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BOOLEAN ALGEBRAS AND RAISING MAPS TO ZERO-DIMENSIONAL SPACES

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ABSTRACT. Let X be a separable metric space and let $\mathcal{F}$ be a family of countably many self-maps of X. Then there is a countable subalgebra $\mathcal{B}$ of the Boolean algebra of regular open subsets of X which is a base for X such that for each $f \in \mathcal{F}$ the function $\Phi_f : \mathcal{B} \to \mathcal{B}$ defined by $\Phi_f(B) = (f^{-1}(B))^{-0}$ is a homomorphism.

0. Definitions. All spaces under discussion are separable metric. If X is a space and $A \subset X$, then A^0 (A^- or $\bar{A}$) denotes the interior (the closure) of A in X. A subset $A \subset X$ is called *regular open* provided that $A = A^{-0}$. The collection of all regular open subsets of X is denoted by $\mathrm{RO}(X)$. If $U, V \in \mathrm{RO}(X)$ put

$$U \wedge V = U \cap V,$$

$$U \vee V = (U \cup V)^{-0},$$

and

$$U' = (X - U)^0.$$

It is well-known, and easy to prove, that $\langle \mathrm{RO}(X), \wedge, \vee, 0, 1, ' \rangle$ with $0 = \emptyset$ and $1 = X$ is a complete Boolean algebra, see e.g. [5, §2].

1. **Introduction.** Let X be a space and let $f : X \to X$ be continuous. The function $\Phi_f : \mathrm{RO}(X) \to \mathrm{RO}(X)$ defined by

$$\Phi_f(U) = f^{-1}(U)^{-0},$$

is a very natural operator from $\mathrm{RO}(X)$ into $\mathrm{RO}(X)$. Unfortunately, simple examples show that Φ_f need not be a homomorphism. This suggests the question whether one can always find a subalgebra $\mathcal{B} \subset \mathrm{RO}(X)$ such that $\Phi_f \mid \mathcal{B} : \mathcal{B} \to \mathrm{RO}(X)$ is a homomorphism. Of course, this question has a very simple answer. If $\mathcal{B} = \{\emptyset, X\}$, then $\mathcal{B}$ is as required. Therefore one should ask the question whether one can always find a *large* subalgebra $\mathcal{B} \subset \mathrm{RO}(X)$ such that $\Phi_f \mid \mathcal{B} : \mathcal{B} \to \mathrm{RO}(X)$ is a homomorphism. A large subalgebra of $\mathrm{RO}(X)$ should be dense in $\mathrm{RO}(X)$ (dense as a subset of $\mathrm{RO}(X)$) and therefore we are interested in subalgebras $\mathcal{B} \subset \mathrm{RO}(X)$ which are a π-base, or even a base, for the topology of X.

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If $\mathscr{B} \subset \mathrm{RO}(X)$ is a subalgebra such that $\Phi_f \mid \mathscr{B} : \mathscr{B} \to \mathrm{RO}(X)$ is a homomorphism, then $\Phi_f \mid \Phi_f(\mathscr{B})$ need not be a homomorphism. If a large subalgebra $\mathscr{B} \subset \mathrm{RO}(X)$ exists such that $\Phi_f \mid \mathscr{B}$ is a homomorphism, then one could ask the question whether one can find a subalgebra $\mathscr{E} \subset \mathrm{RO}(X)$ such that $\Phi_f \mid \mathscr{E}, \Phi_f \mid \Phi_f(\mathscr{E}), \Phi_f \mid \Phi_f(\Phi_f(\mathscr{E})), \ldots$, are all homomorphisms. The aim of this paper is to construct such subalgebras.

1.1 THEOREM. *Let X be a space and let $\mathscr{F}$ be a collection of countably many self-maps of X. Then there is a countable subalgebra $\mathscr{B} \subset \mathrm{RO}(X)$ which is a base for X such that*

(1) *$\Phi_f \mid \mathscr{B}$ is a homomorphism for all $f \in \mathscr{F}$;*
(2) *$\Phi_f(\mathscr{B}) \subset \mathscr{B}$ for all $f \in \mathscr{F}$;*
(3) *if $f \in \mathscr{F}$ is onto then $\Phi_f : \mathscr{B} \to \mathscr{B}$ is a (Boolean algebra) embedding;*
(4) *if $f \in \mathscr{F}$ is a homeomorphism then $\Phi_f : \mathscr{B} \to \mathscr{B}$ is an isomorphism.*

