SPACES WITHOUT REMOTE POINTS

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All spaces considered are completely regular and \( X^* \) denotes \( \beta X - X \). The point \( x \in X^* \) is called a remote point of \( X \) if \( x \notin \text{Cl}_{\beta X} A \) for each nowhere dense subset \( A \) of \( X \). If \( y \in Y \), then the space \( Y \) is said to be extremally disconnected at \( y \) if \( y \notin U \cap V \) whenever \( U \) and \( V \) are disjoint open sets. In this paper we construct two noncompact \( \sigma \)-compact spaces \( X \), one locally compact and one nowhere locally compact, such that \( X \) has no remote points, and in fact such that \( \beta X \) is not extremally disconnected at any point.

Our examples were motivated by the following results from [6]:

(1) \( X \) has remote points if \( X \) has countable \( \pi \)-weight, in particular if \( X \) is separable and first countable, and is not pseudocompact, [6, 1.5]; see also [7] for an earlier consistency result, and [1] for a more general result.

(2) \( \beta X \) is extremally disconnected at each remote point of \( X \), [6, 5.2].

Via the observation that

(3) if \( Y \) is dense in \( Z \), and \( y \in Y \), then \( Y \) is extremally disconnected at \( y \) iff \( Z \) is extremally disconnected at \( y \),

these results and the following imply a nonhomogeneity result, which applies for example to the rationals and the Sorgenfrey line

(4) if \( X \) is a nowhere locally compact nonpseudocompact space which has a remote point and if \( \{ x \in X: X \text{ is not extremally disconnected at } x \} \) is dense in \( X \), e.g. if \( X \) is first countable, then \( X^* \) is not homogeneous because \( X^* \) is extremally disconnected at some but not at all points.

(This is a special case of Frolik’s theorem that \( X^* \) is not homogeneous if \( X \) is not pseudocompact, [8]. The proof of Frolik’s theorem does not yield a simple “because” as in (4). \( X \) is called nowhere locally compact if no point of \( X \) has a compact neighborhood, or, equivalently, if \( X^* \) is dense in \( \beta X \).)

In this paper we produce two closely related examples which show that the condition on the \( \pi \)-weight cannot be omitted altogether in (1), thus answering a question of [6].

Our two examples are rather big: they have cellularity at least \( \omega_3 \). This suggests the question of whether every nonpseudocompact separable space has a remote point. (This would generalize (1).) It follows from a construction in [7] that the answer is affirmative under CH.
**Examples.** There are two noncompact σ-compact spaces $X$, one locally compact and one nowhere locally compact, such that $X$ has no remote points, and in fact such that $\beta X$ is not extremally disconnected at any point.

Because of (3) the nowhere locally compact example shows that the condition on the $\pi$-weight cannot be omitted altogether in the nonhomogeneity result (4). We will show that an older nonhomogeneity proof, involving far points, still applies.

**No remote points.**

A subset $P$ of a space $X$ is called a $P$-set if for each $F_\sigma$-subset $F$ of $X$, if $F \cap P = \emptyset$ then $F \cap P = \emptyset$. A subset $T$ of a space $X$ is called a 2-set if there are disjoint open $U$ and $V$ in $X$ with $T \subseteq U \cap V$.

**Lemma 1.** There is a compact space $U$ such that for each $q \in U$ there is a decreasing $\omega$-sequence $(P^\xi : \xi \in \omega^\lambda)$ of clopen sets such that $\prod_{\xi \in \omega^\lambda} P^\xi$ is a nowhere dense set of $U$ which contains $q$.

□ Give $\omega_2$ the discrete topology. Identify $\omega_2^*$ with the space of free ultrafilters on $\omega_2$. Then

$$U = \{q \in \omega_2^*: |Q| = \omega_2 \text{ for all } Q \subseteq q\},$$

the space of uniform ultrafilters on $\omega_2$, is a closed, hence compact, subspace of $\omega_2^*$ of course. We need the following result due to Čudnovskii and Čudnovskii, [3] and, independently, to Kuen and Prikry, [11], and earlier, but with GCH to Chang [2]:

for each $q \in U$ there is a decreasing $\omega_\lambda$-sequence $\langle Q^\xi : \xi \in \omega_1 \rangle$ in

$(\ast) \quad q$ such that $\prod_{\xi \in \omega_1} Q^\xi = \emptyset$.

