essential as it appears: if ORD^M has cofinality ω, then our theorem remains true. (Similarly, for $M \models$[Peano Arithmetic], if M has cofinality ω then our theorem goes through.) For suppose $(\alpha_n$: $n < \omega)$ is cofinal. Basically, at stage n we construct the p_s^i $(s \in {}^n2,\ i \in 2)$ so that every dense set is intersected, which is Σ_n definable with parameters in $R(\alpha_n)$ (the sets of rank $< \alpha_n$). This argument was previously carried out for arithmetic in Schmerl [4]. It is also shown there (Theorem 1.6) that if $\langle N_\nu\colon \nu \leqslant \alpha \rangle$ is a MacDowell–Specker chain, where $\mathrm{cf}(\alpha) > \omega$, then N_α has only one expansion to a model of predicative second-order extension $\Sigma_\infty^0 - \mathrm{CA}$ of PA. In a more recent paper Schmerl [5] has shown that in fact, if $S \subseteq |N_\alpha|$ and $\{x \in S\colon x <^{N_\alpha} a\}$ is definable in N_α for all $a \in |N_\alpha|$, then S is definable in N_α. (A similar result appears in Theorem 1.5 of [4], but only for regular cardinals α.)

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Orderability from selections: Another solution to the orderability problem

by

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Abstract. We prove that a Tychonov space X is a GO-space iff X admits a certain type of (weak) selection.

0. Introduction. All spaces under discussion are Tychonov.

A space is called *orderable* iff its topology is generated by a linear ordering. In addition, a space is called a *generalized ordered space* (abbreviated GO-space) iff there exists a linear order $\leqslant$ on X such that every point in X has arbitrary small $\leqslant$-convex neighborhoods. It is well known that the class of GO-spaces coincides with the class of subspaces of orderable spaces. As far as we know, the most general characterization of GO-spaces was given by van Dalen & Wattel [1]:

A space X is GO-space iff X possesses an open subbase consisting of two nests.

In this paper we will give quite a different characterization of GO-spaces, namely, we give a characterization in terms of selections. This generalizes results from our paper [3] where the compact case was treated.

1. Preliminaries. Let X be a space and let 2^X denote the hyperspace of nonempty closed subsets of X. A *selection* for X is a map $F\colon 2^X \to X$ such that $F(A) \in A$ for all $A \in 2^X$. Let $X(2)$ denote the 2-fold symmetric product of X, i.e. the subspace of 2^X consisting of all non-empty closed subspaces of X consisting of at most two points. A *weak selection* for X is a map $s\colon X(2) \to X$ such that $s(A) \in A$ for all $A \in X(2)$. It is easy to see that X has a weak selection if and only if there is a map $s\colon X^2 \to X$ such that for all $x, y \in X$,

$$s(x, y) = s(y, x), \tag{1}$$

and

$$s(x, y) \in \{x, y\}. \tag{2}$$

Such a map $s\colon X^2 \to X$ will also be called a *weak selection*.

Michael [2] showed that for a continuum X the following statements are equivalent: (a) X has a selection, (b) X has a weak selection, and (c) X is orderable. In [3], the authors showed that this result is also true without the assumption on connectivity.

Let $s\colon X^2 \to X$ be a weak selection. We call s *locally uniform* provided that for all $x \in X$ and for each neighborhood U of x there is a neighborhood V of x which is contained in U, such that for all $p \in X\backslash U$ and $y \in V$,

$$s(p, y) = p \quad \text{iff} \quad s(p, x) = p$$

(observe that this roughly means that the behaviour of s in the point x determines the behavior of s in some small neighborhood of x). In the remaining part of this paper, we will prove that X is a GO-space iff X has a locally uniform weak selection.

1.1. LEMMA. (a) *Let X be a* GO-*space. Then X admits a locally uniform weak selection,*

(b) *if X is compact and if $s\colon X^2 \to X$ is a weak selection, then s is locally uniform.*

Proof. For (a), observe that $s\colon X^2 \to X$ defined by $s(x, y) = \min\{x, y\}$ is as required. For (b), take $x \in X$ and let U be any open neighborhood of x. Define

$$A = ((X\backslash U) \times \{x\}) \cap s^{-1}(x)$$

and

$$B = ((X\backslash U) \times \{x\}) \cap s^{-1}(X\backslash U),$$

Observe that

$$A = ((X\backslash U) \times \{x\}) \cap s^{-1}(x)$$

and

$$B = ((X\backslash U) \times \{x\}) \cap s^{-1}(X\backslash U),$$

which implies that both A and B are closed. Let W be an open neighborhood of x with $x \in W \subset W^- \subset U$.