To prove this Theorem we construct a countable open basis $\mathscr{U}$ for X such that (1) $\mathscr{U} \subset \mathrm{RO}(X)$, (2) $\mathscr{U}$ is closed under finite unions and finite intersections, (3) if $\mathscr{E} \subset \mathscr{U}$ is finite then $\bigcap_{E \in \mathscr{E}} \bar{E} = (\bigcap \mathscr{E})^-$, (4) if $U \in \mathscr{U}$ and $f \in \mathscr{F}$ then $f^{-1}(\bar{U}) = (f^{-1}(U))^-$, and (5) if $f \in \mathscr{F}$ and $U \in \mathscr{U}$ then $f^{-1}(U) \in \mathscr{U}$. We then let $\mathscr{B}$ be the smallest subalgebra of $\mathrm{RO}(X)$ containing $\mathscr{U}$. The properties of $\mathscr{U}$ are used to prove that $\mathscr{B}$ is as required.

Countable bases are usually constructed by induction using at each step of the induction that only finitely many basis elements have been constructed so far. However, in our situation, if we want to define a certain element of $\mathscr{U}$, say U, we must define it in such a way that we are allowed to add $f^{-n}(U)$ to $\mathscr{U}$ for each $n \in \mathbb{N}$ and $f \in \mathscr{F}$. Consequently, already after the first step of the induction we have a countably infinite collection. This causes some technical problems.

Let X, Y be spaces and let $f: X \to X$ be continuous. The mapping f can be *raised* to Y provided that there exists a continuous surjection $\tau: Y \to X$ and a map $\bar{f}: Y \to Y$ such that $f \circ \tau = \tau \circ \bar{f}$. Anderson [2], showed that any map defined on a compact space X can be raised to a map defined on a compact zero-dimensional space. There is a clever proof of this fact which is due to de Groot and which is simpler than Anderson's argument and which can easily be generalized to the effect that countably many mappings can always be simultaneously raised to a zero-dimensional space, Baayen [3, 3.4.19]. The reason I became interested in Theorem 1.1 is that I tried to understand *why* Anderson's result is true. Obviously, Theorem 1.1 implies Anderson's result. For if Y is the Stone space of $\mathscr{B}$, then Y maps onto X by a natural map τ (X is compact) and the functions $\bar{f}: Y \to Y$ can be defined by

$$\bar{f}(p) = \{B \in \mathscr{B} : \Phi_f(B) \in p\}.$$

For details see Section 3. This gives not only a new proof of Anderson's Theorem but shows also that Y can be chosen in such a way that $\mathrm{RO}(X)$ and $\mathrm{RO}(Y)$ are isomorphic.

2. **Boolean algebras.** In this section we will give the proof of Theorem 1.1. Our results rely on an interesting technique due to Berney [4].

2.1 LEMMA. *Let X be a space and let $A \subset X$ be uncountable. If $<$ is any total ordering on A then there is a point $a \in A$ such that*

$$a \in \{x \in A\colon x < a\}^- \cap \{x \in A\colon a < x\}^-.$$

Proof. If not, then without loss of generality $a \notin \{x \in A\colon x < a\}^-$ for each $a \in A$. Let $\mathcal{U}$ be a countable open basis for X and for each $a \in A$ take $U_a \in \mathcal{U}$ such that $a \in U_a$ and $U_a \cap \{x \in A\colon x < a\} = \emptyset$.

Since A is uncountable, there have to be distinct $a, b \in A$ such that $U_a = U_b$, but this is impossible, since if, e.g. $a < b$ then $a \notin U_b$. □

2.2 REMARK. This Lemma is well-known and the proof is only included for completeness sake. The fact that this Lemma is very useful in constructing special bases, was first observed by Berney [4].

