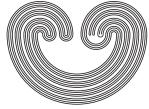
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by

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1. Introduction

All spaces under discussion are separable metric.

The Anderson-Kadec Theorem (see [BP] for background information) that every infinite-dimensional Fréchet space is homeomorphic to the Hilbert space ℓ_2 , suggests the question whether "nice" subsets of Fréchet spaces are always homeomorphic to "nice" subsets of ℓ_2 . As far as I know it is open whether every locally convex real vector space is homeomorphic to a linear subspace of ℓ_2 . Let us consider \mathbb{R}^{∞} to be a vector space over the rationals Q. In this note we will show that there is a linear subspace L of \mathbb{R}^{∞} that is not homeomorphic to a normed vector space over Q.

2. Preliminaries

A (topological) vector space over Q is a topological space X that is a vector space over Q such that the algebraic operations

 $\langle x, y \rangle \rightarrow x + y$, and $x \rightarrow qx$ (q $\in Q$ fixed)

are continuous. A vector space over Q will be called a *rational vector space* from now on. As usual, a rational vector space L is called *normed* if there is a function $|| \cdot || : L + \mathbb{R}^+$ such that

 $|| x + y|| \le || x || + || y ||,$ $|| qx || = |q| \cdot || x ||,$ $|| x || = 0 \leftrightarrow x = 0$

for all x,y ε L, q ε Q, while moreover the metric

$$d(x,y) = ||x - y||$$

generates the topology on L. Observe that $|| x || \in \mathbb{R}^+$ for every x \in L and not, as some might expect, that || x || is always rational.

3. The Construction

In this section we will construct the example that was announced in the introduction.

3.1 Theorem. Let X be a topologically complete vector space over R of dimension at least 2. Then X contains a connected subspace L such that

(1) if $x, y \in L$ and $s, t \in Q$ then $sx + ty \in L$,

(2) if h: $L \rightarrow L$ is any autohomeomorphism then there are $q \in Q \setminus \{0\}$ and $y \in L$ such that h(x) = qx + y, for every $x \in L$,

(3) L intersects every Cantor set in X.

The proof of this result, except for trivial modifications, is the same as the proof of [vM, Theorem 3.1] and will therefore be omitted.

Now let $X = \mathbb{R}^{\infty}$ and let $L \subseteq X$ be as in Theorem 3.1. By (1), L is a rational vector space and we claim that L is as required. Striving for a contradiction, let M be a normed rational vector space and let h: $L \rightarrow M$ be a homeomorphism. By <u>0</u> we will denote the point $(0, 0, \dots) \in \mathbb{R}^{\infty}$. Since M is homogeneous, without loss of generality we may assume that $h(\underline{0}) = 0$. To avoid confusion, the algebraic operations on M will be denoted by \oplus and \cdot , respectively. Define $\gamma: M \neq M$ by $\gamma(x) = x \oplus x$. Then γ is a homeomorphism of M which implies that $\xi = h^{-1}\gamma h$ is a homeomorphism of L. Observe that

$$\xi(\underline{0}) = h^{-1}\gamma h(\underline{0}) = h^{-1}\gamma(0) = h^{-1}(0) = \underline{0}.$$

By (2) there exist $q \in Q$ and $y \in L$ such that

 $\xi(\mathbf{x}) = q\mathbf{x} + \mathbf{y}$

for every $x \in L$. Since $\xi(\underline{0}) = \underline{0}$ it follows that $y = \underline{0}$, whence $\xi(x) = qx$ for every $x \in L$. Let $U = \{x \in M: || x || < 1\}$. Then U is an open neighborhood of 0 in M, whence $h^{-1}(U)$ is an open neighborhood of $\underline{0}$ in L. Choose an open neighborhood V of 0 in R and an $n \in \mathbb{N}$ such that

$$W = (V \times V \times V \times \cdots \times V \times \mathbf{R} \times \mathbf{R} \times \cdots) \cap \mathbf{L} \subseteq \mathbf{h}^{-1}(\mathbf{U}).$$

By (3), there is a point $x \in W \setminus \{\underline{0}\}$ such that $x_i = 0$ for every $i \leq n$. Then $Qx \subseteq W$ and from this we conclude that $\{\xi^n(x): n \in \mathbb{N}\} \subseteq W$. Put y = h(x). Then $\xi^n(h^{-1}(y)) = (h^{-1}\gamma h)^n(h^{-1}(y)) = h^{-1}\gamma^n h h^{-1}(y)$ $= h^{-1}\gamma^n(y) \in W \subseteq h^{-1}(U)$

for every $n \in \mathbb{N}$. Consequently, $\gamma^n(y) \in U$, $n \in \mathbb{N}$. Let $\varepsilon = || y ||$. Observe that $\varepsilon > 0$. Take $n \in \mathbb{N}$ so large that $2^n \varepsilon > 1$. Then $|| \gamma^n(y) || = || 2^n \cdot y || = 2^n || y || = 2^n \varepsilon > 1$, which is a contradiction.

References

- [BP] C. Bessaga and A. Pelczyński, Selected topics in infinite-dimensional topology, PWN, Warsaw, 1975.
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