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A REMARK ON THE RUDIN-KEISLER ORDER OF ULTRAFILTERS

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ABSTRACT. Define a (pre)order on $\beta\omega$ as follows: $p \leqslant q$ iff there is a finite to one function $f \in {}^{\omega}\omega$ such that $\beta f(q) = p$. Let $\leqslant_{RK}$ denote the Rudin-Keisler order on $\beta\omega$. We show that there are points $p,q \in \omega^*$ such that $p \leqslant_{RK} q$, but p and q are $\leqslant$-incomparable.

0. Introduction. In [5], M. E. Rudin showed that if there are points $p,q \in \omega^*$ such that for any finite to one function $f \in {}^{\omega}\omega$ we have that $\beta f(p) \neq \beta f(q)$, then the indecomposable continuum $\mathbf{H}^*$, where $\mathbf{H}$ denotes the half-line $[0,\infty)$, has at least two composants. This result suggests to define a (pre)order $\leqslant$ on $\beta\omega$ as follows:

$$p \leqslant q \text{ iff there is a finite to one } f \in {}^{\omega}\omega \text{ such that } \beta f(q) = p.$$

Observe that this order is quite similar to the Rudin-Keisler order on $\beta\omega$, since this order is defined by

$$p \leqslant_{RK} q \text{ iff there is an } f \in {}^{\omega}\omega \text{ such that } \beta f(q) = p.$$

Intuitively, $\leqslant_{RK}$ is finer than $\leqslant$, and the aim of this note is to make this precise. First observe that if $p \leqslant q$ then $p \leqslant_{RK} q$, but not conversely. For if $p = 0$ and $q \in \omega^*$, then p and q are $\leqslant$-incomparable and $p \leqslant_{RK} q$. This shows that $\leqslant$ is only interesting on ω^*. We will show that there are points $p,q \in \omega^*$ such that $p \leqslant_{RK} q$ but p and q are $\leqslant$-incomparable.

1. Independent matrices and R-points. An indexed family $\{A_j^i \colon i \in I, j \in J\}$ of clopen subsets of ω^* is called a J *by* I *independent matrix* if

– for all distinct $j_0 j_1 \in J$ and $i \in I$ we have that $A_{j_0}^i \cap A_{j_1}^i = \emptyset$,

– for each finite $F \subset I$ and $\varphi \colon F \to J$ it is true that

$$\cap \{A_{\varphi(\alpha)}^{\alpha} \colon \alpha \in F\} \neq \emptyset.$$

This concept is due to K. Kunen. The following Lemma follows immediately from [3, 2.2].

1.1 LEMMA. *There is a* 2^{ω} *by* 2^{ω} *independent matrix of clopen subset of* ω^*.

A point $x \in \omega^*$ is called an R-point if there exists an open F_σ $U \subset \omega^*$ so that $x \in U^- - U$ but $x \notin F^-$ for any subset $F \subset U$ of cardinality less than 2^{ω}. We use independent matrices to construct $2^{2^{\omega}}$ R-points in ω^*. The method of proof is the same as in [4, 2.1].

1.2 THEOREM. *There are* $2^{2^{\omega}}$ R-*points in* ω^*.

PROOF. Let $\{C_n : n < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^*. Put $C = \cup_{n<\omega} C_n$. For each $n < \omega$ let $\{A^i_\alpha(n) : i < \omega, \alpha < 2^{\omega}\}$ be a 2^{ω} by ω independent matrix of clopen subsets of C_n (Lemma 1.1). Put

$$\mathcal{F} = \{F \subset C : \forall\, n < \omega\, \forall\, i \leqslant n\, \exists\, \alpha < 2^{\omega} \text{ such that } A^i_\alpha(n) \subset F\}.$$

Notice that whenever $\mathcal{G} \subset \mathcal{F}$ has cardinality n then $\cap \mathcal{G}$ intersects C_i for each $i \geqslant n\text{-}1$. Let $D \subset C$ have cardinality less than 2^{ω}. For each $n < \omega$ and $i \leqslant n$ choose $\alpha(n,i) < 2^{\omega}$ such that $A^i_{\alpha(n,i)}(n) \cap D = \emptyset$ and put

$$F = \cup_{n<\omega} \cup_{i\leqslant n} A^i_{\alpha(n,i)}(n).$$

Then $F \in \mathcal{F}$ and $F \cap D = \emptyset$. Since F is clopen (in C) and since disjoint closed subsets of C have disjoint closures in ω^* [1, 14.27], we conclude that $F^- \cap G^- = \emptyset$. Consequently, each point of $\cap_{F\in\mathcal{F}} F^-$ is an R-point of ω^*. We claim that $\cap_{F\in\mathcal{F}} F^-$ has cardinality $2^{2^{\omega}}$. For each $p \in \omega^*$ take

$$x_p \in \cap_{F\in\mathcal{F}} F^- \cap \cap_{P\in p} (\cup_{n\in P} C_n)^-.$$

Then $\{x_p : p \in \omega^*\}$ has cardinality $2^{2^{\omega}}$.

