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CLOSED IMAGES OF TOPOLOGICAL GROUPS

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ABSTRACT

A consequence of a result of Uspenskiĭ is that every space is an open image of a homogeneous space. This suggests the question whether every space is a closed image of a homogeneous space. We will show that every separable metrizable space is a closed image of a separable metrizable topological group. In addition we show that there are homogeneous Borel subsets of **R** that are not a perfect image of a topological group.

0. INTRODUCTION

Recently, Uspenskiĭ [12] showed that for every space X there is a homogeneous space Y such that $X \times Y$ is homeomorphic to Y. Among other things, this shows that every space is an open image of a homogeneous space. This suggests the question whether every space is a closed image of a homogeneous space. This question seems relevant not only by Uspenskiĭ's result but also by the fact that every topologically complete separable metrizable space is a closed image of the space of irrational numbers (which is a topological group), [3]. We will show that *every* separable metrizable space is a closed image of a separable metrizable topological group. It is easy to see that not every space is a perfect image of a homogeneous space. An example of such a space is the topological sum of the rationals Q and a one point space. The question naturally arises whether every *homogeneous* space is a perfect image of a topological group. It is easy to see that every homogeneous topologically complete separable metrizable space is a perfect image of one of the following topological groups: $\mathbf{Z}$ (the integers), C (the Cantor set), $\mathbf{Z} \times C$ and $\mathbf{P}$ (the irrational numbers). We will show that there are homogeneous Borel subsets of $\mathbf{R}$ that are not a perfect image of a topological group. This seems delicate since if X and A are separable and metrizable and if $f: X \to A$ is closed then Vaĭnšteĭn's Lemma ([13]; [4], 4.4.16) implies that $f^{-1}(a)$ is compact for all but countably many $a \in A$.

1. CLOSED IMAGES OF TOPOLOGICAL GROUPS

In this section we will show that every separable and metrizable space is a closed image of a topological group. In addition, we will present some classes of spaces which are even a perfect image of a topological group.

We will find it convenient to identity C with $\{0, 1\}^{\mathbf{N}}$. Observe that C is a topological group under coordinatewise addition (mod 2). The identity of C, namely the point having all coordinates equal to 0, for simplicity will be denoted by 0. Observe that C is *Boolean*, i.e. that $x + x = 0$ for every $x \in C$. The following result is probably well-known.

Lemma 1.1. *Let* $n \in \mathbf{N}$. *The set* $A_n = \{x \in C^n : x_1 + \ldots + x_n = 0\}$ *is closed and nowhere dense in* C^n

Proof. That A_n is closed is clear. Take $x \in A_n$ and let U be a neighborhood of x. Without loss of generality, $U = \prod_{i=1}^{n} U_i$, where U_i is a clopen neighborhood of x_i in C, for all $1 \leqslant i \leqslant n$. Since each U_i depends only on finitely many coordinates, there is an $m \in \mathbf{N}$ such that every point $y \in C^n$ with the property that y_i and x_i $(1 \leqslant i \leqslant n)$ agree on their first m coordinates, belongs to U. Let $y \in C^n$ be such a point for which the $(m + 1)$-th coordinate of y_1 equals 1 and the $(m + 1)$-th coordinates of the other y_i's all equal 0. Then $y \in U$ and clearly $y_1 + \ldots + y_n \neq 0$. ∎

Corollary 1.2. *There is a Cantor set* $K \subseteq C$ *with the following property: if* $x_1, \ldots, x_n \in K$, $n \in \mathbf{N}$, *are distinct then* $x_1 + \ldots + x_n \neq 0$.

Proof. Observing that C is dense in itself, this is an immediate consequence of Lemma 1.1 and Mycielski [6], Theorem 1. ∎

It is possible to give an explicit description of a Cantor set such as in Corollary 1.2. However, the existence proof from Mycielski's theorem is much shorter than this explicit construction (and verification). Our use of Mycielski's theorem to construct an independent Cantor set is not the first construction of this type, see e.g. Mauldin [5] and Pol [11].

Corollary 1.3. *There is a Cantor set* $K \subseteq C$ *with the following property: if* $x_1, \ldots, x_n \in K$, $n > 1$, *are distinct then* $x_1 + \ldots + x_n \notin K$.

