon \text{int}\, B_\alpha^0 \cap \text{int}\, B_\beta^0 \neq \varnothing\}$ and $G = [A]^2 - F$. Since X is ccc, there is a set $\tilde{A} \in [A]^\kappa$ with $[\tilde{A}]^2 \subset F$, etc.)

We conclude that the family $\{(B_\alpha^0, B_\alpha^1)\colon \alpha \in A_0\}$ is strongly 2-independent. In the remaining part of this section we will not only need that such a family exists but also how it was constructed.

6.3. Lemma. *For each $2 \leq n < \omega$ there is a set $F_n \in [A_0]^\kappa$ such that the family $\{(B_\alpha^0, B_\alpha^1)\colon \alpha \in F_n\}$ is strongly n-independent.*

Proof. Put $F_2 = A_0$ and assume that we have found F_{n-1} for certain $n > 2$. We will construct F_n by a ramification argument similar to the one used in the proof of Lemma 6.2.

Claim. If $B \in [F_{n-1}]^\kappa$ and if $f: \{1,2,\ldots,n\} \to 2$ then there is a set $\tilde{B} \in [B]^\kappa$ such that whenever $\gamma_1 < \gamma_2 < \cdots < \gamma_n$ and each $\gamma_i \in \tilde{B}$, then $\operatorname{int} \bigcap_{1 \leq i \leq n} B_{\gamma_i}^{f(i)} \neq \varnothing$. For each $\nu \leq \omega_1$ put $S_\nu = \{s: \nu \to [2^\omega]^{n-1}\}$. Let $R_0 = B$ and let $A_s \subset R_s$ be a maximal subset such that whenever $\gamma_1 < \gamma_2 < \cdots < \gamma_n$ and each $\gamma_i \in A_s$, then $\operatorname{int} \bigcap_{1 \leq i \leq n} B_{\gamma_i}^{f(i)} \neq \varnothing$ (for each $s \in \bigcup\{S_\nu: \nu \leq \omega_1\}$ for which R_s is defined). While constructing the sets R_s and A_s we will assume that each A_s has cardinality at most 2^ω. We will derive a contradiction.

Let $\nu \leq \omega_1$ be an ordinal such that R_s has been defined for all s of length $< \nu$. We define R_s for each $s \in S_\nu$ as follows:

Case 1. If ν is a limit and $s \in S_\nu$ then $R_s = \bigcap_{\eta < \nu} R_{s \restriction \eta}$.

Case 2. If ν is a successor, and $s \in S_{\nu-1}$, then we define $R_{[s,F]}$ for all $F \in [B]^{n-1}$ at once. Let $\tilde{R}_s = \{x \in R_s: x > \sup A_s\}$. If $\tilde{R}_s = \varnothing$ then we define $R_{[s,F]} = \varnothing$ for all $F \in [B]^{n-1}$. So, let us assume that $\tilde{R}_s \neq \varnothing$. If $|A_s| < n-2$, then A_s is not maximal since we can add each point of $\tilde{R}_s$ to A_s. Hence $|A_s| \geq n-1$. Since by assumption $|A_s| \leq 2^\omega$, we can list A_s as $\{p_\xi^s: \xi < 2^\omega\}$. Since $|A_s| \geq n-1$ for each $x \in \tilde{R}_s$ we can find $p_{\xi_1}^s, p_{\xi_2}^s, \ldots, P_{\xi_{n-1}}^s \in A_s$ with $p_{\xi_1}^s < p_{\xi_2}^s < \cdots < p_{\xi_{n-1}}^s$ such that $\operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{p_{\xi_i}^s}^{f(i)} \cap B_x^{f(n)} = \varnothing$. Define a function $\phi_s: \tilde{R}_s \to [2^\omega]^{n-1}$ in such a way that whenever $\phi_s(x) = \{\gamma_1, \gamma_2, \ldots, \gamma_{n-1}\}$ with $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-1}$ then $\operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{\gamma_i}^{f(i)} \cap B_x^{f(n)} = \varnothing$. For each $F \in [2^\omega]^{n-1}$ define $R_{[s,F]}$ by

$$R_{[s,F]} = \{x \in \tilde{R}_s: \phi_s(x) = F\}.$$

As in the proof of Lemma 6.2, $|\bigcup\{A_s: \text{length } s < \omega_1\}| \leq 2^\omega < \kappa$ and consequently we can take $y \in \kappa$ such that $y > \sup \bigcup\{A_s: \text{length } s < \omega_1\}$. Put $S(y) = \{s \in \bigcup S_\nu: y \in R_s\}$. Since $y > \sup A_s$ for each s of length $< \omega_1$, by using precisely the same technique as in Lemma 6.2, we find that for some $s_0 \in S(y)$ we have that length $s_0 = \omega_1$.