If $(p, x) \in A$ then, by continuity of s, we can find a neighborhood $B(p)$ of p and a neighborhood $V(p, x)$ of x such that

$$B(p) \times V(p, x) \subset s^{-1}(W).$$

By compactness we can find finitely many $p_1, \ldots, p_n \in X\backslash U$ such that

$$A \subset \bigcup_{i=1}^{n} B(p_i) \times V(p_i, x).$$

Similarly, we can find $q_1, \ldots, q_m \in X\backslash U$ and neighborhoods $C(q_i)$ of q_i and neighborhoods $U(q_i, x)$ of x such that

$$B \subset \bigcup_{i=1}^{m} C(q_i) \times U(q_i, x) \subset s^{-1}(X\backslash W^-).$$

Define

$$V = \bigcap_{i=1}^{n} V(p_i, x) \cap \bigcap_{i=1}^{m} U(q_i, x).$$

It is clear that V is as required.

Having the results of [3] in mind, in view of Lemma 1.1(b), it now seems easy to find a characterization of GO-spaces in terms of weak selections. Let X be a space having a locally uniform weak selection s. Find a compactification γX of X such that s can be extended to a weak selection $s^-\colon (\gamma X)^2 \to \gamma X$. Then, by [3], γX is orderable, whence X is a GO-space. Unfortunately, this procedure does not work, as the next example shows.

1.2. EXAMPLE. There is a GO-space X and a locally uniform weak selection s for X such that there does not exist a compactification γX of X with the property that s can be extended to a weak selection $s^-\colon (\gamma X)^2 \to \gamma X$.

Let $X = Z$ and define $s\colon Z^2 \to Z$ by

$$s(n, m) = \begin{cases} \min\{n, m\} & \text{if } n \neq -m, \\ \max\{n, m\} & \text{if } n = -m. \end{cases}$$

We claim that X and s are as required. First observe that s is locally uniform since X is discrete. Let γX be any compactification of X and take a point $\infty_1 \in N^-\backslash N$. Let $s^-\colon (\gamma X)^2 \to \gamma X$ be a weak selection extending s.

CLAIM. *If $t \in N^-\backslash N$ then $t = \infty_1$.*

If not, then there obviously exists disjoint sets $E, F \subset N$ with $\infty_1 \in E^-$, $t \in F^-$ and $E^- \cap F^- = \emptyset$. Take $n \in E$ arbitrarily. Since

$$(n, t) \in \{(n, m) |\ m \in F \,\&\, m > n\}^-,$$

and since $s(n, m) = n$ for all $m \in F$ with $m > n$, by continuity of s, we conclude that $s^-(n, t) = n$. This implies that

$$s^-(\infty_1, t) = \infty_1 \quad \text{for} \quad (\infty_1, t) \in \{(n, t) |\ n \in E\}^-.$$

The same argument yields $s^-(\infty_1, t) = t$, whence $\infty_1 = t$.

We conclude that N has a unique limit point ∞_1, and similarly we find that $X\backslash N$ has a unique limit point ∞_2.

Since $\lim_{n\to\infty} (-n, n) = (\infty_2, \infty_1)$, we find that

$$s^-(\infty_2, \infty_1) = \lim_{n\to\infty} n = \infty_1.$$

Similarly, since

$$\lim_{n\to\infty} (-n, n+1) = (\infty_2, \infty_1),$$

it follows that

$$s^-(\infty_2, \infty_1) = \lim_{n\to\infty} -n = \infty_2.$$

We conclude that $\infty_1 = \infty_2$ and hence that γX is simply the one point compactification of X. The point at infinity will now be called ∞.

Since $\lim_{n\to\infty}(1, n) = (1, \infty)$, we find that $s^-(1, \infty) = 1$. On the other hand, $\lim_{n\to\infty}(-n, 1) = (\infty, 1)$, which implies that $s^-(\infty, 1) = \infty$. Since $s^-(1, \infty) = s^-(\infty, 1)$, we have derived a contradiction.

2. A characterization of locally uniform selections. In this section we will prove that for weak selections, the internal property of being locally uniform is equivalent to one which is, in a sense, external. This reformulation of local uniformness is needed to make the concept applicable to prove the announced characterization of GO-spaces.