Proof. Let K be as in Corollary 1.2. Take distinct $x_1, \ldots, x_n \in K$, $n > 1$. If $x_1 + \ldots + x_n = x_i$ for certain $1 \leqslant i \leqslant n$, then $x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n = 0$, contradicting Corollary 1.2. Define $x_{n+1} = x_1 + \ldots + x_n$. We see that the $x_1, \ldots, x_{n+1}$ are distinct. If $x_{n+1} \in K$ then

$$x_1 + \ldots + x_{n+1} = x_{n+1} + x_{n+1} = 0,$$

contradicting again Corollary 1.2. We conclude that $x_{n+1} \notin K$. ∎

We will now present the main result in this section.

Theorem 1.4. *For every separable metrizable space* X *there exists a subgroup* G *of* C *admitting a closed mapping onto* X.

Proof. First observe that X is a perfect image of a zero-dimensional separable metrizable space ([4], 4.4.J). We can therefore without loss of generality assume that X is zero-dimensional. Let K be the Cantor set of Corollary 1.3. Since X is zero-dimensional, X is homeomorphic to a subspace of K. So we may assume without loss of generality that X is a subspace of K. Let $G \subseteq C$ be the subgroup of C generated by X.

Observe that

$$G = \{x_1 + \ldots + x_n : x_i \in X \text{ for all } i \leqslant n,\ n \in \mathbf{N}\}$$

(use that C is Boolean). We first claim that X is closed in G. This follows if we show that $K \cap G = X$. Indeed, take $x \in K \cap G$. Find $n \in \mathbf{N}$ and $x_i \in X$ for all $1 \leqslant i \leqslant n$ such that $x = x_1 + \ldots + x_n$. If $x_i = x_j$ for certain distinct $i, j \leqslant n$, then $x_i + x_j = 0$, whence we can assume that the $x_1, \ldots, x_n$ are distinct. Since $x \in K$, by Corollary 1.3, $n = 1$. We conclude that $x \in X$.

Engelking [3] showed that if Y is a zero-dimensional separable metrizable space and if $Z \subseteq Y$ is closed then there is a closed retraction from Y onto Z. Since X is closed in G, this result implies that X is a closed image of G. ∎

As observed in the introduction, it is easy to construct spaces that are not a perfect image of a homogeneous space. For a homogeneous space it seems rather difficult to decide whether it is a perfect image of a topological group. In the following proposition we will give some classes of spaces that are a perfect image of a topological group.

Proposition 1.5. *Let X be a separable metrizable space which is either compact, or homogeneous and topologically complete, or homogeneous and σ-compact, or nowhere locally compact and σ-compact. Then X is a perfect image of a topological group.*

Proof. If X is compact, then X is an image of C. Suppose therefore that X is homogeneous and topologically complete. Let Y be a zero-dimensional space admitting a perfect irreducible mapping onto X, say $f: Y \to X$ ([4], 3.1.C). If X has an isolated point then $X \approx \{0\}$ or $X \approx \mathbf{Z}$, so X is then a perfect image of one of the spaces $C, \mathbf{Z}$. If X has no isolated points, then Y is dense in itself. If X is compact then $Y \approx C$. If X is locally compact but not compact then $Y \approx \mathbf{Z} \times C$. If X is not locally compact then X is nowhere locally compact, so Y is nowhere locally compact and topologically complete, whence $Y \approx \mathbf{P}$.

Now assume that X is nowhere locally compact and σ-compact. In van Mill and Woods [10] it was shown that X is a perfect and irreducible image of $Q \times C$, where Q denotes the space of rational numbers.

Finally, assume that X is homogeneous and σ-compact. If X is not locally compact, then X is nowhere locally compact, whence nowhere locally compact and σ-compact, and with such X we dealt already. If X is locally compact, then X is topologically complete, so again we are in a previous case. ∎

Corollary 1.6. *Let X be a homogeneous separable metrizable space which is either topologically complete or σ-compact. Then X is a perfect image of a topological group.* ∎

Let X be a separable metrizable space which is not topologically complete and not σ-compact, and which can be written as $X = A \cup B$, where A is topologically complete and B is σ-compact. In the next section we will show that such an X is *not* a perfect image of a topological group. This shows that, in a sense, Corollary 1.6 is best possible since homogeneous spaces of this type exist.