Now put, for each $\xi < \omega_1$,

$$F_\xi = \{p_\alpha^{s_0 \restriction \xi}: \alpha \in s_0(\xi+1)\} \subset A_{s_0 \restriction \xi}.$$

Let $F_\xi = \{\gamma_1^\xi, \ldots, \gamma_{n-1}^\xi\}$ with $\gamma_1^\xi < \gamma_2^\xi < \cdots < \gamma_{n-1}^\xi$. Notice that if $\xi < \eta < \omega_1$ then

(6) $\max F_\xi < \min F_\eta$, and

(7) if $x \in F_\eta$ then $\operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{\gamma_i^\xi}^{f(i)} \cap B_x^{f(n)} = \varnothing$.

In order to find a contradiction, we have to distinguish several cases.

Case 1. $f(i) = 0$ for each $1 \leq i \leq n-1$ and $f(n) = 1$.

Since $\{(B_\alpha^0, B_\alpha^1): \alpha \in F_{n-1}\}$ is strongly $(n-1)$-independent, we have that $\operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{\gamma_i^0}^0 \neq \varnothing$. Consequently, $\operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{\gamma_i^0}^0 \in \mathfrak{S}_{\gamma_1^1}$ and consequently, by (3), $\operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{\gamma_i^0}^0 \cap B_{\gamma_1^1}^1 \neq \varnothing$, contradiction.

Case 2. $f(i) = 1$ for each $1 \leq i \leq n-1$ and $f(n) = 0$.

By Lemma 6.2, $B_{\gamma_1^1}^0 \not\subset (\bigcup_{1 \leq i \leq n-1} U_{\gamma_i^0})^-$, or, equivalently, $\operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{\gamma_i^0}^1 \cap B_{\gamma_1^1}^0 \neq \varnothing$, contradiction (precisely because of this we had to include Lemma 6.2).

Case 3. Not Case 1 and not Case 2.

For each $\xi < \omega_1$ put $H_\xi = \operatorname{int} \bigcap_{1 \leq i \leq n-1} B_{\gamma_i^\xi}^{f(i)}$. Since $\{(B_\alpha^0, B_\alpha^1): \alpha \in F_{n-1}\}$ is strongly $(n-1)$-independent we have that $H_\xi \neq \varnothing$ for each $\xi < \omega_1$. Obviously, $H_\xi \cap H_\eta = \varnothing$ if $\xi < \eta < \omega_1$. This contradicts X being ccc.

Since there are only finitely many $f\colon \{1,2,\dots,n\} \to 2$, by using the claim, it is now easy to find F_n from F_{n-1}. $\square$

6.4. REMARK. Lemmas 6.2 and 6.3 were proved by a similar ramification technique. We could have proved Lemmas 6.2 and 6.3 also at the same time using only one ramification instead of two. The proof is then heavily obscured by technical and notational difficulties. For readability, we have therefore chosen the above approach.

6.5. REMARK. It is easily seen that our results can also be proved for higher cardinals.

We have completed the proof of the following

6.6. THEOREM. *Let X be a ccc space of weight greater than 2^ω. Then for each $1 \leqslant n < \omega$ there is a strongly n-independent family in X of cardinality $(2^\omega)^+$.*

Notes for §6. All results in this section are new.

The ramificaion technique used in the proof of Lemmas 6.2 and 6.3 is similar to the one used in Juhász [**J**, A4.8].

I originally proved Theorem 6.6 for spaces of weight greater than 2^{2^ω} using the partition relation $(2^\alpha)^+ \to (\alpha^+)^2_\alpha$ for $\alpha = 2^\omega$. In addition, I knew how to prove Theorem 6.6 from the following:

$(*)$ If X is ccc and has weight greater than 2^ω, then there is a strongly 2-independent family in X of cardinality $(2^\omega)^+$.

I asked Charley Mills whether $(*)$ is true and he showed me that my ideas could easily be used to prove $(*)$ from the relation $(2^\omega)^+ \to ((2^\omega)^+, \omega_1)$. Combining this result with ours gave a proof of Theorem 6.6. The proof was rather complicated and later I found the easier proof presented here, in which we use an idea in Charley's proof of $(*)$ which had not occurred to us and which turned out to be very useful.

