Arch. Math., Vol. 40, 159–169 (1983)

0003-889X/83/4002-0159 $ 3.70/0

An almost fixed point theorem for metrizable continua

By

JAN VAN MILL

0. Introduction. Let X be a topological space. A collection $\mathscr{A}$ of closed subsets of X has the *n-intersection property* ($n > 1$) provided that any subfamily of $\mathscr{A}$ of n elements has nonempty intersection. Whenever a system $\mathscr{A}$ of closed sets is maximal with respect to the n-intersection property we say that $\mathscr{A}$ is a *maximal n-centered system*. The motivation for this paper arose from the observation that if X is a metrizable continuum and if $f\colon X \to X$ is continuous then there is a maximal 2-centered system $\mathscr{M}$ so that $\mathscr{M} = \{A \in 2^X : f^{-1}(A) \in \mathscr{M}\}$ (see [4, Th. 4] and [9, II 4.5]) (M. van de Vel has recently given an elementary, though nontrivial, proof of a similar assertion for arbitrary continua; see [8], and also [7]). This result suggests the question whether one can obtain a similar result for maximal n-centered systems with $n > 2$. Obviously we cannot expect the same result for systems of closed sets maximal with respect to the finite intersection property, for if $\mathscr{F}$ is such a system and $\mathscr{F} = \{A \in 2^X : f^{-1}(A) \in \mathscr{F}\}$ then $\bigcap \mathscr{F}$ is a fixed point of f. Unfortunately the same method of proof as in the maximal 2-centered system case does not work for $n > 2$. In fact, we do not know whether the result can be generalized but we can prove that for each $n > 2$ there is a maximal n-centered system $\mathscr{A}$ so that $\mathscr{A}$ and $\{A \in 2^X : f^{-1}(A) \in \mathscr{A}\}$ are as "close" as we please. To make this explicit we first have to define a notion of closeness for systems of closed sets. There is a very natural way to do this. It turns out, see Section 1, that each maximal n-centered system $\mathscr{A}$ in X is closed when regarded to be a subset of the hyperspace 2^X of X. Hence such a system is a point of 2^{2^X}. In the same way $\bar{f}(\mathscr{A}) = \{A \in 2^X : f^{-1}(A) \in \mathscr{A}\}$ is a point of 2^{2^X}. Since 2^{2^X} can be metrized, in a natural way, by the Hausdorff distance it is natural to say that $\mathscr{A}$ and $\bar{f}(\mathscr{A})$ have distance ε in case $\mathscr{A}$ and $\bar{f}(\mathscr{A})$ have distance ε in 2^{2^X}. Let us say that a maximal n-centered system $\mathscr{A}$ is finitely generated whenever there is a finite subset $F \subset X$ so that $\{A \cap F : A \in \mathscr{A}\}$ is n-centered. Our main result is that if X is a metric continuum and if $f\colon X \to X$ is continuous then for each $\varepsilon > 0$ and $n \geqq 2$ there is a finitely generated maximal n-centered system $\mathscr{A}$ in X so that $\mathscr{A}$ and $\bar{f}(\mathscr{A})$ have distance less than ε in 2^{2^X}. By an example we show that for finitely generated maximal n-centered systems this result is best possible.

I am indebted to Marcel van de Vel for some helpful suggestions.

1. Preliminaries. Let (X, d) be a compact metric space. Put

$$U_\varepsilon(A) = \{x \in X : d(x, A) < \varepsilon\},$$
$$B_\varepsilon(A) = \{x \in X : d(x, A) \leqq \varepsilon\}$$

for all $\varepsilon \geqq 0$ and $A \subset X$.

The *hyperspace* 2^X of X is the space of all nonempty closed subsets of X topologized by the Hausdorff distance d_H, i.e.

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset U_\varepsilon(B) \ \&\ B \subset U_\varepsilon(A)\}.$$

The hyperspace of 2^X is metrized by d_{HH} which is denoted, for simplicity, by ϱ. For information concerning hyperspaces see Nadler [3].

1.1. Lemma. *Let $n \geqq 2$ and let $\mathscr{A}$ be a maximal n-centered system in X. Then $\mathscr{A} \subset 2^X$ is closed.*

Proof. Suppose that $B \notin \mathscr{A}$. By maximality of $\mathscr{A}$ there is a subfamily $\mathscr{E}$ of $\mathscr{A}$ of cardinality $n - 1$ so that $B \cap \bigcap \mathscr{E} = \emptyset$. Take $\varepsilon > 0$ so that

$$U_\varepsilon(B) \cap \bigcap \mathscr{E} = \emptyset.$$

Then $\{D \in 2^X : d_H(D, B) < \varepsilon\}$ is a neighborhood of B in 2^X which misses $\mathscr{A}$. □

If $\mathscr{A} \subset 2^X$ put $\mathscr{A}^\dagger = \{B \in 2^X : \exists A \in \mathscr{A} \text{ with } A \subset B\}$. The proof of the following simple lemma is left to the reader.

