A PATHOLOGICAL HOMOGENEOUS SUBSPACE OF THE REAL LINE

A.J.M. van ENGELEN & J. van MILL

1. INTRODUCTION

All spaces under discussion are separable metric.

A zero-dimensional space is called *strongly homogeneous* provided that all nonempty clopen subspaces are hemeomorphic. Strongly homogeneous spaces behave very well; for example, they have the pleasant property that any homeomorphism between closed and nowhere dense sets can be extended to a homeomorphism of the whole space, [4]. As a consequence, all strongly homogeneous spaces are homogeneous (we encourage the reader to find a direct elementary proof of this corollary). Observe that a strongly homogeneous nowhere locally compact space has the property that all nonempty open subspaces are homeomorphic.

From the above observations it is clear that a strongly homogeneous space has "many" autohomeomorphisms. Many familiar subspaces of the real line are strongly homogeneous, for example, the rationals, the irrationals and the Cantor set. R.D. Anderson [1] has shown that, in particular, the autohomeomorphism group of a srongly homogeneous space is algebraically simple.

The aim of this note is to construct a very pathological exmple of a strongly homogeneous subspace X of the real line $\mathbb{R}$. The space X is pathological since it contains a countable dense subset $D \subset X$ such that $Y = X \setminus D$ is rigid, i.e. has only one autohomeomorphism, namely the identity. In fact, we prove a little bit more, i.e., if $h: Y \to Y$ is an embedding (not necessarily surjective), then h = identity.

A space X is called *strongly locally homogeneous* if it has an open base $\mathcal{U}$ such that for each $U \in \mathcal{U}$ and points $x,y \in U$, there exists a homeomorphism $h: X \to X$ with $h(x) = y$ and $h \mid X \setminus U$ equal to the identity. The most obvious examples of strongly locally homogeneous spaces are locally euclidean spaces and zero-dimensional homogeneous spaces. Clearly, every connected strongly locally homogenous space is homogeneous.

By a result of Anderson, Curtis and van Mill [2], if X is strongly locally homogeneous and topologically complete, and if $D \subset X$ is countable and dense, then $Y = X \setminus D$ is strongly locally homogeneous and Y has the property that for every countable subset E of Y we have that $Y \approx Y \setminus E$ (i.e., Y is homeomorphic to $Y \setminus E$). Our example shows that in this theorem the assumption of topological completeness is essential. In fact, both conclusions are false in general since our example has the property that for some countable dense set D the complement of D admits no non-trivial embeddings. As we will show, our example is even Baire which shows that the above cited result of Anderson, Curtis and van Mill, in a sense, is the best possible.

2. PRELIMINARIES

If X is a space then Auth(X) denotes the group of autohomeomorphisms of X. The domain and range of a function f will be denoted by dom(f) and range(f), respectively.

Let Y be a fixed dense in itself, topologically complete space and let $\Phi \subset \mathrm{Auth}(Y)$ be a countable subgroup. For all $x \in Y$ let $V(x) = \{h(x) : h \in \Phi\}$. Define

$$\mathcal{F} = \{f : \mathrm{dom}(f) \text{ and } \mathrm{range}(f) \text{ are } G_\delta\text{-subsets of } Y \text{ and } f : \mathrm{dom}(f) \to \mathrm{range}(f) \text{ is a homeomorphism}\},$$

$$\mathcal{G} = \{f \in \mathcal{F} : |\{x \in \mathrm{dom}(f) : f(x) \notin V(x)\}| = 2^{\aleph_0}\},$$

respectively.

The following result was proven in detail for the special case $Y = \mathbb{R}^2$ in van Mill [5, section 3]. The reader can easily check that the only thing used in the proof was that $\mathbb{R}^2$ is topologically complete and dense in itself. We therefore state Theorem 2.1 without proof.