7. The nowhere of weight $\leqslant 2^\omega$ case: Part 2. In this section we show that if X is a nonpseudocompact space which is nowhere of weight $\leqslant 2^\omega$, then X^* contains a point which is a weak P-point of βX. We use the results from §§5 and 6.

We first begin with an application of Theorem 6.6 to the theory of Čech-Stone compactifications.

7.1. THEOREM. *Let X be the sum of countably many compact ccc spaces of weight greater than 2^ω. Then X^* can be mapped onto $I^{(2^\omega)^+}$.*

PROOF. Let $X = \Sigma_{n<\omega} X_n$. By Theorem 6.6 for each $1 \leqslant n < \omega$ we can find an n-independent family $\{(B^0_\alpha(n), B^1_\alpha(n))\colon \alpha < (2^\omega)^+\}$ in X_n. For each $\alpha < (2^\omega)^+$ put

$$F^i_\alpha = \bigcup_{1 \leqslant n < \omega} B^i_\alpha(n) \quad \text{and} \quad G^i_\alpha = \left(\mathrm{cl}_{\beta X} F^i_\alpha\right) - F^i_\alpha.$$

It is easily seen that the family $\{(G^0_\alpha, G^1_\alpha)\colon \alpha < (2^\omega)^+\}$ is independent, i.e. $G^0_\alpha \cap G^1_\alpha = \varnothing$ for each $\alpha < (2^\omega)^+$ and whenever $F \subset (2^\omega)^+$ is finite and $f\colon F \to 2$, then $\bigcap_{\alpha \in F} G^{f(\alpha)}_\alpha \neq \varnothing$. This implies, as is well known, that some closed subset of X^* maps onto $2^{(2^\omega)^+}$ and since this space maps onto $I^{(2^\omega)^+}$, by the Tietze extension theorem, our claim follows. $\square$

We now come to the main result in this section.

7.2. THEOREM. *Let X be a nonpseudocompact space which is nowhere of weight $\leq 2^\omega$. Then βX has a weak P-point which is a point of X^*.*

PROOF. By beginning this proof in the same way as the proof of Theorem 5.3 we see that it suffices to prove the theorem for locally compact and σ-compact X. For each $n < \omega$ take a compact nonempty regular closed set $V_n \subset X$ such that

(a) if $n \neq m$ then $V_n \cap V_m = \varnothing$, and

(b) for any $E \subset \omega$, $\bigcup_{n \in E} V_n$ is closed in X.

Case 1. There is a set $E \in [\omega]^\omega$ such that for each $n \in E$ there is a point $x_n \in \operatorname{int} V_n$ which has a ccc neighborhood in X.

So, without loss of generality, each V_n is ccc. Let $W_n \subset \operatorname{int} V_n$ be any nonempty regular closed set and let $\{E_n: n < \omega\}$ be a partition of ω in countably many infinite sets. Put

$$X_n = \bigcup_{i \in E_n} V_i \quad \text{and} \quad Y_n = \bigcup_{i \in E_n} W_n.$$

Notice that, by normality of X, $\operatorname{cl}_{\beta X} X_n = \beta X_n$, for each $n < \omega$. By Theorem 7.1, for each $n < \omega$, there is a continuous surjection f_n: $Y_n^* \to I^{(2^\omega)^+}$ and by the Tietze extension theorem we can extend this map to a map $\bar{f}_n$: $\beta X_n \to I^{(2^\omega)^+}$. Since $I^{(2^\omega)^+}$ is ccc and nowhere separable, by Lemma 4.1, there is a nice filter $\mathcal{F}$ on $\Sigma_{n<\omega} Y_n^*$ such that for any countable $D \subset \Sigma_{n<\omega} \beta X_n$ there is some $F \in \mathcal{F}$ which misses the closure of D in $\Sigma_{n<\omega} \beta X_n$. By Theorem 4.3 there is a weak P-point $x \in (\Sigma_{n<\omega} Y_n^*)^*$ such that $x \in \bigcap_{F \in \mathcal{F}} \operatorname{cl}_{\beta(\Sigma_{n<\omega} Y_n^*)} F$. Since X^* is an F-space, by Lemma 4.2, $x \in X^*$. We claim that x is as required. Let $H \subset \beta X - \{x\}$ be countable. Since, by Lemma 2.1, $(\bigcup_{n<\omega} X_n)^*$ is a P-set of X^*, $x \notin \overline{H}_0$, where $H_0 = H \cap (X^* - (\bigcup_{n<\omega} X_n)^*)$. So, without loss of generality, $H \subset X \cup (\bigcup_{n<\omega} X_n)^*$. Since $F = X - \bigcup_{n<\omega} \operatorname{int} V_n$ and $\bigcup_{n<\omega} Y_n$ have disjoint closures in βX and since $x \in (\bigcup_{n<\omega} Y_n)^-$, we may assume that, without loss of generality,