Throughout, βX denotes the Čech-Stone compactification of a space X. If $s\colon X^2 \to X$ is a weak selection then, for all $x \in X$, we define

$$A_x = \{y \in X \mid s(x, y) = x\} \quad \text{and} \quad B_x = \{y \in X \mid s(x, y) = y\},$$

respectively. (Formally we have to supply both A_x and B_x with an additional index s; since from the context it will always be clear which weak selection we mean, for notational simplicity we suppress the index s). Observe that both A_x and B_x are closed, that $A_x \cup B_x = X$ and finally that $A_x \cap B_x = \{x\}$.

2.1. LEMMA. *Let X be a space and let $s\colon X^2 \to X$ be a weak selection. If $p \in \beta X\backslash X$ and $x \in X$ then either $p \notin \mathrm{Cl}_{\beta X} B_x$ or $p \notin \mathrm{Cl}_{\beta X} A_x$.*

Proof. Suppose to the contrary that

$$p \in \mathrm{Cl}_{\beta X} B_x \cap \mathrm{Cl}_{\beta X} A_x.$$

Let $Z \subset X$ be a zero-set containing a neighborhood of x such that $p \notin \mathrm{Cl}_{\beta X} Z$.

Define $B = B_x \cup Z$ and $A = A_x \cup Z$. We claim that both A and B are zero-sets of X. Indeed, let $f\colon X \to [0, 1]$ be continuous such that $f^{-1}(0) = Z$. Define $g\colon X \to [0, 1]$ by

$$g(q) = \begin{cases} f(q) & (q \notin B_x), \\ 0 & (q \in B_x). \end{cases}$$

An easy check shows that g is continuous and that $g^{-1}(0) = B$. Consequently, B is a zero-set and similarly, A is a zero-set. Since

$$p \notin \mathrm{Cl}_{\beta X} Z = \mathrm{Cl}_{\beta X}(B \cap A) = \mathrm{Cl}_{\beta X} B \cap \mathrm{Cl}_{\beta X} A,$$

we may assume, without loss of generality, that $p \notin \mathrm{Cl}_{\beta X} B$. Since $B_x \subset B$, this shows that $p \notin \mathrm{Cl}_{\beta X} B_x$.

We now come to the main result of this section.

2.2. THEOREM. *Let X be a space and let $s\colon X^2 \to X$ be a weak selection. The following statements are equivalent:*

(1) *s is locally uniform,*

(2) *for all $p \in \beta X\backslash X$, s can be extended to a weak selection $s^-\colon (X \cup \{p\})^2 \to X \cup \{p\}$.*

Proof. Suppose first that $s\colon X^2 \to X$ is locally uniform. Take $p \in \beta X\backslash X$ arbitrarily. Define $t\colon (X \cup \{p\})^2 \to (X \cup \{p\})$ by

$$\begin{aligned} &t(p, p) = p, \\ &t(a, b) = t(b, a) = s(a, b) \quad \text{for} \quad a, b \in X, \\ &t(p, a) = t(a, p) = p \quad \text{if} \quad p \in \mathrm{Cl}_{\beta X} B_a \text{ and } a \in X, \\ &t(p, a) = t(a, p) = a \quad \text{if} \quad p \in \mathrm{Cl}_{\beta X} A_a \text{ and } a \in X. \end{aligned}$$

By Lemma 2.1, t is well defined. Since, modulo continuity, t is clearly a weak selection which extends s, we only need to verify continuity. To this end, put $Y = X \cup \{p\}$. It is clear that we only need to show that $t^{-1}(\mathrm{Cl}_Y Z)$ is closed in Y^2 for an arbitrary zero-set $Z \subset X$. Take $\langle a, b\rangle \notin t^{-1}(\mathrm{Cl}_Y Z)$. Since

$$t^{-1}(\mathrm{Cl}_Y Z) \cap X^2 = s^{-1}(Z) \cap X^2$$

and since X^2 is open in Y^2, by continuity of s we find that if $\langle a, b\rangle \in X^2$ then some neighborhood of $\langle a, b\rangle$ in Y^2 misses $t^{-1}(\mathrm{Cl}_Y Z)$. Therefore, without loss of generality, assume e.g. that $p = a$.

Case 1. $t(p, b) = p$ and $b \in X$. By Lemma 2.1, $p \notin \mathrm{Cl}_{\beta X} A_b$. Find a zero-set $S \subset X$ such that $\mathrm{Cl}_{\beta X} S$ is a neighborhood of p (in βX) which misses $\mathrm{Cl}_{\beta X} A_b \cup \mathrm{Cl}_{\beta X} Z$. By the local uniformness of s we can find a neighborhood V of b contained in $X\backslash S$ such that for all $x \in S$ and $v \in V$,

$$s(b, x) = b \Leftrightarrow s(v, x) = v.$$

Then $(\mathrm{Cl}_Y S) \times V$ is a neighborhood of $\langle p, b\rangle$ which misses $t^{-1}(\mathrm{Cl}_Y Z)$ (this needs some justification which we leave to the reader).