1.2. Lemma. *Let $\mathscr{F}, \mathscr{G} \subset 2^X$ be closed so that $\mathscr{F} = \mathscr{F}^\dagger$ and $\mathscr{G} = \mathscr{G}^\dagger$. Then*

$$\varrho(\mathscr{F}, \mathscr{G}) = \inf\{\varepsilon > 0 : (\forall F \in \mathscr{F} : B_\varepsilon(F) \in \mathscr{G}) \ \&\ (\forall G \in \mathscr{G} : B_\varepsilon(G) \in \mathscr{F})\}. \quad \square$$

Define an operator $F \colon 2^{2^X} \to 2^{2^X}$ by $F(\mathscr{A}) = \mathscr{A}^\dagger$.

1.3. Lemma. *F is well defined and continuous.*

Proof. Suppose that $B \notin \mathscr{A}^\dagger$. For each $A \in \mathscr{A}$ let $U(A)$ be an open set in X so that $A \cap U(A) \neq \emptyset$ and $U(A)^- \cap B = \emptyset$. By the compactness of $\mathscr{A}$ there is a finite subcollection $\mathscr{E} \subset \mathscr{A}$ so that for each $A \in \mathscr{A}$ we have that $A \cap U(E) \neq \emptyset$ for some $E \in \mathscr{E}$. Then

$$\{C \in 2^X : C \cap U(E)^- = \emptyset \text{ for each } E \in \mathscr{E}\}$$

is an open neighborhood of B which misses $\mathscr{A}^\dagger$. Hence $\mathscr{A}^\dagger$ is closed.

The continuity of F is an easy exercise which is left to the reader (use Lemma 1.2). □

1.4. Corollary. $\{\mathscr{A} \in 2^{2^X} : \mathscr{A} = \mathscr{A}^\dagger\}$ *is closed in* 2^{2^X}. □

2. Finitely generated maximal *n*-centered systems. Let X be a compact metric space. For each $n \geqq 2$ define

$$\mathscr{L}_n(X) = \{\mathscr{A} \in 2^{2^X} : \mathscr{A} \text{ is } n\text{-centered and } \mathscr{A} = \mathscr{A}^\dagger\}.$$

2.1. Lemma. *$\mathscr{L}_n(X)$ is closed in 2^{2^X}.*

This Lemma easily follows from Corollary 1.4.

Define a function $\mu: \mathscr{L}_n(X)^{n+1} \to \mathscr{L}_n(X)$ by

$$\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) = \{B \in 2^X : \exists i \leqq n+1 \text{ so that } B \in \mathscr{A}_j \text{ for all } j \neq i\}.$$

It is clear that μ is well-defined.

2.2. Lemma. *μ is continuous.*

Proof. Take $\mathscr{A}_i \in \mathscr{L}_n(X)$ $(i \leqq n+1)$ and let $\mathscr{E}_m^i$ $(m \in \mathbb{N})$ be a sequence of points in $\mathscr{L}_n(X)$ converging to $\mathscr{A}_i$ $(i \leqq n+1)$. Choose $\varepsilon > 0$ and let $m_0 \in \mathbb{N}$ be so that

$$\varrho(\mathscr{E}_m^i, \mathscr{A}_i) < \varepsilon$$

for all $m \geqq m_0$ and $i \leqq n+1$. Fix $m \geqq m_0$ and take $E \in \mu(\mathscr{E}_m^1, \ldots, \mathscr{E}_m^{n+1})$. Then there is an $i \leqq n+1$ so that $E \in \mathscr{E}_m^j$ for all $j \neq i$. Since $\varrho(\mathscr{E}_m^j, \mathscr{A}_j) < \varepsilon$, by Lemma 1.2, it follows that $B_\varepsilon(E) \in \mathscr{A}_j$ for all $j \neq i$ and consequently

$$B_\varepsilon(E) \in \mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}).$$

In the same way one shows that if

$$A \in \mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) \quad \text{then} \quad B_\varepsilon(A) \in \mu(\mathscr{E}_m^1, \ldots, \mathscr{E}_m^{n+1}).$$

By Lemma 1.2 we conclude that

$$\varrho(\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}), \mu(\mathscr{E}_m^1, \ldots, \mathscr{E}_m^{n+1})) \leqq \varepsilon,$$

which proves continuity of μ. $\square$

Notice that μ is *symmetric*, i.e.

$$\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) = \mu(\mathscr{A}_{\pi(1)}, \ldots, \mathscr{A}_{\pi(n+1)})$$

for each permutation π of $\{1, 2, \ldots, n+1\}$. Also $\mu(\mathscr{A}, \ldots, \mathscr{A}) = \mathscr{A}$ for each $\mathscr{A} \in \mathscr{L}_n(X)$. Hence μ is a $(n+1)$-*mean* in the sense of Aumann [1].

The following Lemma is a trivial though fundamental observation.