$$H \subset \bigcup_{n<\omega} \operatorname{int} V_n \cup \left(\bigcup_{n<\omega} X_n \right)^*.$$

Take $F \in \mathcal{F}$ which misses the closure of $H \cap \bigcup_{n<\omega} \operatorname{int} V_n$ in $\Sigma_{n<\omega} \beta X_n$. Since $\Sigma_{n<\omega} \beta X_n$ is normal, being σ-compact, and clearly C^*-embedded in βX we conclude that

$$\overline{F} \cap \left(H \cap \bigcup_{n<\omega} \operatorname{int} V_n \right)^- = \varnothing.$$

Hence, we may assume that $H \cap X = \varnothing$. Since, by Lemma 4.1, $(\bigcup_{n<\omega} Y_n)^*$ is a P-set of $(\bigcup_{n<\omega} X_n)^*$ and since, by construction, x is a weak P-point of $(\bigcup_{n<\omega} Y_n)^*$, we conclude that $x \notin (H \cap (\bigcup_{n<\omega} X_n)^*)^-$, i.e. $x \notin \overline{H}$.

Case 2. All but finitely many V_n are nowhere ccc.

Now use the same technique as in the proof of Theorem 5.3. □

7.3. REMARK. Notice that we have used Theorem 5.3 to prove Theorem 7.2 and that for the actual construction of the weak P-point we used the same "ccc nowhere separable technique" twice: first to find an appropriate nice filter, and second to extend this nice filter to a weak P-point.

7.4. REMARK. If we generalize the results in §6 to higher cardinals, using the same technique as in Theorem 7.1 we get the following result:

If X is a nonpseudocompact space of cellularity at most κ which is nowhere of weight less than or equal to 2^κ, then X^* contains a compact subset which can be mapped onto $I^{(2^\kappa)^+}$.

Notes for §7. All results in this section are new.

8. A result under $BF(2^\omega)$. Let X be the sum of countably many nonempty compact spaces, say X_n ($n < \omega$). If each X_n has weight at most 2^ω and, in addition, satisfies the countable chain condition, then, under $BF(2^\omega)$, X has a nice filter $\mathcal{F}$ such that whenever $D \subset X$ is nowhere dense, then $D \cap F = \varnothing$ for some $F \in \mathcal{F}$.

If $f, g \in \omega^\omega$ then we write, as usual, $g \leqslant_* f$ iff $\{n < \omega: g(n) > f(n)\}$ is finite. A subset $G \subset \omega^\omega$ is called *bounded* if there is an $f \in \omega^\omega$ such that $g \leqslant_* f$ for any $g \in G$. By $BF(2^\omega)$ we mean the statement that each subset of ω^ω of cardinality less than 2^ω is bounded. $BF(2^\omega)$ is known to be consistent with the usual axioms of set theory and follows easily from MA.

It is easy to see that $BF(2^\omega)$ follows from CH, the Continuum Hypothesis, since obviously no countable subset of ω^ω is unbounded. Therefore, the reader not familiar with MA or $BF(2^\omega)$ can simply assume CH in this section.

We now come to the main result in this section.

8.1. THEOREM [$BF(2^\omega)$]. *Let $X = \Sigma_{n<\omega} X_n$, here each X_n is a compact ccc space of weight at most 2^ω. Then there is a nice filter $\mathcal{F}$ on X such that for each nowhere dense $D \subset X$ there is an $F \in \mathcal{F}$ which misses D.*

PROOF. For each $n < \omega$ let $\{\{A_m^\alpha(n): m < \omega\}: \alpha < 2^\omega\}$ enumerate all families of pairwise disjoint nonempty regular closed subsets of X_n, the union of which is dense. Notice that there are only 2^ω such families. Let $\mathfrak{D}$ be the collection of nowhere dense subsets of X. We obviously may assume that we have indexed the $A_m^\alpha(n)$'s in such a way that for all $D \in \mathfrak{D}$ there is an $\alpha < 2^\omega$ such that $\bar{D} \cap \bigcup_{n<\omega} \bigcup_{m<\omega} A_m^\alpha(n) = \varnothing$.