2.3 LEMMA. *Let X be a space and let $\mathcal{F}$ be a countable collection of maps from X into $[0, 1]$. In addition, let $B \subset X$ be open. If $\mathcal{G} \subset \mathcal{F}$ is finite, then $\{\delta \in (0, 1)\colon \bigcap_{f \in \mathcal{G}} (f^{-1}[0, \delta))^- \cap \bar{B} \neq (\bigcap_{f \in \mathcal{G}} f^{-1}[0, \delta) \cap B)^-\}$ is countable.*

Proof. We induct on the cardinality of $\mathcal{G}$. If $|\mathcal{G}| = 0$, then there is nothing to prove. Therefore assume the lemma to be true for all $\mathcal{G} \subset \mathcal{F}$ of cardinality n and let $\mathcal{G} \subset \mathcal{F}$ have cardinality $n+1$. Assume that the set

$$A = \left\{\delta \in (0, 1)\colon \bigcap_{f \in \mathcal{G}} (f^{-1}[0, \delta))^- \cap \bar{B} \neq \left(\bigcap_{f \in \mathcal{G}} f^{-1}[0, \delta) \cap B\right)^-\right\}$$

is uncountable and for each $\delta \in A$ take a point

$$a(\delta) \in \left(\bigcap_{f \in \mathcal{G}} (f^{-1}[0, \delta))^- \cap \bar{B}\right) - \left(\bigcap_{f \in \mathcal{G}} f^{-1}[0, \delta) \cap B\right)^-.$$

Claim. Take $f_0 \in \mathcal{G}$. Then $\{\delta \in A\colon f_0(a(\delta)) \neq \delta\}$ is countable.

Suppose, to the contrary, that $\{\delta \in A\colon f_0(a(\delta)) \neq \delta\} = A_0$ is uncountable. Clearly $f_0(a(\delta)) < \delta$ for each $\delta \in A_0$. Consequently, $a(\delta) \in f_0^{-1}[0, \delta)$. If for some $\delta \in A_0$,

$$a(\delta) \in \left(\bigcap_{f \in \mathcal{G} - \{f_0\}} f^{-1}[0, \delta) \cap B\right)^-,$$

then, since $f_0^{-1}[0, \delta)$ is open and contains $a(\delta)$,

$$a(\delta) \in \left(\bigcap_{f \in \mathcal{G} - \{f_0\}} f^{-1}[0, \delta) \cap B \cap f_0^{-1}[0, \delta)\right)^- = \left(\bigcap_{f \in \mathcal{G}} f^{-1}[0, \delta) \cap B\right)^-,$$

which is impossible. We therefore conclude that

$$a(\delta) \in \left(\bigcap_{f \in \mathcal{G} - \{f_0\}} (f^{-1}[0, \delta))^- \cap \bar{B}\right) - \left(\bigcap_{f \in \mathcal{G} - \{f_0\}} f^{-1}[0, \delta) \cap B\right)^-$$

for each $\delta \in A_0$. But this contradicts our inductive assumptions.

Put $\tilde{A}=\{\delta\in A: f(a(\delta))=\delta$ for each $f\in\mathcal{G}\}$. By the claim $\tilde{A}$ is uncountable. Let $F=\{a(\delta): \delta\in\tilde{A}\}$ and define an order $<$ on F by putting

$$a(\delta_0)<a(\delta_1) \quad \text{iff} \quad \delta_0<\delta_1.$$

Notice that $a(\delta_0)\neq a(\delta_1)$ iff $\delta_0\neq\delta_1$ so that this order is well-defined. By Lemma 2.1 we can find $a(\delta_0)\in F$ such that

$$a(\delta_0)\in\{a(\delta)\in F: \delta<\delta_0\}^{-}.$$

Let V be any neighborhood of $a(\delta_0)$ and take $a(\delta_1)\in F$ with $\delta_1<\delta_0$ such that $a(\delta_1)\in V$. Since $f(a(\delta_1))=\delta_1<\delta_0$ for each $f\in\mathcal{G}$, we have that

$$a(\delta_1)\in V\cap\bigcap_{f\in\mathcal{G}} f^{-1}[0,\delta_0)$$

and consequently, since $a(\delta_1)\in\bar{B}$,

$$V\cap\bigcap_{f\in\mathcal{G}} f^{-1}[0,\delta_0)\cap B\neq\varnothing.$$

We conclude that $a(\delta_0)\in(\bigcap_{f\in\mathcal{G}} f^{-1}[0,\delta_0)\cap B)^{-}$, which is a contradiction. □

2.4 LEMMA. *Let X be a space and let $\mathcal{F}$ be a countable collection of maps from X into $[0,1]$. In addition, let $B\subset X$ be regular open. If $G\subset\mathcal{F}$ is finite, then $\{\delta\in(0,1): \bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta)\cup B\notin \mathrm{RO}(X)\}$ is countable.*