2. HOMOGENEOUS SPACES AND PERFECT IMAGES OF TOPOLOGICAL GROUPS

In this section we will give various examples of homogeneous spaces which are not a perfect image of a topological group. Our main tool is the following

Theorem 2.1. *Let X be a separable and metrizable space having the following properties:*

(1) *X is not topologically complete and not σ-compact.*

(2) *X can be written as $X = A \cup B$, where A is topologically complete and B is σ-compact.*

Then X is not a perfect image of a topological group.

Proof. We may assume that X is a subset of some compact metrizable space $\widetilde{X}$. Then (2) implies that X is a Borel set of $\widetilde{X}$. It is clear that (1) implies that X is not a G_δ-subset of $\widetilde{X}$. By a well-known result of Hurewicz [4], there is a compact subset $K \subseteq \widetilde{X}$ such that $K \cap X \approx Q$ and $K \setminus A \approx \mathbf{P}$. We conclude that X contains a closed copy of Q, say E. Now assume that there is a topological group G which admits a perfect mapping, say f, onto X. We will derive a contradiction. Observe that $f^{-1}(E \cup B)$ is σ-compact, whence the subgroup G' of G generated by $f^{-1}(E \cup B)$ is also σ-compact. Since X is not σ-compact, $f(G') \neq X$ from which we conclude that $G' \neq G$. Take a point $p \in G' \setminus G$. The translation $g_p\colon G \to G$ defined by $g_p(x) = px$, is a homeomorphism of G such that $F = g_p(f^{-1}(E))$ misses $f^{-1}(B)$. Since E is closed in X. F is closed in G. In addition, since $F \cap f^{-1}(B) = \phi$, we conclude that F can be mapped by a perfect map onto a closed subspace of the complete space A. Therefore, F can be mapped by a perfect map onto a topologically complete space as well as a space which is not topologically complete, namely Q. As easy application of [4], 3.9.10, yields the desired contradiction. ∎

Now let $X = \{\langle x, y\rangle \in \mathbf{R}^2\colon x \in Q \text{ and } y \in \mathbf{P}\}$. It can be shown that X is homogeneous and by Theorem 2.1 it follows that X is not a perfect image of a topological group.

In [1], van Douwen showed that there is a unique (= unique up to homeomorphism) space T having the following properties:

(1) T is a zero-dimensional separable metrizable space,

(2) T is the union of a topologically complete and a countable subspaces,

(3) T is nowhere topologically complete and nowhere σ-compact.

From this characterization it follows that all nonempty clopen subspaces of T are homeomorphic to T, whence the zero-dimensionality of T easily implies that T is homogeneous (in fact, homeomorphisms between closed and nowhere dense subsets of T can be extended to autohomeomorphisms of T, [7]). Consequently, T is another example of a homogeneous space which is not a perfect image of a topological group. The weaker statement that T does not admit the structure of a topological group is due to van Douwen [1]; ideas in his proof of this fact were used in the proof of Theorem 2.1. Since van Douwen's paper has not been written yet, the interested reader should consult van Engelen and van Mill [2] for a (new) proof of the characterization of the space T.

In [8], I showed that there is a unique space S having the following properties:

(1) S is a zero-dimensional separable metrizable space,

(2) S is the union of a topologically complete and a σ-compact subspace,

(3) S is nowhere σ-compact and nowhere the union of a complete and a countable subspace.

The characterization again implies that S has the property that all of its nonempty clopen subspaces are homeomorphic, whence S is homogeneous. By Theorem 2.1 it again follows that S is not a perfect image of a topological group.

In [9], I constructed a partition of the real line $\mathbf{R}$ into two homeomorphic homogeneous subsets which do not admit the structure of a topological group. It can be shown that these spaces also have the property that they are not a perfect image of a topological group. The proof of this, however, is much more complicated than the easy proof of Theorem 2.1. The problem with these spaces is not that they are homogeneous, but that they are not a perfect image of a topological group. Interestingly, S and T behave in a different way. Proving homogeneity is complicated for these spaces, not that they are not a perfect image of a topological group.

Finally, let us pose the following problems:

(1) *Is every space a closed image of a homogeneous space?*

(2) *Is there a topological group which is not a perfect image of an Abelian topological group?*

Since every compact group is dyadic, every compact group is a continuous image of an Abelian compact group. So a space such as in (2) cannot be compact.

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