2.3. Lemma. *Take $\mathscr{A}, \mathscr{B} \in \mathscr{L}_n(X)$. Then $\mu(\mathscr{A}, \mathscr{A}, \ldots, \mathscr{A}, \mathscr{B}) = \mathscr{A}$.*

It is precisely this observation which makes our construction work.

As noted in the introduction, a maximal n-centered system $\mathscr{A}$ in X is called *finitely generated* if there is a finite set $F \subset X$ so that

$$\{A \cap F : A \in \mathscr{A}\}$$

is n-centered. There are many maximal n-centered systems $\mathscr{A}$ in X which are finitely generated. For example, let $F \subset X$ be a set of precisely $n+1$ points. Then

$$\{A \in 2^X : |F - A| \leqq 1\}$$

is a finitely generated maximal n-centered system in X.

Let $\wedge_n(X) = \{\mathscr{A} \in \mathscr{L}_n(X) : \mathscr{A}$ is a finitely generated maximal n-centered system$\}$. If $\mathscr{A} \in \wedge_n(X)$ then a finite set $F \subset X$ is called a *center* for $\mathscr{A}$ provided that $\{A \cap F : A \in \mathscr{A}\}$ is n-centered.

2.4. Lemma. *Let* $\mathscr{A} \in \wedge_n(X)$.

(i) *If* F *is a center for* $\mathscr{A}$ *then* $\{A \cap F : A \in \mathscr{A}\} \subset \mathscr{A}$.

(ii) *If* F *and* G *are centers for* $\mathscr{A}$ *then so is* $F \cap G$.

Proof. For (i), take $A \in \mathscr{A}$ so that $A \cap F \notin \mathscr{A}$. By maximality of $\mathscr{A}$ there is a subfamily $\mathscr{E}$ of $\mathscr{A}$ of cardinality $n-1$ so that

$$(A \cap F) \cap \bigcap \mathscr{E} = \emptyset.$$

However, this contradicts the fact that F is a center for $\mathscr{A}$.

Now assume that both F and G are centers for $\mathscr{A}$. By (i) $\{A \cap F : A \in \mathscr{A}\} \subset \mathscr{A}$. Hence, also by (i), $\{(A \cap F) \cap G : A \in \mathscr{A}\} \subset \mathscr{A}$. This proves that

$$\{A \cap (F \cap G) : A \in \mathscr{A}\}$$

is n-centered, i.e. $F \cap G$ is a center for $\mathscr{A}$. □

We will show that if X is a continuum and if $\overline{\wedge}_n(X)$ denotes the closure of $\wedge_n(X)$ is 2^{2^X} then $\mu[\overline{\wedge}_n(X)^{n+1}] = \overline{\wedge}_n(X)$, where μ is defined above. We first need a simple Lemma.

2.5. Lemma. *Let* $\mathscr{A}_i \in \wedge_n(X)$ *and let* F_i *be a center for* $\mathscr{A}_i$ $(i \leqq n+1)$. *If* $\{F_1, \ldots, F_{n+1}\}$ *is pairwise disjoint then* $\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) \in \wedge_n(X)$.

Proof. We will first show that $\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1})$ is a maximal n-centered system. Suppose not, then there is some $B \in 2^X$ so that

$$\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) \cup \{B\}$$

is n-centered but $B \notin \mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1})$. Without loss of generality $B \notin \mathscr{A}_1$ and $B \notin \mathscr{A}_2$. Take $E_i^1 \in \mathscr{A}_1$ and $E_i^2 \in \mathscr{A}_2$ $(i \leqq n-1)$ so that

$$\bigcap_{i=1}^{n-1} E_i^1 \cap B = \emptyset \quad \text{and} \quad \bigcap_{i=1}^{n-1} E_i^2 \cap B = \emptyset.$$

Lemma 2.4(i) implies that we may assume that $E_i^1 \subset F_1$ and $E_i^2 \subset F_2$ for each $i \leqq n-1$. Define

$$G_i = E_i^1 \cup E_i^2 \cup \bigcup \{F_j : 3 \leqq j \leqq n+1 \text{ and } j \neq i+2\}$$

for all $i \leqq n-1$. Clearly $G_i \in \mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1})$. Since $\{F_1, \ldots, F_{n+1}\}$ is pairwise disjoint,

$$\bigcap_{i=1}^{n-1} G_i \cap B = \bigcap_{i=1}^{n-1} (E_i^1 \cup E_i^2) \cap B = \emptyset,$$

which contradicts the fact that $\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) \cup \{B\}$ is n-centered.