We plan to choose, for each $\alpha < 2^\omega$, a function h_α: $\omega \to \omega$. We then define our filter $\mathcal{F}$ to be generated by the collection $\{\bigcup_{n<\omega} \bigcup_{j\leqslant h_\alpha(n)} A_j^\alpha(n): \alpha < 2^\omega\}$. Observe that this collection consists of closed sets. So the idea is to select the h_α's to ensure that the filter is nice.

Let $h_0(n) = n$ for each $n < \omega$. Suppose we have defined h_γ for each $\gamma < \alpha < 2^\omega$ such that for any finite sequence $\gamma_1 < \gamma_2 < \cdots < \gamma_k < \alpha$, there is an $N < \omega$ such that for each $n \geqslant N$,

$$\bigcap_{1\leqslant i\leqslant k} \bigcup_{j\leqslant h_{\gamma_i}(n)} A_j^{\gamma_i}(n) \neq \varnothing .$$

This is the condition we require to ensure we get a nice filter.

Let us now construct h_α. For each $E \in [\alpha]^{<\omega}$ we define a function g_E as follows. Let E be the sequence $\gamma_1 < \gamma_2 < \cdots < \gamma_k$.

$$g_E(n) = \begin{cases} 0 \ \text{ if } \bigcap_{1\leqslant i\leqslant k} \bigcup_{j\leqslant h_{\gamma_i}(n)} A_j^{\gamma_i}(n) = \varnothing, \\ \min\left\{p < \omega: A_p^\alpha(n) \cap \bigcap_{1\leqslant i\leqslant k} \bigcup_{j\leqslant h_{\gamma_i}(n)} A_j^{\gamma_i}(n) \neq \varnothing\right\} \quad \text{otherwise.} \end{cases}$$

Since $|\{g_E\colon E \in [\alpha]^{<\omega}\}| \leq |[\alpha]^{<\omega}| < 2^\omega$ we may choose, by $BF(2^\omega)$, a function $f \in \omega^\omega$ such that for each $E \in [\alpha]^{<\omega}$ the set $\{n < \omega\colon g_E(n) > f(n)\}$ is finite. Define $h_\alpha = f$. We claim that h_α is as required. Let $E = \{\gamma_i\colon 1 \leq i \leq k, \gamma_i < \alpha\}$ be some finite subset of α. By induction hypothesis, there is an $N < \omega$ such that

$$\bigcap_{1 \leq i \leq k} \bigcup_{j \leq h_{\gamma_i}(n)} A_j^{\gamma_i}(n) \neq \varnothing$$

for each $n \geq N$. Therefore, for each $n \geq N$,

$$A^\alpha_{g_E(n)}(n) \cap \bigcap_{1 \leq i \leq k} \bigcup_{j \leq h_{\gamma_i}(n)} A_j^{\gamma_i}(n) \neq \varnothing .$$

By definition of h_α, there is an $N_1 < \omega$ such that $h_\alpha(n) \geq g_E(n)$ for each $n \geq N_1$. Therefore, for $n \geq \max\{N, N_1\}$,

$$\bigcup_{j \leq h_\alpha(n)} A_j^\alpha(n) \cap \bigcap_{1 \leq i \leq k} \bigcup_{j \leq h_{\gamma_i}(n)} A_j^{\gamma_i}(n) \neq \varnothing .$$

This completes the induction. □

Notes for §8. The interesting Theorem 8.1 is due to Dow [**D**].

9. Nonhomogeneity of βX and X^*. Let X be any nonpseudocompact space. If we assume $BF(2^\omega)$, then there is a point $x \in X^*$ such that $x \notin \overline{D}$ for each countable and nowhere dense $D \subset \beta X - \{x\}$.

In this last section we present the main result in this paper.

9.1. THEOREM [$BF(2^\omega)$]. *Let X be any nonpseudocompact space. There is a point $x \in X^*$ such that $x \notin \overline{D}$ for each countable and nowhere dense $D \subset \beta X - \{x\}$.*

PROOF. It is clear that we only need to show the result for locally compact and σ-compact X (cf. the proof of Theorem 5.3). For each $n < \omega$ take a compact nonempty regular closed set $V_n \subset X$ such that:

(a) if $n \neq m$ then $V_n \cap V_m = \varnothing$, and

(b) for any $E \subset \omega$, $\bigcup_{n \in E} V_n$ is closed in X.