That there are $2^{2^{\omega}}$ R-points in ω^* also follows from [3]. A point p of a space X is called κ-OK provided that, for any sequence of neighborhoods $\{U_n : n < \omega\}$ of p, there are neighborhoods $\{V_\alpha : \alpha < \kappa\}$ of p such that, for all $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$, we have that $\cap_{i=1}^{n} V_{\alpha_i} \subset U_n$. Kunen [3] showed that 2^{ω}-OK points exist in ω^*, and the referee of an earlier version of this paper observed the following:

1.3 LEMMA. *Let* $p \in \omega^*$ *be a* 2^{ω}-OK *point which is not a* P-*point. Then* p *is an* R-*point.*

PROOF. Since p is not a P-point, there is an open F_σ $U \subset \omega^*$ such that $p \in U^- - U$. Let $U = \cup_{n<\omega} C_n$, where each C_n is clopen. Since p is 2^{ω}-OK, there is a sequence of clopen neighborhoods $\{A_\alpha : \alpha < 2^{\omega}\}$ of p such that whenever $F \subset 2^{\omega}$ has cardinality n then

(*) $\cap_{\alpha\in F} A_\alpha \subset (\omega^* - C_n)$.

Suppose that $D \subset U$ has cardinality less than 2^ω. If $A_\alpha \cap D \neq \emptyset$ for all $\alpha < 2^\omega$, then there is a $d \in D$ such that the set $\{\alpha < 2^\omega \colon d \in A_\alpha\}$ is infinite. Since $d \in C_n$ for certain $n < \omega$, this easily contradicts (*).

2. **The construction.** A closed subset $A \subset \omega^*$ is called an R-*set* if there is an open F_σ $U \subset \omega^*$ such that $A \subset U^- - U$ and $A \cap F^- = \emptyset$ for each subset $F \subset U$ of cardinality less than 2^ω. If $A \subset X$ is closed, then $\chi(A,X)$ denotes the least cardinality of a neighborhood basis of A.

2.1 THEOREM. *Let A be a family of 2^ω R-sets in ω^*. If $\{C_n\colon n < \omega\}$ is a family of countably many nonempty clopen subsets of ω^*, then for each $n < \omega$ there is a point $x_n \in C_n$ such that $\cup A \cap \{x_n\colon n < \omega\}^- = \emptyset$.*

PROOF. List A as $\{A_\alpha\colon \alpha < 2^\omega\}$. By induction, for each $\alpha < 2^\omega$ we will construct for each $n < \omega$ a nonempty closed subset $F^n_\alpha \subset C_n$ such that:

(1) $(\cup_{n<\omega} F^n_\alpha)^- \cap A_\alpha = \emptyset$,

(2) $\chi(F^n_\alpha, \omega^*) \leqslant |\alpha|\cdot\omega$ for each $n < \omega$,

(3) if $\kappa < \alpha$ and $n < \omega$ then $F^n_\alpha \subset F^n_\kappa$.

Let $U \subset \omega^*$ be an open F_σ which witnesses the fact that A_0 is an R-set. Define $E = \{n < \omega\colon C_n - U^- \neq \emptyset\}$ and for each $n \in E$ choose a nonempty clopen $E_n \subset C_n - U^-$. For all $n \notin E$, pick a point $t_n \in C_n \cap U$. Since A_0 is an R-set, $A_0 \cap \{t_n\colon n \notin E\}^- = \emptyset$. Consequently, we can find for any $n \notin E$ a clopen neighborhood E_n of t_n such that $E_n \subset C_n$ while moreover $(\cup_{n\notin E} E_n)^- \cap A_0 = \emptyset$. For each $n < \omega$ define $F^n_0 = E_n$. To show that the F^n_0's defined in this way are as required, it suffices to show that $(\cup_{n<\omega} E_n)^- \cap A_0 = \emptyset$. Since for each $n \in E$, the set E_n misses U, and since disjoint open F_σ's in ω^* have disjoint closures (this follows from the fact that ω^* is an F-space, [1, 14.27]), $(\cup_{n\in E} E_n)^- \cap A_0 \subset (\cup_{n\in E} E_n)^- \cap U^- = \emptyset$. Since by construction $(\cup_{n\notin E} E_n)^- \cap A_0 = \emptyset$, we see that $(\cup_{n<\omega} E_n)^- \cap A_0 = \emptyset$.