As usual, let $\hat{A}$ denote $U \cap \hat{A}$ (closure in $\beta \omega_2$), for $A \subseteq \omega_2$. For a given $q \in U$ let $\langle Q^\xi : \xi \in \omega_1 \rangle$ be as in $(\ast)$, and define $\langle P^\xi : \xi \in \omega_1 \rangle$ by $P^\xi = \hat{Q}^\xi$ for $\xi \in \omega_1$. Clearly $\langle P^\xi : \xi \in \omega_1 \rangle$ is a decreasing $\omega_1$-sequence of clopen subsets of $U$ such that $P = \bigcap_{\xi \in \omega_1} P^\xi$ contains $q$. Now recall that $\{\hat{B} : B \subseteq \omega_2$ and $|B| = \omega_2\}$, being the collection of all nonempty clopen subsets of $U$, is a base for $U$. Consider any $B \subseteq \omega_2$ with $|B| = \omega_2$. There is an $\eta \in \omega_1$ with $|B - Q^\eta| = \omega_2$. Then $\emptyset \neq (B - Q^\eta) = \hat{B} - \hat{Q}^\eta \subseteq \hat{B} - P$. It follows that $P$ is nowhere dense. □

**Remark.** Instead of $\omega_1$ we can take any regular cardinal $\kappa$, and then $U$ will be the space of uniform ultrafilters on $\kappa^+$. 
Clearly Lemma 1 implies that there is a compact space which is covered by the collection of its nowhere dense closed $P$-sets. Since evidently each 2-set is nowhere dense the following is a stronger assertion.

**Lemma 2.** There is a compact space $H$ such that for each $q \in H$ there is a closed $P$ in $H$ with $q \in P$ such that $P$ is both a $P$-set and a 2-set.

Let $U$ be as in Lemma 1, and let $H = U \times U$. Consider any $q_0, q_1 \in U$. For $i \in 2$ choose a decreasing $\omega_1$-sequence $\langle P_{i, \xi} : \xi \in \omega_1 \rangle$ of clopen sets in $U$ such that $P_i = \bigcap_{\xi \in \omega_1} P_{i, \xi}$ is a nowhere dense subset of $U$ which contains $q_i$. Then $P_0 \times P_1$ is a nowhere $P$-set in $H$ which contains $\langle q_0, q_1 \rangle$. We show that $P_0 \times P_1$ is also a 2-set.

For $i \in 2$ define an open $V_{i, \xi}$ with recursion on $\xi \in \omega_1$ by

$$V_{i, \xi} = (U - P_{i, \xi}) - \left( \bigcup_{\eta \in \xi} V_{i, \eta} \right) \left( \bigcup_{\nu \in \omega} V_{i, \eta} = \emptyset \text{ of course} \right).$$

Then evidently $(\bigcup_{\eta \in \xi} V_{i, \eta})^\complement = U - P_{i, \xi}$ for $i \in 2$ and $\xi \in \omega_1$. Since $P_0$ and $P_1$ are nowhere dense it follows that

$$(\dagger) \quad \left( \bigcup_{\xi \in \omega_1} V_{i, \xi} \right)^\complement = (U - P_i)^\complement = U, \quad \text{for } i \in 2.$$  

Define open subsets $W_0$ and $W_1$ of $H$ by

$$W_0 = \bigcup_{\xi \in \omega_1} P_{0, \xi} \times V_{1, \xi} \quad \text{and} \quad W_1 = \bigcup_{\xi \in \omega_1} V_{0, \xi} \times P_{1, \xi}.$$  

Then $W_0 \cap W_1 = \emptyset$ since if $\xi \leq \eta < \omega_1$ then $V_{i, \xi} \subseteq U - P_{i, \xi} \subseteq U - P_{i, \eta}$, for $i \in 2$ (so that $(P_{0, \xi} \times V_{1, \xi}) \cap (V_{0, \eta} \times P_{1, \xi}) = \emptyset$ for all $\xi, \eta \in \omega_1$). To prove that $P_0 \times P_1 \subseteq W_0 \cap W_1$ we have only to prove that $P_0 \times P_1 \subseteq W_0$, because of symmetry. We have

$$W_0 \supseteq \bigcup_{\xi \in \omega_1} \left( \left( \bigcap_{\eta \in \omega_1} P_{0, \eta} \right) \times V_{1, \xi} \right) = P_0 \times \bigcup_{\xi \in \omega_1} V_{1, \xi},$$

hence $\overline{W_0} \supseteq P_0 \times U \supseteq P_0 \times P_1$ as required.  