Proof. The proof is similar to the proof of Lemma 2.3. We induct on $|\mathcal{G}|$. Assume the Lemma to be true for all $\mathcal{G}\subset\mathcal{F}$ of cardinality n and let $\mathcal{G}\subset\mathcal{F}$ have cardinality $n+1$. Assume that the set

$$A=\left\{\delta\in(0,1): \bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta)\cup B\notin\mathrm{RO}(X)\right\}$$

is uncountable and for each $\delta\in A$ take a point

$$(*) \qquad a(\delta)\in\left(\bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta)\cup B\right)^{-0}-\left(\bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta)\cup B\right).$$

Claim. Take $f_0\in\mathcal{G}$. Then $A_0=\{\delta\in A: f_0(a(\delta))\neq\delta\}$ is countable.

Suppose that A_0 is uncountable. Clearly $f_0(a(\delta))>\delta$ for each $\delta\in A_0$. Consequently,

$$a(\delta)\in\left(\bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta)\cup B\right)^{-0}-f_0^{-1}[0,\delta]\subset\left(\bigcup_{f\in\mathcal{G}-\{f_0\}} f^{-1}[0,\delta)\cup B\right)^{-0}.$$

Since obviously $a(\delta)\notin\bigcup_{f\in\mathcal{G}-\{f_0\}} f^{-1}[0,\delta)\cup B$, we conclude that

$$a(\delta)\in\left(\bigcup_{f\in\mathcal{G}-\{f_0\}} f^{-1}[0,\delta)\cup B\right)^{-0}-\left(\bigcup_{f\in\mathcal{G}-\{f_0\}} f^{-1}[0,\delta)\cup B\right)$$

for each $\delta\in A_0$. But this contradicts our inductive assumptions.

Put $\tilde{A}=\{\delta\in A\colon f(a(\delta))=\delta$ for each $f\in\mathcal{G}\}$. By the claim $\tilde{A}$ is uncountable. Let $F=\{a(\delta)\colon \delta\in\tilde{A}\}$ and define an order $<$ on F by putting

$$a(\delta_0)<a(\delta_1) \quad \text{iff} \quad \delta_0<\delta_1.$$

By Lemma 2.1 we can find $a(\delta_0)\in F$ such that

$$a(\delta_0)\in\{a(\delta)\in F\colon \delta_0<\delta\}^-.$$

By $(*)$ we can choose $a(\delta_1)\in\mathcal{F}$ with $\delta_0<\delta_1$ and

$$a(\delta_1)\in\left(\bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta_0)\cup B\right)^{-0}$$

and since $f(a(\delta_1))=\delta_1$ for each $f\in\mathcal{G}$ this implies that

$$a(\delta_1)\in\left(\bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta_0)\cup B\right)^{-0}-\bigcup_{f\in\mathcal{G}} f^{-1}[0,\delta_0]\subset B^{-0}=B,$$

since B is regular open. But this contradicts $(*)$. □

2.5 Lemma. *Let X be a space and let $\mathcal{F}$ be a countable collection of maps from X into* $[0,1]$. *Then* $\{\delta\in(0,1)\colon \exists f\in\mathcal{F}$ *such that* $(f^{-1}[0,\delta))^-\neq f^{-1}[0,\delta]\}$ *is countable.*

Proof. If not, then we can find an uncountable subset $A\subset(0,1)$ and an $f\in\mathcal{F}$ such that for any $\delta\in A$ there is a point

$$a(\delta)\in f^{-1}[0,\delta]-(f^{-1}[0,\delta))^-.$$

Clearly $f(a(\delta))=\delta$ for each $\delta\in A$, so, by Lemma 2.1, we can find $\delta_0\in A$ such that

$$a(\delta_0)\in\{a(\delta)\colon \delta\in A \text{ and } \delta<\delta_0\}^-.$$

Let V be any neighborhood of $a(\delta_0)$ and take $\delta<\delta_0$ such that $a(\delta)\in V$. Then

$$a(\delta)\in V\cap f^{-1}[0,\delta_0)$$

and since V was arbitrary, this implies that $a(\delta_0)\in(f^{-1}[0,\delta_0))^-$. Contradiction. □

If $\mathcal{E}$ is a collection of sets, then $\vee.\wedge.\mathcal{E}$ denotes the *ring* generated by $\mathcal{E}$, i.e. the collection of all finite unions of finite intersections of elements of $\mathcal{E}$.