<u>THEOREM 2.1</u>. *For each* $f \in \mathcal{F}$ *there is a point* $x_f \in \mathrm{dom}(f)$ *such that if* $X = \bigcup_{f \in \mathcal{F}} V(x_f)$, *then*

(3) $f(x_f) \notin X$,

(2) *if* $f, g \in \mathcal{F}$ *are distinct, then* $V(x_f) \cap V(x_g) = \emptyset$,

(3) *if* $\mathcal{D}$ *is a family of countably many nowhere dense subsets of* X *and if* $U \subset X$ *is open and nonempty, then* $|U \setminus \bigcup\mathcal{D}| = 2^{\aleph_0}$. □

3. THE EXAMPLE

Let $Q \subset \mathbb{R}$ be the set of rational numbers and put $Q' = Q \setminus \{0\}$. Put

$$\Phi = \{h \in \mathrm{Auth}(\mathbb{R}) : \exists q \in Q\ \exists s \in Q'\ \forall x \in \mathbb{R} : h(x) = s \cdot x + q\}.$$

Observe that Φ is a countable subgroup of $\mathrm{Auth}(\mathbb{R})$ and that for all $x \in \mathbb{R}$ the set $V(x)$, defined in section 2, is equal to $Q' \cdot x + Q$. Let $Y = \mathbb{R}$ and let X be as in Theorem 2.1. We claim that X is as required.

LEMMA 3.1.
(1) *If* $K \subset \mathbb{R}$ *is a Cantor set, then both* $K \cap X \neq \emptyset$ *and* $K \cap (\mathbb{R} \setminus X) \neq \emptyset$,
(2) *if* $x \in \mathbb{R} \setminus X$ *then* $V(x) \subset \mathbb{R} \setminus X$.

PROOF. Let $K \subset \mathbb{R}$ be a Cantor set and let $L \subset \mathbb{R}$ be a Cantor set disjoint from $Q' \cdot K + Q$. If $h: K \to L$ is any homeomorphism, then $h \in G$ and consequently, by 2.1 (1), $X \cap K \neq \emptyset$. Since also $h^{-1} \in G$, again by 2.1 (1), $K \cap (\mathbb{R} \setminus X) \neq \emptyset$. This proves (1) and the trivial proof of (2) is left to the reader. □

Fix a point $x \in \mathbb{R} \setminus X$.

LEMMA 3.2.
(1) X *is zero-dimensional,*
(2) *if* $a,b,c,d \in V(x)$ *and if* $a < b$ *and* $c < d$ *then there is a homeomorphism* $h: X \to X$ *such that* $h([a,b] \cap X) = [c,d] \cap X$.

PROOF. (1) follows immediately since $V(x)$ is dense. For (2), find a point $s \in Q'$ and a point $q \in Q$ such that the homeomorphism $\tilde{h}: \mathbb{R} \to \mathbb{R}$ defined by $\tilde{h}(t) = s \cdot t + q$ has the property that $\tilde{h}([a,b]) = [c,d]$. Then $\tilde{h} \in \Phi$ and consequently, by the definition of X, we see that $\tilde{h} = h \mid X$ is as required. □

COROLLARY 3.3. X *has the property that all of its nonempty open subsets are homeomorphic; in particular,* X *is strongly homogeneous.*

PROOF. Let $U \subset X$ be nonempty and open. By 3.1 (1), U is not compact. Consequently, we can find a countably infinite collection $\mathcal{L}$ of nonempty intervals such that
(1) If $L \in \mathcal{L}$ then $\{\min L, \max L\} \subset V(x)$,
(2) if $L,M \in \mathcal{L}$ are distinct, then $(L \cap X) \cap (M \cap X) = \emptyset$, and

(3) $U = \bigcup_{L \in \mathcal{L}} L \cap X$.

By 3.2 (2), if $L, M \in \mathcal{L}$ are distinct, then $(L \cap X) \approx (M \cap X)$. In addition, for each $L \in \mathcal{L}$ we have that $L \cap X$ is a clopen subset of X. Therefore, again by 3.2 (2), we find that

$$U \approx ([x,x+1] \cap X) \times \mathbb{N}.$$

We therefore conclude that all nonempty open subspaces of X are homeomorphic. □

By 2.1 (2), we can find a countable dense set $D \subset X$ such that

(1) if $d,e \in D$ are distinct, then $d \notin V(e)$ (and $e \notin V(d)$).

(2) $D \cap \{x_f : f \in \mathcal{G}\} = \emptyset$.

We claim that $Y = X \setminus D$ admits no nontrivial embeddings.