Suppose that we have completed the construction for each $\mu < \alpha < 2^\omega$. For each $n < \omega$ define $G_n = \cap_{\mu<\alpha} F^n_\mu$ and notice that the compactness of ω^* implies that $\chi(G_n, \omega^*) \leqslant |\alpha|\cdot\omega$. Let $U \subset \omega^*$ be an open F_σ which witnesses the fact that A_α is an R-set. Put $E = \{n < \omega\colon G_n \cap U^- \neq \emptyset\}$ and for each $n \in E$ let $\{V^n_\rho\colon \rho < |\alpha|\cdot\omega\}$ be a neighborhood basis for G_n. For each $n \in E$ and $\rho < |\alpha| \cdot \omega$ pick a point in $V^n_\rho \cap U$ and let Z be the set of points obtained in this way. Then $|Z| < 2^\omega$ and consequently, $Z^- \cap A_\alpha = \emptyset$. Let C be a clopen neighborhood of A_α which misses Z^-. Define $F^n_\alpha = G_n$ if $n \notin E$ and $F^n_\alpha = G_n - C$ if $n \in E$. Notice that $F^n_\alpha \neq \emptyset$ for each $n < \omega$. We only need to check (1). This is trivial however, since $\cup_{n \notin E} F^n_\alpha$ is an F_σ which does not intersect U^-, so again by the fact that ω^* is an F-space it follows that $(\cup_{n \notin E} F^n_\alpha)^- \cap U^- = \emptyset$.

For each $n < \omega$ take a point $x_n \in \cap_{\alpha < 2^\omega} F^n_\alpha$. Then the sequence $\{x_n : n < \omega\}$ is clearly as required.

2.2 LEMMA. *Let* $f \in {}^\omega\omega$ *be finite to one. If* $p \in \omega^*$ *is an* R-*point, then* $\beta f^{-1}(\{p\})$ *is an* R-*set.*

PROOF. Observe that $\beta f \colon \beta\omega \to \beta\omega$ is an open map which maps ω^* into ω^*. If $p \notin \beta f(\beta\omega)$, then there is nothing to prove, so, without loss of generality, assume that f is onto. Let U be an open F_σ in ω^* which witnesses the fact that p is an R-point. Since βf is an open map, $Z = \beta f^{-1}(\{p\})$ is contained in $V^- - V$, where $V = \beta f^{-1}(U)$. Notice that V is an open F_σ of ω^* since $\beta f(\omega^*) = \omega^*$. If $D \subset V$ has cardinality less than 2^ω, then $p \notin \beta f(D)^-$, which implies that $Z \cap D^- = \emptyset$. This implies that Z is an R-set.

2.3 COROLLARY. *For each* R-*point* $x \in \omega^*$ *the set* $\{y \in \omega^* : x \not\leqslant y$ *and* $y \not\leqslant x\}$ *has cardinality* 2^{2^ω}.

PROOF. For each finite to one function $f \in {}^\omega\omega$ put $A_f = \beta f^{-1}(\{x\})$. By Lemma 2.2 each A_f is an R-set. Therefore, by Theorem 2.1, $|\omega^* - \cup\{A_f : f \in {}^\omega\omega$ is finite to one$\}| = 2^{2^\omega}$. Since obviously $|\{p : p \leqslant x\}| \leqslant 2^\omega$ we find that $\{y \in \omega^* : x \not\leqslant y$ and $y \not\leqslant x\}$ has cardinality 2^{2^ω}.

We now come to the main result of this note.

2.4 THEOREM. *For each* R-*point* $x \in \omega^*$ *htere is a point* $y \in \omega^*$ *such that* $x \not\leqslant y$ *but* x *and* y *are* $\leqslant$-*incomparable.*

PROOF. Let $f \in {}^\omega\omega$ be a function such that for all $n < \omega$ we have that the set $f^{-1}(\{n\})$ is infinite. For each $n < \omega$, let $\{E^i_n : i < \omega\}$ be a family of countably many (faithfully indexed) nonempty clopen subsets of $\beta f^{-1}(\{n\}) \cap \omega^*$. By Theorem 2.1 we may pick for all $i,n < \omega$ a point $x^i_n \in E^i_n$ such that $\{x^i_n : i,n < \omega\}^-$ does not intersect

$$\cup\{\beta g^{-1}(\{x\}) : g \in {}^\omega\omega \text{ is finite to one}\}.$$

For each $i < \omega$, let $S_i = \{x_n^i : n < \omega\}$. Observe that $\beta f(S_i) = \omega$, which implies that $S_i^- \cap \beta f^{-1}(\{x\}) \neq \emptyset$. If $i \neq j$ then $\cup_{n<\omega} E_n^i \cap \cup_{n<\omega} E_n^j = \emptyset$, and therefore have disjoint closures since ω^* is an F-space. We conclude that if $i \neq j$ then $S_i^- \cap S_j^- = \emptyset$. Since $S_i^- \cap \beta f^{-1}(\{x\}) \neq \emptyset$ for all $i < \omega$, this implies that

$$\{x_n^i : i,n < \omega\}^- \cap \beta f^{-1}(\{x\})$$

is infinite, hence must have cardinality 2^{2^ω} (since it contains a copy of $\beta\omega$). The proof can now be completed by precisely the same argument as in the proof of Corollary 2.3.

3. **Remarks.** We have seen that finding two $\leqslant$-incomparable points is not too difficult. To find two $\not\leqslant$-incomparable points is highly nontrivial, for details see Kunen [2]. We do not know whether our construction can be adapted to give a new, and hopefully elementary, proof of Kunen's result.

Let us finally remark that the referee of an earlier version of this paper has asked whether it is consistent that all points in ω^* are R-points.

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