Hence $\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1})$ is maximal. It is clear that $\bigcup_{i=1}^{n+1} F_i$ is a center for

$$\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}), \quad \text{hence} \quad \mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) \in \wedge_n(X). \qquad \square$$

2.6. Corollary. *Let X be a continuum. Then $\mu[\overline{\lambda}_n(X)^{n+1}] = \overline{\lambda}_n(X)$.*

Proof. Clearly $\overline{\lambda}_n(X) \subset \mu[\overline{\lambda}_n(X)^{n+1}]$. Choose $\varepsilon > 0$ and let $\delta > 0$ be so that if $\varrho(\mathscr{A}_i, \mathscr{A}_i') < \delta$ for each $i \leqq n+1$ then

$$\varrho(\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}),\ \mu(\mathscr{A}_1', \ldots, \mathscr{A}_{n+1}')) < \varepsilon .$$

Suppose that $\mathscr{L} = \mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1})$ where $\mathscr{A}_i \in \overline{\lambda}_n(X)$ for each $i \leqq n+1$. For each $i \leqq n+1$ take $\mathscr{E}_i \in \lambda_n(X)$ so that $\varrho(\mathscr{E}_i, \mathscr{A}_i) < \frac{1}{2}\delta$. In addition, let E_i be a center for $\mathscr{E}_i$. By induction we will construct for each $i \leqq n+1$ a point $\mathscr{E}_i' \in \lambda_n(X)$ with center E_i' such that

(a) $\varrho(\mathscr{E}_i, \mathscr{E}_i') < \frac{1}{2}\delta$ for all $i \leqq n+1$;

(b) $E_i' \cap \bigcup\limits_{j<i} E_j' = \emptyset$.

Define $\mathscr{E}_1' = \mathscr{E}_1$ and $E_1' = E_1$. Suppose that $\mathscr{E}_j'$ and E_j' are defined for all $j < i \leqq n+1$. Suppose that $E_i = \{e_1, \ldots, e_m\}$ where $e_k \neq e_l$ if $k \neq l$. Choose $0 < \delta_0 < \frac{1}{2}\delta$ so that the family

$$\{U_{\delta_0}(e_k) : k \leqq m\}$$

is pairwise disjoint. For each $k \leqq m$ take a point $y_k \in U_{\delta_0}(e_k) - \bigcup\limits_{j<i} E_j'$. Define

$$\mathscr{F} = \{F \subset \{y_1, \ldots, y_m\} : \{e_k : y_k \in F\} \in \mathscr{E}_i\}$$

and

$$\mathscr{F}' = \{A \in 2^X : \exists F \in \mathscr{F} \text{ with } F \subset A\}.$$

It is clear that $\mathscr{F}' \in \lambda_n(X)$ and that $\{y_1, \ldots, y_m\}$ is a center for $\mathscr{F}'$. Define $\mathscr{E}_i' = \mathscr{F}'$ and $E_i' = \{y_1, \ldots, y_m\}$. It is clear that our inductive hypotheses are satisfied.

Notice that $\varrho(\mathscr{A}_i, \mathscr{E}_i') < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta$ so that

$$\varrho(\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}),\ \mu(\mathscr{E}_1', \ldots, \mathscr{E}_{n+1}')) < \varepsilon .$$

Since $\{E_i' : i \leqq n+1\}$ is pairwise disjoint, $\mu(\mathscr{E}_1', \ldots, \mathscr{E}_{n+1}') \in \lambda_n(X)$ (Lemma 2.5). We conclude that $\varrho(\mathscr{L}, \lambda_n(X)) < \varepsilon$. Therefore $\mathscr{L} \in \overline{\lambda}_n(X)$. $\square$

3. Some topological properties of $\overline{\lambda}_n(X)$. In this section we give some topological properties of $\overline{\lambda}_n(X)$ which are of crucial importance throughout the remaining part of this paper. Our main result is that $\overline{\lambda}_n(X)$ is connected if X is. The proof uses a similar trick as in [5, Th. 2.1].

Throughout the remaining part of this section X is a compact connected metric space and μ is defined as in Section 2.

A subset $A \subset \overline{\lambda}_n(X)$ is called *convex* if for all $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \in A$ and $\mathscr{B} \in \overline{\lambda}_n(X)$ we have that

$$\mu(\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n, \mathscr{B}) \in A .$$

For each $A \in 2^X$ put $A^+ = \{\mathscr{A} \in \overline{\lambda}_n(X) : A \in \mathscr{A}\}$.

3.1. Lemma. *If $A \in 2^X$ then A^+ is convex.*

Proof. Take $\mathscr{A}_1, \ldots, \mathscr{A}_n \in A^+$ and $\mathscr{B} \in \overline{\lambda}_n(X)$. Since

$$\{C \in 2^X : C \in \mathscr{A}_i \text{ for all } i \leqq n\} \subset \mu(\mathscr{A}_1, \ldots, \mathscr{A}_n, \mathscr{B})$$

by definition of μ we conclude that $A \in \mu(\mathscr{A}_1, \ldots, \mathscr{A}_n, \mathscr{B})$, or, equivalently,

$$\mu(\mathscr{A}_1, \ldots, \mathscr{A}_n, \mathscr{B}) \in A^+. \quad \square$$

3.2. Lemma. *If $\mathscr{F} \in \lambda_n(X)$ and if G is a center for $\mathscr{F}$ then*

$$\{\mathscr{F}\} = \bigcap \{A^+ : A \in \mathscr{A}\}, \quad \textit{where} \quad \mathscr{A} = \{F \cap G : F \in \mathscr{F}\}.$$