<u>LEMMA 3.4</u>. *If* $h: Y \to Y$ *is an embedding, then* h = identity.

<u>PROOF</u>. The technique used in this proof is similar to the one in [5, 3.4 and 3.5].

<u>CLAIM</u>. $|\{y \in Y : h(y) \notin V(y)\}| < 2^{\aleph_0}$.

Suppose that this is not true. By the classical Lavrentieff Theorem [3], there are G_δ-subsets $S,T \subset \mathbb{R}$ with $Y \subset S$ and range(h) $\subset T$ such that h can be extended to a homeomorphism $f: S \to T$. Then $f \in \mathcal{G}$ and consequently, $x_f \in Y$. However, since f extends h, $h(x_f) = f(x_f) \notin X$, by 2.1 (1). This obviously is a contradiction.

For each $s \in Q'$ and $q \in Q$ put

$$A^s_q = \{y \in Y : h(y) = s \cdot y + q\}.$$

Observe that each A^s_q is closed in Y. Let B^s_q be the closure of A^s_q in X. Since D is countable, by the claim and by 2.1 (3), the union of the B^s_q's having nonempty interior in X is dense in X.

Take $s \in Q'$ and $q \in Q$ such that B^s_q has nonempty interior in X. Since Y is dense in X and since $B^s_q \cap Y = A^s_q$, this implies that A^s_q is not nowhere dense in Y. Since D is dense in X, the set

$$E = \{\tfrac{1}{s}(d-q) : d \in D\}$$

is also dense in X. Suppose that either $s \neq 1$ or $q \neq 0$. Then obviously $E \subset Y$ and consequently we can find a point $d \in D$ such that $\frac{1}{s}(d-q) \in A_q^s$. Since

$$h(\frac{1}{s}(d-q)) = d \notin Y,$$

this is a contradiction. We conclude that $s = 1$ and $q = 0$.

This argument shows that there is only one B_q^s with nonempty interior, namely B_0^1. Since B_0^1 is dense, h = identity. □

REMARK 3.5. In [6] it was shown that there is a homogenenous subset $A \subset \mathbb{R}$ such that

(1) $A \approx \mathbb{R} \setminus A$, and

(2) A does not admit the structure of a topological group.

It can be shown that there is also a countable dense subset $D \subset A$ such that $A \setminus D$ is rigid. However, A is not strongly homogenenous. So if one is willing to sacrifice strong homogeneity, it is possible to construct X as above with the additional curious property that $X \approx \mathbb{R} \setminus X$. With similar arguments as in [6] it can be shown that our space X constructed above also has the property that it is not a topological group.

REMARK 3.6. If one performs the above construction in the plane, one gets an example of a one dimensional, connected and locally connected space Z such that

(1) Z is strongly locally homogeneous,

(2) if $K \subset Z$ is compact, then $Z \approx K \setminus Z$,

(3) there is a countable dense set $D \subset Z$ such that $Z \setminus D$ does not admit a nontrivial embedding.

For details, see [5]. The argument given there that a space such as Z is strongly locally homogeneous can also be used to show that $Z \setminus \{pt\} \approx Z$ and with a little bit more work that $Z \setminus$ compact $\approx Z$ (use that Z contains no Cantor sets (Lemma 3.1 (1)). The details of working this out are left to the reader.

REFERENCES

[1] ANDERSON, R.D., *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. 80 (1958) 955-963

[2] ANDERSON, R.D., CURTIS, D.W. and van MILL. J., *A fake topological Hilbert space*, Trans. Amer. Math. Soc. 272 (1982) 311-321.

[3] LAVRENTIEFF, M., *Contribution à la théorie des ensembles homéomorphes*, Fund. Math. 6 (1924) 149-160.

[4] van MILL, J., *Characterization of some zero-dimensional separable metric spaces*, Trans. Amer. Math. Soc. 264 (1981) 205-215.

[5] van MILL, J., *Strong local homogeneity does not imply countable dense homogeneity*, Proc. Amer. Soc. 84 (1982) 143-148.

[6] van MILL, J., *Homogeneous subsets of the real line which do not admit the structure of a topological group*, Indag. Math. 85 (1982) 37-43.