2.6 Theorem. *Let X be a space and let $\mathcal{F}$ be a collection of countably many self-maps of X. Then there is a countable open basis $\mathcal{U}$ for X such that*

(1) $\mathcal{U}\subset\mathrm{RO}(X)$;
(2) *$\mathcal{U}$ is closed under finite unions and finite intersections*;
(3) *if $\mathcal{E}\subset\mathcal{U}$ is finite then $\bigcap_{E\in\mathcal{E}}\bar{E}=(\bigcap\mathcal{E})^-$*;
(4) *if $U\in\mathcal{U}$ and $f\in\mathcal{F}$ then $f^{-1}(\bar{U})=(f^{-1}(U))^-$*;
(5) *if $U\in\mathcal{U}$ and $f\in\mathcal{F}$ then $f^{-1}(U)\in\mathcal{U}$.*

Proof. Without loss of generality assume that $f \circ g \in \mathcal{F}$ for all $f, g \in \mathcal{F}$ and that $\mathrm{id}_X \in \mathcal{F}$. Enumerate $\mathcal{F}$ as $\{f_n: n \in \mathbb{N}\}$. Let $\mathcal{B}$ be a countable open basis for X and for any $B_0, B_1 \in \mathcal{B}$ with $\bar{B}_0 \subset B_1$ choose a Urhsohn function $g: X \to I$ such that $g(\bar{B}_0) = 0$ and $g(X - B_1) = 1$. Let $\{g_k: k \in \mathbb{N}\}$ denote the set of functions chosen in this way

For each $k, n \in \mathbb{N}$ let $g_{k,n} X \to I$ be defined by

$$g_{k,n} = g_k \circ f_n.$$

By induction on k we will construct a point $\delta_k \in (0, 1)$ such that if $\mathcal{E} = \vee . \wedge . \{g_{l,n}^{-1}[0, \delta_l): l < k, n \in \mathbb{N}\}$ and $\mathcal{F} = \vee . \wedge . \{g_{k,n}^{-1}[0, \delta_k): n \in \mathbb{N}\}$ then

(1′) $\bar{E} \cap \bar{F} = \overline{E \cap F}$ for all $E \in \mathcal{E} \cup \mathcal{F}$ and $F \in \mathcal{F}$;

(2′) $E \cup F \in \mathrm{RO}(X)$ for all $E \in \mathcal{E}$ and $F \in \mathcal{F}$;

(3′) $(g_{k,n}^{-1}[0, \delta_k))^- = g_{k,n}^{-1}[0, \delta_k]$ for each $n \in \mathbb{N}$.

Assume that we have completed the construction for all $l < k$ so that (1′)–(3′) hold. Let $\mathcal{H}$ be the set of all functions $\{g_{k,n}: n \in N\}$, along with all finite sups of such functions. Choose $\delta_k = \delta$ by 2.3–2.5 so that for all $B \in \mathcal{E}$, for all finite $G \cup \{g\} \subset \mathcal{H}$ the following conditions hold:

(a) $\bigcap_{f \in G} (f^{-1}[0, \delta))^- \cap \bar{B} = (\bigcap_{f \in G} f^{-1}[0, \delta) \cap B)^-$,

(b) $\bigcup_{f \in G} f^{-1}[0, \delta) \bigcup B \in \mathrm{RO}(X)$,

(c) $(g^{-1}[0, \delta))^- = g^{-1}[0, \delta]$.

Now if G is a finite subset of $\mathcal{H}$, then it is easy to check that $(\bigcap_{g \in G} g^{-1}[0, \delta_k))^- = \bigcap_{g \in G} g^{-1}[0, \delta_k]$. Now (1′)–(3′) are easily checked, using finite sups for (2′). Put

$$\mathcal{G} = \{g_{k,n}^{-1}[0, \delta_k): k, n \in \mathbb{N}\}.$$

If $g_{k,n}^{-1}[0, \delta_k) \in \mathcal{G}$ and if $f_p \in \mathcal{F}$ then

$$f_p^{-1} g_{k,n}^{-1}[0, \delta_k) = f_p^{-1} f_n^{-1} g_k^{-1}[0, \delta_k) = f_q^{-1} g_k^{-1}[0, \delta_k) = g_{k,q}^{-1}[0, \delta_k)$$