Case 1. There is a set $E \in [\omega]^\omega$ such that for each $n \in E$ there is a point $x_n \in \operatorname{int} V_n$ which has a ccc neighborhood in X.

So, without loss of generality, each V_n is ccc. If countably many of the V_n's have weight greater than 2^ω, then apply a similar technique as in the proof of Theorem 7.2. It then follows that X^* contains a point which is even a weak P-point of βX. If not, then without loss of generality, each V_n has weight at most 2^ω. Let $\mathfrak{F}$ be a nice filter on $\bigcup_{n<\omega} V_n$ as described in Theorem 8.1. Clearly $\mathfrak{F}$ "avoids" all nowhere dense subsets of X. Now apply Theorem 4.3 on the filter $\mathfrak{F}$.

Case 2. Not Case 1.

Then without loss of generality, each V_n is nowhere ccc. Now apply the same technique as in the proof of Theorem 5.3. □

9.2. REMARK. Notice that we did not only use Theorem 8.1 in the proof of the above result, but also Theorems 5.3 and 7.2.

Notes for §9. Theorem 9.1 is new.

10. Remarks. A point in X^* can be identified with an ultrafilter of zero-sets on X. Hence if we wish to construct a point in X^* we must consider collections of closed sets in X with the finite intersection property. It turns out that the finite intersection property is much too complicated. It is easier to consider collections of sets with the property that any n of them intersect, where $1 \leqslant n < \omega$, i.e. n-linked collections. Sometimes it is possible to construct large n-linked families with a certain property for each $1 \leqslant n < \omega$ although it is not possible to construct a similar family with the finite intersection property. For example, if $X = [0, 1]$ then for each $1 \leqslant n < \omega$ there is an n-linked family of closed sets $\mathcal{F}_n$ "avoiding" all nowhere dense sets in $[0, 1]$ [**CS**, **vD**$_3$], while obviously a centered collection with this property does not exist. Yet, the existence of such collections easily implies that on the space $\omega \times X$ there is a collection of closed sets with the finite intersection property avoiding all nowhere dense subsets of $\omega \times X$; simply put

$$\mathcal{F} = \{F \subset \omega \times X: F \text{ is closed and } F \cap (\{n\} \times X) \in \mathcal{F}_n \text{ for each } 1 \leqslant n < \omega\}.$$

There is much evidence that this way of constructing filters, i.e. by considering n-linked families for each $1 \leqslant n < \omega$, is implicit in the proofs of several recent results in the last years; see e.g. [**CS**, **D**, **DvM**, **vD**$_1$, **vD**$_3$, **vDvM**$_2$, **vM**$_1$–**vM**$_3$, **K**$_2$] and this paper. I think this tells us an important fact. It was precisely because of this that I found Theorem 6.6.

By a method of proof similar to the one used in the proof of Theorem 7.2 the reader can easily show that Theorem 9.1 is true in ZFC under the following hypothesis:

(*) *If* $X = \Sigma_{n<\omega} X_n$, *where each* X_n *is a compact ccc space of weight at most* 2^ω, *then there is a nice filter* $\mathcal{F}$ *on* X *such that whenever* $D \subset X$ *is countable and nowhere dense, then* $\overline{D} \cap F = \varnothing$ *for some* $F \in \mathcal{F}$.

It is hard to guess whether (*) is true in ZFC. On the one hand, one feels that since X is "small" one could easily run into a set theoretic problem, while on the other hand countable nowhere dense sets are "very small" and therefore are easy to avoid, which might indicate that a theorem could be possible. I do not know what to guess.

10.1. *Question. Is* (*) *true in ZFC*?

Remarks added in May 1981. Professor I. Juhász has kindly informed me that the results in §6 of the present paper can also be derived from a result of Šapirovskii; see, e.g., I. Juhász, *Cardinal functions in topology–ten years later*, Math. Centre Tracts **123** (1980), Corollary 3.20. Professor M. Hušek has kindly informed me that the results in §6 also follow from one of his theorems; see M. Hušek, *Convergence versus character in compact spaces*, Colloq. Math. Soc. János Bolyai, vol. 23, Topology (Budapest, 1978), Corollary 4, pp. 647–651.

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