Proof. Suppose not. Take $\mathscr{B} \in \bigcap \{A^+ : A \in \mathscr{A}\} - \{\mathscr{F}\}$. Then $\mathscr{B} \neq \mathscr{F}$ and consequently there is a $B \in \mathscr{B}$ such that $B \notin \mathscr{F}$. By maximality of $\mathscr{F}$ there are $A_1, A_2, \ldots, A_{n-1} \in \mathscr{A}$ (Lemma 2.4 (i)) so that $B \cap A_1 \cap \cdots \cap A_{n-1} = \emptyset$. Since $\mathscr{B}$ is n-centered, by Lemma 2.1, there must be an $i \leqq n-1$ so that $A_i \notin \mathscr{B}$, or, equivalently, $\mathscr{B} \notin A_i^+$. Contradiction. $\square$

Define an embedding $\varphi\colon X \to 2^{2^X}$ by $\varphi(x) = \{A \in 2^X : x \in A\}$. Notice that $\varphi[X] \subset \lambda_n(X)$. Now inductively define subspaces Z_i ($i \in \mathbb{N}$) of $\overline{\lambda}_n(X)$ as follows:

$$Z_1 = \varphi[X], \quad Z_{i+1} = \mu[Z_i^{n+1}].$$

Notice that $Z_i \subset Z_{i+1} \subset \overline{\lambda}_n(X)$ for all $i \in \mathbb{N}$ (Corollary 2.6). Put $Z = \bigcup_{i=1}^{\infty} Z_i$.

3.3. Lemma. *Let $\mathscr{A}$ be a finite n-centered system of closed subsets of X. Then*

$$\bigcap \{A^+ : A \in \mathscr{A}\} \cap Z \neq \emptyset.$$

Proof. We will prove this by induction on $|\mathscr{A}|$. If $|\mathscr{A}| \leqq n$ then there is nothing to prove. Suppose that the statement is true for all n-centered systems $\mathscr{A}$ of closed subsets of X of cardinality at most i, where $n \leqq i$. Let $\{A_1, \ldots, A_{i+1}\} \subset 2^X$ be n-centered. By induction hypothesis, for each $j \leqq n+1$ there is a point

$$\mathscr{A}_j \in \bigcap \{A_k^+ : k \leqq i+1 \;\&\; k \neq j\} \cap Z.$$

Take $l \in \mathbb{N}$ so that $\{\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}\} \subset Z_l$. Since $|A_k^+ \cap \{\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}\}| \geqq n$ by the convexity of the sets A_k^+ (Lemma 3.1) it follows that

$$\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) \in \bigcap_{k=1}^{i+1} A_k^+.$$

By definition, $\mu(\mathscr{A}_1, \ldots, \mathscr{A}_{n+1}) \in Z_{l+1} \subset Z$. We conclude that

$$\bigcap_{k=1}^{i+1} A_k^+ \cap Z \neq \emptyset. \quad \square$$

We can now prove the main result in this section.

3.4. Theorem. *Let X be a metric continuum. Then $\overline{\lambda}_n(X)$ is connected.*

Proof. Define Z as above. Since Z_1 is connected, by the continuity of μ it follows that Z_i is connected for all $i \in \mathbb{N}$, hence Z is connected. We claim that $\lambda_n(X) \subset Z$ which shows that $\overline{\lambda}_n(X)$ is connected since $\lambda_n(X)$ is dense in $\overline{\lambda}_n(X)$.

Take $\mathscr{A} \in \overline{\lambda}_n(X)$ and let F be a center for $\mathscr{A}$. Put $\mathscr{E} = \{A \cap F : A \in \mathscr{A}\}$. Then $\mathscr{E}$ is n-centered and since $\{\mathscr{A}\} = \bigcap \{E^+ : E \in \mathscr{E}\}$ (Lemma 3.2), by Lemma 3.3 it follows that $\mathscr{A} \in Z$. $\square$

For $n = 2$ this result is known, see [9, III.4.1].

We now turn to compactness properties of $\overline{\lambda}_n(X)$.

The reader might wonder why we work with $\overline{\lambda}_n(X)$ and not with $\lambda_n(X)$, i.e. the space with underlying set the set of all maximal n-centered systems topologized by regarding it to be a subspace of 2^{2^X}. The following lemma explains this.

3.5. Lemma. *Let X be a metric compactum with at least one non-isolated point. Then $\lambda_n(X)$ is compact iff $n = 2$.*

Proof. Take $\mathscr{A}_n \in \lambda_2(X)$ $(n \in \mathbb{N})$ so that $\mathscr{A}_n \to \mathscr{A}$. It is clear that $\mathscr{A}$ is 2-centered. Suppose that $\mathscr{A} \notin \lambda_2(X)$. Take $D \in 2^X$ so that $\mathscr{A} \cup \{D\}$ is 2-centered while $D \notin \mathscr{A}$. There is an $\varepsilon > 0$ so that $B_\varepsilon(D) \notin \mathscr{A}$. Since $\mathscr{A}_n \to \mathscr{A}$ there is an $m \in \mathbb{N}$ such that $B_\varepsilon(D) \notin \mathscr{A}_n$ for all $n \geqq m$. The maximality of $\mathscr{A}_n$ now implies that

$$E = X - U_\varepsilon(D) \in \mathscr{A}_n \quad (n \geqq m).$$

Since $\mathscr{A}_n \to \mathscr{A}$ we conclude that $E \in \mathscr{A}$. This is a contradiction however since $E \cap D = \emptyset$.