for certain $q \in \mathbb{N}$. This implies that $f_n^{-1}(\mathcal{G}) \subset \mathcal{G}$ for all $n \in \mathbb{N}$. If we put $\mathcal{U} = \vee . \wedge . \mathcal{G}$, then $\mathcal{U}$ is as required. The only thing we have to check is (4). To this end, take $k_1, \ldots, k_n \in \mathbb{N}$ and for each $1 \le i \le n$ let $F_i \subset \mathbb{N}$ be finite. Put

$$U = \bigcap_{i=1}^{n} \bigcap_{j \in F_i} g_{k_i,j}^{-1}[0, \delta_{k_i}).$$

Take $p \in \mathbb{N}$. Then

$$\begin{aligned} f_p^{-1}(\bar{U}) &= f_p^{-1}\left(\left(\bigcap_{i=1}^{n} \bigcap_{j \in F_i} g_{k_i,j}^{-1}[0, \delta_{k_i})\right)^-\right) \\ &= f_p^{-1}\left(\bigcap_{i=1}^{n} \bigcap_{j \in F_i} g_{k_i,j}^{-1}[0, \delta_{k_i}]\right) \quad \text{(by (1′) and (3′))} \\ &= \bigcap_{i=1}^{n} \bigcap_{j \in F_i} f_p^{-1} g_{k_i,j}^{-1}[0, \delta_{k_i}] \end{aligned}$$

$$= \bigcap_{i=1}^{n} \bigcap_{j\in F_i} (f_j \circ f_p)^{-1} g_{k_i}^{-1}[0, \delta_{k_i}]$$

$$= \bigcap_{i=1}^{n} \bigcap_{j\in F_i} ((f_j \circ f_p)^{-1}(g_{k_i}^{-1}[0, \delta_{k_i}))^{-}) \quad \text{(by (3'))}$$

$$= \left(\bigcap_{i=1}^{n} \bigcap_{j\in F_i} (f_j \circ f_p)^{-1} g_{k_i}^{-1}[0, \delta_{k_i})\right)^{-} \quad \text{(by (1'))}$$

$$= \left(\bigcap_{i=1}^{n} \bigcap_{j\in F_i} f_p^{-1} f_j^{-1} g_{k_i}^{-1}[0, \delta_{k_i})\right)^{-}$$

$$= \left(f_p^{-1}\left(\bigcap_{i=1}^{n} \bigcap_{j\in F_i} f_j^{-1} g_{k_i}^{-1}[0, \delta_{k_i})\right)\right)^{-}$$

$$= \left(f_p^{-1}\left(\bigcap_{i=1}^{n} \bigcap_{j\in F_i} g_{k_i,j}^{-1}[0, \delta_{k_i})\right)\right)^{-}$$

$$= (f_p^{-1}(U))^{-}.$$

This easily implies that $\mathcal{U}$ satisfies (1), (2), (3), (4) and (5). □

That each space has a basis which satisfies (3) was first shown by Aarts [1] and Steiner and Steiner [6]. We are now prepared to give the proof of Theorem 1.1.

2.7 **Proof of Theorem 1.1.** We may assume that $\varphi^{-1} \in \mathcal{F}$ for all homeomorphisms $\varphi \in \mathcal{F}$. Let $\mathcal{U}$ be as in Theorem 2.6 and let $\mathcal{B} = \langle\langle \mathcal{U} \rangle\rangle$, the smallest Boolean subalgebra of RO(X) containing $\mathcal{U}$. We claim that $\mathcal{B}$ is as required. Take $f \in \mathcal{F}$ and define $F: \mathcal{U} \to \mathcal{U}$ by $F(U) = f^{-1}(U)$. Take $U_1, U_2, \ldots, U_n \in \mathcal{U}$ and $V_1, V_2, \ldots, V_k \in \mathcal{U}$. Then

$$(*) \quad \bigwedge_{1\le i\le n} F(U_i) \wedge \bigwedge_{1\le j\le k} (F(V_j))' = \bigcap_{1\le i\le n} f^{-1}(U_i) \cap \bigcap_{1\le j\le k} (X - \overline{f^{-1}(V_j)})$$

$$= \bigcap_{1\le i\le n} f^{-1}(U_i) \cap \bigcap_{1\le j\le k} (X - f^{-1}(\bar{V}_j)) \quad \text{(by (4) of 2.6)}$$

$$= \bigcap_{1\le i\le n} f^{-1}(U_i) \cap \bigcap_{1\le j\le k} (f^{-1}(X - \bar{V}_j))$$

$$= f^{-1}\left(\bigcap_{1\le i\le n} U_i \cap \bigcap_{1\le j\le k} (X - \bar{V}_j)\right) = f^{-1}\left(\bigwedge_{1\le i\le n} U_i \wedge \bigwedge_{1\le j\le k} V_j'\right).$$