**Remark 2.** We do not know if the space $U$ of Lemma 1 can be used for the space $H$ of Lemma 2. We are indebted to the referee for pointing out that the set $P = \bigcap_{\xi \in \omega_1} P_{\xi}$ obtained in Lemma 1 is not a 2-set: $P$ has character $\omega_1$, but in $U$ the closure of every open $F_{\omega_1}$-set ($\equiv$ union of $\omega_1$ many closed sets) is easily seen to be open, [CoN, Thm. 14.9], which implies that no closed set in $U$ of character $\omega_1$ is a 2-set. To see this let $F$ be a closed set in $U$ of character $\omega_1$ and let $V$ and $W$ be disjoint open sets
in $U$ such that $F \subseteq \overline{V}$. Since $F$ has character $\omega_1$ there is an open $F_{\omega_1}$-set $T \subseteq V$ such that $\overline{T} \cap \overline{F} \neq \emptyset$. Now $\overline{T} \cap W = \emptyset$ since $T \cap W = \emptyset$, and $\overline{T}$ is clopen. It follows that $F \not\subseteq \overline{W}$.

**Subremark.** It is at least consistent that $U = U(\omega_2)$ has a closed $P$-set that is a 2-set. There is a closed nowhere dense $P \subseteq U$ which is a $P_{\omega_2}$-set (≡ for every $F_{\omega_2}$-set $F$ in $U$, if $F \cap P = \emptyset$ then $\overline{F} \cap P = \emptyset$), namely $\bigcap \{ C : C \subseteq \omega_2 \text{ is a cub}\}$, and if $2^{\omega_2} = \omega_3$ then every nowhere dense $P_{\omega_3}$-set in $U$ (or in any space of weight $\omega_3$) is a 2-set. However, if $2^{\omega_2} = \omega_3$, then $U$ is not covered by the collection of its nowhere dense closed $P_{\omega_3}$-sets, by [10, 1.1].

**Remark 3.** After this paper had been written another proof of Lemma 1 was discovered by Kunen, van Mill and Mills: the space of nondecreasing functions $\omega_2 \to \omega_1 + 1$, [10, 3.1]. It is easy to see that the $P$-sets obtained there are 2-sets. The example of Lemma 2 has the additional feature that each $P$-set has character $\omega_1$.

**Remark 4.** The above remarks suggest the question of whether there is a compact space which is covered by the collection of its closed nowhere dense $P$-sets but which has no nonempty closed $P$-set which is also a 2-set. This question can be answered quite easily. Let $E$ be the projective cover of the example of Lemma 1, i.e. $E$ is the unique extremally disconnected compact space that admits an irreducible map, say $\pi$, onto $U$. As is well known, $\pi^{-1}(D)$ is nowhere dense in $E$ iff $D$ is nowhere dense in $E$. Since it is easily seen that $\pi^{-1}(P)$ is a $P$-set of $E$ iff $P$ is a $P$-set of $U$, we conclude that $E$ can be covered by nowhere dense closed $P$-sets. Since $E$ is extremally disconnected, there are no nonempty 2-sets in $E$. The following question however remains open:

**Question.** Is there (in ZFC) a compact space which is covered by the collection of its closed nowhere dense $P$-sets but which has no nonempty nowhere dense $P_{\omega_2}$-set?

**Lemma 3.** Let $K$ be a compact space, and let $P$ be a $P$-set in $K$. Furthermore, let $Y$ be a countable space, let $\pi : K \times Y \to K$ be the projection, and let $\beta \pi : \beta(K \times Y) \to K$ be the Stone extension of $\pi$. Then for each $x \in \beta(K \times Y)$, if $\beta \pi(x) \in P$ then $x \in (P \times Y)^-^\circ$.

□ Consider any $x \in \beta(K \times Y) - (P \times Y)^-^\circ$. Let $V$ be a closed neighborhood of $x$ which misses $P \times Y$. Then $x \in ((K \times Y) \cap V)^-^\circ$, hence

$$\beta \pi(x) \in (\beta \pi^-^\circ((K \times Y) \cap V))^-^\circ = (\pi^-^\circ((K \times Y) \cap V))^-^\circ.$$
Also, $\pi^{-1}((K \times Y) \cap V)$ is an $F_\sigma$ (since $(K \times Y) \cap V$ is $\sigma$-compact) in $K$ which misses the $P$-set $P$, hence $(\pi^{-1}((K \times Y) \cap V))^{-} \cap P = \emptyset$. Consequently $\beta\pi(x) \not\in P$.