Let x be a non-isolated point of X and let x_n $(n \in \mathbb{N})$ be a sequence converging to x. We assume that $x_i = x_j$ iff $i = j$. We will only show that $\lambda_3(X)$ is not compact. The proof that $\lambda_n(X)$ is not compact for all $n \geqq 3$ is similar. Define

$$\mathscr{A}_n = \{A \in 2^X : |A \cap \{x, x_{n+2}, x_1, x_2\}| \geqq 3\}.$$

Then $\mathscr{A}_n \in \lambda_3(X)$ and the sequence $\{\mathscr{A}_n\}_n$ converges in 2^{2^X} to $\mathscr{A}$, where

$$\mathscr{A} = \{A \in 2^X : \{x, x_1\} \subset A \vee \{x, x_2\} \subset A\}.$$

It is clear that $\mathscr{A} \notin \lambda_3(X)$ since $\mathscr{A} \cup \{\{x\}\}$ is 3-centered while $\{x\} \notin \mathscr{A}$. This shows that $\lambda_3(X)$ is not compact. $\square$

4. Mixers. A map $\mu\colon X^{n+1} \to X$ is called an *n-mixer* provided that

$$\mu(x, x, \ldots, x, y) = \mu(x, x, \ldots, x, y, x) = \mu(x, x, \ldots, x, y, x, x) = \cdots = x$$

for all $x, y \in X$. This concept, for $n = 2$, is due to van Mill and van de Vel [6]. Notice that the function $\mu\colon \overline{\lambda}_n(X)^{n+1} \to \overline{\lambda}_n(X)$ described in Section 2 is an n-mixer.

4.1. Lemma. *Let X be a metric continuum with an n-mixer μ. Then X is locally connected.*

Proof. Let $U \subset X$ be open and let $K \subset U$ be a component. We will show that K is open. Take $x \in K$. Then

$$(\{x\} \times \{x\} \times \cdots \times \{x\} \times X) \cup (\{x\} \times \{x\} \times \cdots \times \{x\} \times X \times \{x\}) \\ \cup \cdots \subset \mu^{-1}[U]$$

and by the compactness of X there is a neighborhood V of x so that

$$E(V) = (V \times V \times \cdots \times V \times X) \cup (V \times V \times \cdots \times V \times X \times V) \cup \cdots \subset \mu^{-1}[U].$$

By the connectedness of X it follows that $E(V)$ is connected. Consequently, $x \in V \subset \mu[E(V)] \subset K$. Hence K is open. $\square$

4.2. Remark. The technique of proof in Lemma 4.1 is the same as in [6, 1.1].

Let μ be an n-mixer. A subset $A \subset X$ is called *μ-convex* if for all $x_1, x_2, \ldots, x_{n+1} \in A$ and for each permutation π of $\{1, 2, \ldots, n+1\}$ it is true that

$$\mu(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n+1)}) \in A.$$

We use the standard representations

$$S^n = \left\{(x_0, \ldots, x_n) \in R^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\right\},$$

$$B^{n+1} = \left\{(x_0, \ldots, x_n) \in R^{n+1} : \sum_{i=0}^{n} x_i^2 \leqq 1\right\}.$$

The following Lemma is inspired by van Mill and van de Vel [6, Th. 1.3].

4.3. Lemma. *Let X be a compact metric space and let $\mu\colon X^{n+1} \to X$ be an n-mixer. Then for each μ-convex set $A \subset X$ and $i \geqq 1$ and mapping $f\colon S^i \to A$ there is a map $\bar{f}\colon B^{i+1} \to A$ which extends f.*

Proof. Before we prove the Lemma we first verify the following usefull Fact (compare [6, Lemma 1.2]).

Fact. If $x_i^1, x_i^2, \ldots, x_i^n, y_i$ $(i \in \mathbb{N})$ are points of X such that the sequences $(x_i^j)_{i \in \mathbb{N}}$ $(j \leqq n)$ all converge to $a \in X$, then for each permutation π of $\{1, 2, \ldots, n\}$ and $j \leqq n$ the sequence

$$(\mu(x_i^{\pi(1)}, x_i^{\pi(2)}, \ldots, x_i^{\pi(j)}, y_i, x_i^{\pi(j+1)}, \ldots, x_i^{\pi(n)}))_{i \in \mathbb{N}}$$

converges to a.