Consequently, if $\bigwedge_{1\le i\le n} U_i \wedge \bigwedge_{1\le j\le k} V_j' = 0$ then $\bigwedge_{1\le i\le n} F(U_i) \wedge \bigwedge_{1\le j\le k} (F(V_j))' = 0$. This implies that there is a unique homomorphism $\Psi: \mathcal{B} \to \mathcal{B}$ that extends F, [5, 2.15]. We claim that $\Psi(B) = \Phi_f(B) = f^{-1}(B)^{-0}$ for each $B \in \mathcal{B}$. First observe that since $\mathcal{U}$ is closed under finite unions,

$$\bigvee_{1\le i\le n} U_i = \bigcup_{1\le i\le n} U_i$$

for all $U_i \in \mathcal{U}$ $(1 \le i \le n)$. This implies that if

$$\mathcal{G} = \{U \wedge V' \colon U, V \in \mathcal{U}\},$$

then for each $B \in \mathcal{B}$ there are finitely many $G_1, G_2, \ldots, G_n \in \mathcal{G}$ such that

$$B = \bigvee_{1 \le i \le n} G_i.$$

Claim. If $n \in \mathbb{N}$ and $G_i \in \mathcal{G}$ for all $1 \le i \le n$ then $f^{-1}(\bigcup_{1 \le i \le n} G_i)$ is dense in $\Psi(\bigvee_{1 \le i \le n} G_i)$.

If $n = 1$, then there is nothing to prove since by formula $(*)$ it is true that

$$\Psi(G) = f^{-1}(G)$$

for all $G \in \mathcal{G}$ (observe that Ψ extends F). So assume the Claim to be true for n and take $G_1, G_2, \ldots, G_{n+1} \in \mathcal{G}$. Since $\Psi(\bigvee_{1 \le i \le n+1} G_i) = \Psi(\bigvee_{1 \le i \le n} G_i) \vee \Psi(G_{n+1})$ and since $\Psi(\bigvee_{1 \le i \le n} G_i) \cup \Psi(G_{n+1})$ is clearly dense in $\Psi(\bigvee_{1 \le i \le n} G_i) \vee \Psi(G_{n+1})$, by induction hypothesis,

$$f^{-1}\Big(\bigcup_{1 \le i \le n} G_i\Big) \cup f^{-1}(G_{n+1}) = f^{-1}\Big(\bigcup_{1 \le i \le n} G_i\Big)$$

is dense in $\Psi(\bigvee_{1 \le i \le n} G_i) \vee \Psi(G_{n+1}) = \Psi(\bigvee_{1 \le i \le n+1} G_i)$.

Now take $B \in \mathcal{B}$. We claim that

$$\Psi(B) = \Phi_f(B).$$

Assume this is not the case. Find $G_1, \ldots, G_n \in \mathcal{G}$ such that $B = \bigvee_{1 \le i \le n} G_i$. By the Claim, $f^{-1}(\bigcup_{1 \le i \le n} G_i)$ is dense in $\Psi(B)$.

Since $f^{-1}(\bigcup_{1 \le i \le n} G_i) \subset f^{-1}(\bigvee_{1 \le i \le n} G_i)$,

$$\Psi(B) = f^{-1}\Big(\bigcup_{1 \le i \le b} G_i\Big)^{-0} \subset \Big(f^{-1}\Big(\bigvee_{1 \le i \le n} G_i\Big)\Big)^{-0},$$

and consequently,

$$E = \Big(f^{-1}\Big(\bigvee_{1 \le i \le n} G_i\Big)\Big)^{-0} - \Big(f^{-1}\Big(\bigcup_{1 \le i \le n} G_i\Big)\Big)^{-} \neq \varnothing.$$