**Corollary 1.** If $K$ is a compact space which is covered by nowhere dense $P$-sets, then $K \times Y$ has no remote points, for each countable space $Y$.

**Corollary 2.** If $K$ is a compact space which is covered by $P$-sets which are $2$-sets, then $\beta(K \times Y)$ is not extremally disconnected at any point, for each countable space $K$.

The key observation is that if $D$ is dense in a space $X$, then the closure in $X$ of each $2$-set in $D$ is a $2$-set in $X$.

If $H$ is as in Lemma 2, if $\omega$ is the integers and if $Q$ is the rationals, then our examples are $H \times \omega$ and $H \times Q$.

**Far points.**

A point $p$ of $X^*$ is called a far (or $\omega$-far) point of $X$ if $p \not\in \overline{D}$ for each (countable) closed discrete subset $D$ of $X$. Clearly, if $X$ has no isolated points then each remote point of $X$ is a far point; the converse of this is generally false, [6,4.8]. There is a nonhomogeneity result involving far points, or $\omega$-far points, similar to (4) of the introduction, but less attractive since it involves $X^{**} = (X^*)^*$. If $X$ is nowhere locally compact, and is not countably compact, and has a far ($\omega$-far) point, then $X^*$ is not homogeneous because for some but not for all $x \in X^*$ there is a (countable) closed discrete $D$ in the space $X^{**}$ such that $x \in \overline{D}$, [5, 2,4.3].

One might hope that our examples can be used to answer the question of [5] of whether every noncompact Lindelöf space has an $\omega$-far point (which would be a far point). (It is easy to see that every normal nonLindelöf space has an $\omega$-far point, [5, 4.3].) This is not the case: both our examples have far points. This follows from the following result.

**Theorem.** If $X$ has a countably infinite discrete collection $K$ of compact subspaces without isolated points, and if $X$ is normal, or, more generally, if $K$ can be separated by a discrete open family, then $X$ has a far point.

Before we proceed to the proof we point out an attractive corollary:

**Corollary.** Every locally compact (or, more generally, Čech-complete) nonpseudocompact space has a far-point.
If $X$ is nonpseudocompact it has a countably infinite family $\mathcal{U}$ consisting of nonempty open sets. By a well-known tree argument one finds for each $U \in \mathcal{U}$ a compact $K_U \subseteq U$ that admits a continuous map $f_U$ onto the Cantor discontinuum $\omega^2$. For $U \in \mathcal{U}$ choose a compact $L_U \subseteq K_U$ such that $f_U \upharpoonright L_U$ is an irreducible map onto $\omega^2$, then $L_U$ has no isolated points.

**Proof of Theorem.** First recall that $\mathbb{R}$ has a far point, by an elegant argument due to Eberlein [7, Thm. 1.3]. It follows that $Y = U\mathcal{K}$ has a far point. As in the proof of the Corollary, each member of $\mathcal{K}$ admits a (necessarily closed) map onto the Cantor discontinuum, hence on the closed unit interval. Since $\mathcal{K}$ is countably infinite it follows that $Y$ admits a closed map onto $\mathbb{R}$. The Stone extension $\beta f$ of $f$ maps $\phi Y$ onto $\beta \mathbb{R}$, hence there is $y \in Y^*$ such that $\beta f(y)$ is a far point of $\mathbb{R}$. Since $f^{-1}D$ is closed discrete in $\mathbb{R}$ for each closed discrete $D$ in $Y$ this $y$ is a far point of $Y$, cf. [5, §2, Fact 3].

We now point out that

\begin{equation}
(*) \quad \text{For any two disjoint closed } F \text{ and } G \text{ in } X, \text{ if } F \subseteq Y \text{ then } \\
\text{Cl}_{\beta X} F \cap \text{Cl}_{\beta X} G = \emptyset.
\end{equation}

The proof is similar to the known case, [9, 3L], that $\mathcal{K}$ consists of singletons. From ($*$) we see that $\text{Cl}_{\beta X} Y = \beta Y$. Since $Y$ is closed in $X$ it follows that $X$ contains a far point of $Y$. This point is a far point of $X$ since, by ($*$), for each closed discrete subset $D$ of $Y$ we have $\text{Cl}_{\beta X}(D - Y) \cap \text{Cl}_{\beta X} Y = \emptyset$.

**Remark 5.** Dow [4] has shown that every separable nonpseudocompact space has a remote point under MA.

**Remark 6.** After this paper was written there has been much progress on the question of whether every Lindelöf space has a far point: It is known that the answer is affirmative under MA, [12, 9.1].

**References**


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