Let U be a neighborhood of a. As in the proof of Lemma 4.1 we can find a neighborhood V of a such that

$$E(V) = (V \times V \times \cdots \times V \times X) \cup (V \times V \times \cdots \times V \times X \times V) \cup \cdots \subset \mu^{-1}[U].$$

Let $i_0 \in \mathbb{N}$ be such that $x_i^j \in V$ for all $i \geqq i_0$ and $j \leqq n$. Now, if $i \geqq i_0$ the point

$$z_i = (x_i^{\pi(1)}, x_i^{\pi(2)}, \ldots, x_i^{\pi(j)}, y_i, x_i^{\pi(j+1)}, \ldots, x_i^{\pi(n)})$$

belongs to $E(V)$, whence $\mu(z_i) \in U$.

Now let us proceed to the proof of the Lemma. Suppose that $A \subset X$ is μ-convex and let $f\colon S^i \to A$ be given $(i \geqq 1)$. Take $u_1, \ldots, u_{n+1} \in S^i$ so that $u_i = u_j$ iff $i = j$.

Define, for each $j \leqq n+1$ a function $g_j\colon B^{i+1} \to S^i$ by

$$\begin{cases} g_j(v) = v \quad (v \in S^i), \\ g_j(v) = \text{the unique point of } S^i - \{u_j\} \text{ which lies on the straight line} \\ \qquad\quad \text{through } v \text{ and } u_j \ (v \notin S^i). \end{cases}$$

This leads us to a function

$$g = (g_1, \ldots, g_{n+1}) : B^{i+1} \to (S^i)^{n+1}.$$

Notice that this function is not continuous. Define $\bar{f}\colon B^{i+1} \to X$ as the composition

$$B^{i+1} \xrightarrow{g} (S^i)^{n+1} \to X^{n+1} \xrightarrow{\mu} X,$$

where the map in the middle is $(f, f, \ldots, f)$. Then $\bar{f}$ clearly extends f. Also, by a straightforward application of the Fact, $\bar{f}$ is continuous. Finally, since A is μ-convex, $\bar{f}[B^{i+1}] \subset A$. $\square$

We now come to the main result in this section.

4.4. Theorem. *Let X be a metrizable continuum and let $\mu\colon X^{n+1} \to X$ be an n-mixer. If X has a basis of μ-convex sets then X is an Absolute Retract.*

Proof. Let d be a metric for X and let $\varepsilon > 0$. Let $\mathscr{U}$ be a finite cover of X by μ-convex sets so that $\{\operatorname{int} U : U \in \mathscr{U}\}$ covers X and each $U \in \mathscr{U}$ has diameter at most ε. Let $\mathscr{V}$ be a finite open star refinement of $\{\operatorname{int} U : U \in \mathscr{U}\}$. Define

$$\mathscr{K} = \{K \subset X : (\exists V \in \mathscr{V} : K \text{ is a component of } V)\}.$$

By Lemma 4.1 each $K \in \mathscr{K}$ is open and since $\bigcup \mathscr{K} = X$ there is a finite subcollection $\mathscr{K}' \subset \mathscr{K}$ so that $\bigcup \mathscr{K}' = X$. Let λ be a Lebesgue number for $\mathscr{K}'$.

Now let P be a compact polyhedron and let $P_0 \subset P$ be a subpolyhedron containing all the vertices of P and let $f\colon P_0 \to X$ be continuous so that the partial image of f of any simplex of P has diameter less than λ. Since, by Lemma 4.1, each $K \in \mathscr{K}'$ is a locally compact locally connected and connected metric space we can extend f to a map $g\colon P_1 \to X$ where P_1 is a subpolyhedron of P which contains P_0 and the 1-skeleton of P while moreover the partial image of g of any simplex of P is contained in some $U \in \mathscr{U}$. Using Lemma 4.3 and the fact that each intersection of μ-convex sets is again μ-convex we can extend g to a map $\bar{f}\colon P \to X$ so that for each simplex $\sigma \subset P$ there is a $U \in \mathscr{U}$ with $\bar{f}[\sigma] \subset U$.

By a well known result of Lefschetz [2] it follows that X is an ANR. Since X is Peano continuum (Lemma 4.1), Lemma 4.3 implies that X is C^∞. However, a C^∞ ANR is an AR. $\square$

4.5. Corollary. *Let X be a metrizable continuum. Then $\overline{\curlywedge}_n(X)$ is an Absolute Retract.*

Proof. By Theorem 3.4, $\overline{\curlywedge}_n(X)$ is connected. Therefore, by Theorem 4.2, we only need to show that the n-mixer μ for $\overline{\curlywedge}_n(X)$ described in Section 2 is stable in the sense that there is a basis for $\overline{\curlywedge}_n(X)$ consisting of μ-convex sets. Let d be a metric for X. As in Section 1, ϱ is the induced metric on 2^{2^X}. Take $\mathscr{L} \in \overline{\wedge}_n(X)$ and $\varepsilon > 0$. We claim that

$$U = \{\mathscr{L}' \in \overline{\wedge}_n(X) : \varrho(\mathscr{L}, \mathscr{L}') < \varepsilon\}$$

is μ-convex, which suffices to prove the Corollary.

