

PROPERTY C AND FINE HOMOTOPY EQUIVALENCES

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ABSTRACT. We show that within the class of metric σ -compact spaces, proper fine homotopy equivalences preserve property C, which is a slight generalization of countable dimensionality. We also give an example of an open fine homotopy equivalence of a countable dimensional space onto a space containing the Hilbert cube.

1. Introduction. In this note we shall study the behaviour of some “dimensionality properties” of infinite-dimensional spaces under fine homotopy equivalences. Let us recall that a map $f: X \rightarrow Y$ is a *fine homotopy equivalence* if for every open cover \mathcal{U} of Y there exists a map $g: Y \rightarrow X$ such that $f \circ g$ is \mathcal{U} -homotopic to id_Y and $g \circ f$ is $f^{-1}(\mathcal{U})$ -homotopic to id_X . Let us mention that a closed map $f: X \rightarrow Y$ of an ANR X onto an ANR Y is a fine homotopy equivalence if: (a) all fibers of f are contractible or (b) f is a cell-like map, i.e. f is a proper map with fibers of trivial shape (see [Ha1 and To]). We are interested in countable dimensional spaces (a space X is *countable dimensional* if X is a countable union of finite dimensional sets) and spaces having property C (a metric space X has property C, abbreviated $X \in C$, iff given any sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers, there exists an open cover \mathcal{U} of X such that $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, where \mathcal{U}_n is a pairwise disjoint family with $\text{diam}(U) < \varepsilon_n$ for every $U \in \mathcal{U}_n$, $n \in \mathbf{N}$). Note that each metric, countable dimensional space has property C and that a space containing a topological copy of the Hilbert cube $Q = [-1, 1]^{\infty}$ does not have property C (for details see [Ha2]).

Because fine homotopy equivalences do not raise finite dimension, the following question was posed by D. Henderson and G. Kozłowski.

Question 1. Do cell-like maps, which are fine homotopy equivalences, preserve countable dimension?

In this note we will show that within the class of σ -compact spaces, *proper* fine homotopy equivalences preserve property C and we give an example of an *open* fine homotopy equivalence α of the space $\sigma = \{(x_i) \in l_2: x_i = 0 \text{ for all but finitely many } i\}$ onto the space $\Sigma = \{(x_i) \in l_2: \sum_{i=1}^{\infty} (ix_i)^2 < \infty\}$. The map $\alpha: \sigma \rightarrow \Sigma$ “raises” dimension because σ is countable dimensional but Σ contains the Hilbert cube Q and hence $\Sigma \notin C$.

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2. The main result. In this section we formulate and prove our main result.

2.1. THEOREM. *Let X be a σ -compact metric space with property C and let $f: X \rightarrow Y$ be a proper fine homotopy equivalence of X onto a metric space Y . Then $Y \in C$.*

PROOF. Because every space which is the countable union of compacta with property C, has property C itself, it is enough to prove that each compact subset of Y has property C. Let A be a compact subset of Y and let $B = f^{-1}(A)$. Let ρ be an extension on Y of a given metric on A . Define a compatible metric d on X by the formula $d(x_1, x_2) = \delta(x_1, x_2) + \rho(f(x_1), f(x_2))$, where δ is a compatible metric on X and $x_1, x_2 \in X$. Observe that $\rho(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for every $x_1, x_2 \in X$. Now choose a sequence $\{\varepsilon_n\}_1^\infty$ of positive real numbers. Since $X \in C$, there is an open cover \mathcal{V} of X such that $\mathcal{V} = \bigcup_{n=1}^\infty \mathcal{V}_n$, where \mathcal{V}_n is a pairwise disjoint family consisting of sets of diameter less than $\varepsilon_n/3$. Because B is compact we can choose a finite subfamily \mathcal{V}' of \mathcal{V} which covers B . Let $n_0 = \min\{n \in \mathbf{N}: \mathcal{V}' \subset \bigcup_{m \leq n} \mathcal{V}_m\}$ and let $W = \bigcup \mathcal{V}'$. Then $f(W)$ is a neighborhood of A in Y . Let $g: Y \rightarrow X$ be a map such that $\rho(f \circ g, \text{id}_Y) < \eta$, where $\eta = \frac{1}{3} \circ \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n_0}\}$, and $g(A) \subset W$. Then $\mathcal{Q} = g^{-1}(\mathcal{V}')$ is a cover of A . We will show that the cover \mathcal{Q} has the properties required in the definition of property C for the sequence $\{\varepsilon_n\}_1^\infty$ and the metric ρ . To this end, first observe that $\mathcal{Q} = \bigcup_{n=1}^{n_0} g^{-1}(\mathcal{V}_n \cap \mathcal{V}')$ and that $g^{-1}(\mathcal{V}_n \cap \mathcal{V}')$ is a pairwise disjoint family for $n = 1, 2, \dots, n_0$. Let $V \in \mathcal{V}_n \cap \mathcal{V}'$. We shall prove that $\text{diam}_\rho g^{-1}(V) < \varepsilon_n$. Take $y_1, y_2 \in g^{-1}(V)$ and for $i = 1, 2$ let $x_i = g(y_i)$. Then

$$\begin{aligned} \rho(y_1, y_2) &\leq \rho(y_1, fg(y_1)) + \rho(fg(y_1), fg(y_2)) + \rho(y_2, fg(y_2)) \\ &< 2\eta/3 + \rho(f(x_1), f(x_2)) \\ &\leq 2\eta/3 + d(x_1, x_2) < 2\eta/3 + \varepsilon_n/3 < \varepsilon_n. \end{aligned}$$