Then $f(E) \subset (\bigcup_{1 \le i \le n} G_i)^{-} - (\bigcup_{1 \le i \le n} G_i)$. If $A \subset X$, let $\mathrm{Bd}(A)$ denote the boundary of A. For each $1 \le i \le n$ find $U_i, V_i \in \mathcal{U}$ such that

$$G_i = U_i \cap V_i'.$$

Observe that $\mathrm{Bd}(G_i) \subset \mathrm{Bd}(U_i) \cup \mathrm{Bd}(V_i') = \mathrm{Bd}(U_i) \cup \mathrm{Bd}(V_i)$. Since

$$\Big(\bigcup_{1 \le i \le n} G_i\Big)^{-} - \Big(\bigcup_{1 \le i \le n} G_i\Big) \subset \bigcup_{1 \le i \le n} \mathrm{Bd}(G_i) \subset \bigcup_{1 \le i \le n} (\mathrm{Bd}(U_i) \cup \mathrm{Bd}(V_i))$$

it follows that

$$E \subset f^{-1}(\bigcup_{1\le i\le n} (\mathrm{Bd}(U_i) \cup \mathrm{Bd}(V_i))) = \bigcup_{1\le i\le n} (f^{-1}(\mathrm{Bd}(U_i)) \cup f^{-1}(\mathrm{Bd}(V_i))).$$

Since, by (1) and (4) of Theorem 2.6,

$$f^{-1}(\mathrm{Bd}(U)) = \mathrm{Bd}(f^{-1}(U))$$

for all $U \in \mathcal{U}$, it follows that $\bigcup_{1\le i\le n} (f^{-1}(\mathrm{Bd}(U_i)) \cup f^{-1}(\mathrm{Bd}(V_i)))$ is nowhere dense, which contradicts E being open and nonempty (observe that the union of finitely many nowhere dense sets is nowhere dense). Therefore, $\Psi = \Phi_f$. Now suppose that f is onto. Take distinct $E_0, E_1 \in \mathcal{B}$. Then, without loss of generality, $E_0 \not\subset \bar{E}_1$. Hence, since f is onto, $f^{-1}(E_0 - \bar{E}_1) \neq \emptyset$ and open. This obviously implies that $\Phi_f(E_0) \neq \Phi_f(E_1)$.

Now let $\varphi \in \mathcal{F}$ be a homeomorphism. If $U \in \mathcal{U}$ then also $\varphi(U) \in \mathcal{U}$ since

$$\varphi(U) = (\varphi^{-1})^{-1}(U),$$

and $\varphi^{-1} \in \mathcal{F}$. This easily implies that Φ_φ is an isomorphism. □

3. **Raising maps to zero-dimensional spaces.** In this section we show that the results of Anderson, de Groot and Baayen previously cited, easily follow from Theorem 1.1.

3.1 THEOREM. *Let X be a compact space and let $\mathcal{F}$ be a family of countably many self-maps of X. Then there is a zero-dimensional compact space Y and a continuous surjection $\tau: Y \to X$ such that for any $f \in \mathcal{F}$ there is a map $\bar{f}: Y \to Y$ such that*

(1) $\tau \circ \bar{f} = f \circ \tau$;

(2) *if f is onto then $\bar{f}$ is onto*;

(3) *if f is a homeomorphism then $\bar{f}$ is a homeomorphism.*

Moreover, the function $\tau^{\#}: \mathrm{RO}(Y) \to \mathrm{RO}(X)$ defined by $\tau^{\#}(U) = X - \tau(Y - U)$ is an isomorphism.

Proof. Let $\mathcal{B}$ be as in Theorem 1.1 and let Y be the Stone space of $\mathcal{B}$, i.e. the space of all ultrafilters of $\mathcal{B}$ and observe that Y is a compact zero-dimensional metric space (since $\mathcal{B}$ is countable). Define $\tau: Y \to X$ by

$$\{\tau(p)\} = \bigcap_{B \in p} \bar{B}.$$

As is well-known, since $\mathcal{B}$ is a basis for X, τ is well-defined, continuous and onto. In addition, since $\mathcal{B}$ is a subalgebra of $\mathrm{RO}(X)$, the function $\tau^{\#}: \mathrm{RO}(Y) \to \mathrm{RO}(X)$ defined by $\tau^{\#}(U) = X - \tau(Y - U)$ is an isomorphism. It is now clear how to define the functions $\bar{f}\,(f \in \mathcal{F})$, simply put

$$\bar{f}(p) = \{B \in \mathcal{B} : \Phi_f(B) \in p\}.$$

An easy check shows that $\tau \circ \bar{f} = f \circ \tau$. □

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