Take $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_{n+1} \in U$ and let π: $\{1, 2, \ldots, n+1\} \to \{1, 2, \ldots, n+1\}$ be a permutation. Let $\delta = \max\{\varrho(\mathscr{L}_i, \mathscr{L}) : 1 \leqq i \leqq n+1\}$. Then $\delta < \varepsilon$. Take $L \in \mathscr{L}$. Then $B_\delta(L) \in \mathscr{L}_i$ for all $1 \leqq i \leqq n+1$ and consequently

$$B_\delta(L) \in \mu(\mathscr{L}_{\pi(1)}, \mathscr{L}_{\pi(2)}, \ldots, \mathscr{L}_{\pi(n+1)}).$$

Also, take $E \in \mu(\mathscr{L}_{\pi(1)}, \mathscr{L}_{\pi(2)}, \ldots, \mathscr{L}_{\pi(n+1)})$. There is an index $i \leqq n+1$ so that $E \in \mathscr{L}_{\pi(i)}$. Since

$$\varrho(\mathscr{L}, \mathscr{L}_{\pi(i)}) \leqq \delta$$

we conclude that $B_\delta(E) \in \mathscr{L}$. This implies that

$$\varrho(\mathscr{L}, \mu(\mathscr{L}_{\pi(1)}, \mathscr{L}_{\pi(2)}, \ldots, \mathscr{L}_{\pi(n+1)})) \leqq \delta,$$

i.e. $\mu(\mathscr{L}_{\pi(1)}, \mathscr{L}_{\pi(2)}, \ldots, \mathscr{L}_{\pi(n+1)}) \in U$. $\square$

5. Proof of the main result. We now can prove the main result of this p`per.

5.1. Theorem. *Let X be a metrizable continuum and let f: $X \to X$ be continuous. Then for each $n \geqq 2$ and $\varepsilon > 0$ there is a maximal n-centered system $\mathscr{A}$ of closed subsets of X so that $\varrho(\mathscr{A}, \{B \in 2^X : f^{-1}(B) \in \mathscr{A}\}) < \varepsilon$. In addition, $\mathscr{A}$ can be taken to be finitely generated.*

Proof. Define $\bar{f}$: $\overline{\wedge}_n(X) \to \overline{\wedge}_n(X)$ by

$$\bar{f}(\mathscr{A}) = \{B \in 2^X : f^{-1}(B) \in \mathscr{A}\}.$$

A straightforward check shows that $\bar{f}$ is continuous. By Corollary 4.5, $\overline{\wedge}_n(X)$ is an AR and therefore, as is well known, has the fixed point property. Let $\mathscr{A} \in \overline{\wedge}_n(X)$ be a fixed point of $\bar{f}$. Since $\wedge_n(X)$ is dense in $\overline{\wedge}_n(X)$ we can find $\mathscr{L} \in \wedge_n(X)$ so that $\mathscr{L}$ and $\bar{f}(\mathscr{L})$ are as close as we please. $\square$

This result, for finitely generated maximal n-centered systems, is best possible. Indeed, let T: $S^1 \to S^1$ be a translation through an irrational angle. It is clear that for each finite $F \subset S^1$ there is a $k \geqq 1$ so that $T^k[F] \cap F = \emptyset$. It is now routine to check that $\bar{T}: \overline{\wedge}_n(S^1) \to \overline{\wedge}_n(S^1)$, where $\bar{T}$ is defined as in the proof of Theorem 5.1, has no fixed point belonging to $\wedge_n(S^1)$.

References

[1] G. AUMANN, Über Räume mit Mittelbildungen. Math. Ann. **119**, 210–215 (1943).
[2] S. LEFSCHETZ, On locally-connected and related sets. Ann. Math. **35**, 118–129 (1934).
[3] S. B. NADLER, Hyperspaces of sets. New York 1978.
[4] J. VAN MILL, The superextension of the closed unit interval is homeomorphic to the Hilbert cube. Fund. Math. **CIII**, 151–175 (1979).
[5] J. VAN MILL and A. SCHRIJVER, Subbase characterizations of compact topological spaces. Gen. Top. Appl. **10**, 183–201 (1979).

[6] J. van Mill and M. van de Vel, On an internal property of Absolute Retracts. Top. Proc. **4**, 193–200 (1979).
[7] M. van de Vel, Superextensions and Lefschetz fixed point structures. Fund. Math. CIV, 33–48 (1979).
[8] M. van de Vel, Some elementary proofs in fixed point theory. Proc. Kon. Ned. Akad. Wet. **83**, 437–444 (1980).
[9] A. Verbeek, Superextensions of topological spaces. MC tract **41**, Amsterdam 1972.

Eingegangen am 23. 4. 1980

Anschrift des Autors:

Jan van Mill
Department of Mathematics
Vrije Universiteit
De Boelelaan 1081
Amsterdam
The Netherlands