We conclude that $\text{diam}_\rho g^{-1}(V) < \varepsilon_n$. Observe that the cover $\mathcal{Q}' = \mathcal{Q} \cup \emptyset$ has the properties required in the definition of property C for the sequence $\{\varepsilon_n\}_{n=1}^\infty$ and the metric ρ . \square

REMARK. In the proof of the theorem we used only the fact that the map is approximately right invertible, i.e. given an open cover \mathcal{Q} of Y there exists a map $g: Y \rightarrow X$ such that $f \circ g$ is \mathcal{Q} -close to id_Y .

G. Kozłowski [K o] proved that a proper map $f: X \rightarrow Y$ between ANR's is a fine homotopy equivalence iff f is a hereditary shape equivalence, i.e., $\text{Sh}(f^{-1}(A)) = \text{Sh}(A)$ for each compact set A in Y . This result is used in the proof of the following

2.2. COROLLARY. *Let X be a σ -compact space with property C and let $f: X \rightarrow Y$ be a hereditary shape equivalence. Then Y has property C.*

PROOF. Without losing generality, we can assume that X and Y are compact. By the Freudenthal Expansion Theorem, see e.g. Borsuk [B o], X is the inverse limit of finite dimensional ANR's, say $X = \varprojlim \{X_n, f_n\}$, with each X_n an ANR. Let M be the infinite mapping cylinder of the sequence $\{X_n, f_n\}$ with a copy of X attached at its end. Then $M \in \text{ANR}$ and $M \in C$ (observe that we added a countable dimensional set to X). Let $\mathcal{G}_f = \{f^{-1}(y) : y \in Y\} \cup \{\text{points}\}$, then \mathcal{G}_f is a cell-like decomposition of M . Let $p_f: M \rightarrow M/\mathcal{G}_f$ be the quotient map. Because f is a hereditary shape

equivalence, $M/\mathcal{G}_f \in \text{ANR}$ and p_f is a fine homotopy equivalence [Ko]. By Theorem 2.1, $M/\mathcal{G}_f \in C$ and since Y embeds in M/\mathcal{G}_f , $Y \in C$. \square

3. The example. In this section we construct an example of an open fine homotopy equivalence of σ onto Σ .

3.1. EXAMPLE. There exists a map $\alpha: \sigma \rightarrow \Sigma$ such that:

- (1) α is “onto”,
- (2) α is open,
- (3) point inverses of α are homeomorphic to σ ,
- (4) α is a fine homotopy equivalence.

PROOF. Let $\beta: K \rightarrow Q$ be an open map of the universal Menger curve K onto the Hilbert cube such that $\beta^{-1}(q)$ is homeomorphic to K for each $q \in Q$ (see [An]). Let 2_f^K and 2_f^Q denote the hyperspaces of finite subsets of K and Q , respectively. By [Cu], 2_f^K is homeomorphic to σ and 2_f^Q is homeomorphic to Σ . Let $\alpha: 2_f^K \rightarrow 2_f^Q$ be the map defined by $\alpha(\{k_1, k_2, \dots, k_n\}) = \{\beta(k_1), \beta(k_2), \dots, \beta(k_n)\}$. Then α satisfies (1)–(4). The conditions (1) and (2) are satisfied because the map α is open and onto. We will check (3). Take distinct $q_1, q_2, \dots, q_n \in Q$, arbitrarily. Observe that

$$\alpha^{-1}(\{q_1, q_2, \dots, q_n\}) = \{A_1 \cup A_2 \cup \dots \cup A_n : A_i \subset \beta^{-1}(q_i) \text{ is finite and nonempty}\} \\ \approx \underbrace{2_f^K \times 2_f^K \times \dots \times 2_f^K}_n \approx \sigma^n \approx \sigma.$$

It is not hard to check that α is a UV^∞ -map, i.e., given $y \in \Sigma$ and a neighborhood U of y , there is a neighborhood $V \subset U$ of y such that $\alpha^{-1}(V)$ is contractible in $\alpha^{-1}(U)$. By [Ha1], α is a fine homotopy equivalence.

3. Questions. At the end of this note, we state some open problems.

Question 2. Let $f: X \rightarrow Y$ be a closed fine homotopy equivalence such that $X \in \text{ANR}$. If $X \in C$, does it follow that $Y \in C$? This is true for σ -compact X .

Question 3. Let $f: W \rightarrow V$ be an affine map of a σ -compact convex subset $W \subset l_2$ onto a subset $V \subset l_2$. If $W \in C$, does it follow that $V \in C$?

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