Case 2. $t(p, b) = b$ and $b \in X$. This case can be treated analogously.

Case 3. $p = a$ and $p = b$. Then $p \notin \mathrm{Cl}_Y Z$. So find a neighborhood U of p in Y such that $U \cap \mathrm{Cl}_Y Z = \emptyset$. Then $U \times U$ is a neighborhood of $\langle p, p\rangle$ in Y^2 which misses $t^{-1}(\mathrm{Cl}_Y Z)$.

Assume next that the weak selection $s\colon X^2 \to X$ is such that for every $p \in \beta X\backslash X$ there is a weak selection $t_p\colon (X \cup \{p\})^2 \to X \cup \{p\}$ extending s. We claim that s is locally uniform. To this end, let $x \in X$ and let U be a neighborhood of x in X. Choose a neighborhood W of x in X with $W^- \subset U$. Define

$$B = \{p \in \beta X\backslash X \mid t_p(p, x) = p\} \cup B_x$$

and

$$A = \{p\in\beta X\backslash X|\ t_p(p, x)=x\}\cup A_x.$$

Applying Lemma 2.1 and using the continuity of the map t_p for an arbitrary $p\in\beta X\backslash X$, the reader can easily check that

$$B=\mathrm{Cl}_{\beta X} B_x \quad \text{and} \quad A=\mathrm{Cl}_{\beta X} A_x.$$

Define

$$B' = B\cap \mathrm{Cl}_{\beta X}(X\backslash U) \quad \text{and} \quad A' = A\cap \mathrm{Cl}_{\beta X}(X\backslash U),$$

respectively. Take $p\in B'$. If $p\in X$ let $t_p=s$. By continuity of t_p, we can find a neighborhood $B(p)$ of p in βX and a neighborhood $V(p, x)$ of x in βX such that

$$(B(p)\cap(X\cup\{p\}))\times(V(p, x)\cap(X\cup\{p\}))\subset t_p^{-1}(X\backslash W).$$

By compactness of B' we can find finitely many points $q_1, \dots, q_m\in A'$ and neighborhoods $C(q_i)$ of q_i in βX and neighborhoods $U(q_i, x)$ of x in βX such that

$$A'\subset \bigcup_{j=1}^{m} C(q_j),$$

while moreover for each $1\leqslant j\leqslant m$ we have that

$$\langle C(q_j)\cap(X\cup\{q_j\})\rangle\times\langle U(q_j, x)\cap(X\cup\{q_j\})\rangle\subset t_p^{-1}(W).$$

Now define

$$V=\langle\bigcap_{i=1}^{n} V(p_i, x)\rangle\cap\langle\bigcap_{j=1}^{m} U(q_j, x)\rangle\cap X.$$

Then V is a neighborhood of x in X which is contained in U such that for all $p\in X\backslash U$ and $v\in V$ we have that

$$s(p, x)=x\Leftrightarrow s(p, v)=v.$$

Consequently, V is as required.

2.3. Remark. If s: $X^2\to X$ is a locally uniform weak selection and if $p\in\beta X\backslash X$ then by the previous result, s can be extended to a weak selection s^-: $(X\cup\{p\})^2\to X\cup\{p\}$. Simple examples show that the weak selection s^- need not be locally uniform.

2.4. Remark. Observe that Example 1.2 shows that the condition in Theorem 2.2(2) is best possible.

3. The construction. In this section we will prove the announced characterization of GO-spaces. If s: $X^2\to X$ is a weak selection and $x\in X$ then A_x and B_x are defined as in Section 2.

The proof of Theorem 3.1 below follows closely, but not literally the ideas in [3], Theorem 1.1, except for the last part where we have to give an additional argument that the total ordering $\leqslant$ we construct has the property that the open $\leqslant$-convex sets form a base for the topology. For the reader's convenience we give the proof in full detail.

3.1. Theorem. *Let X be a space. Then the following statements are equivalent.*

(1) *X has a locally uniform weak selection,*

(2) *X is a GO-space.*

Proof. The implication (2)$\Rightarrow$(1) was established in Lemma 1.1(a), so it suffices to prove that (1)$\Rightarrow$(2). To this end, let s: $X^2\to X$ be a locally uniform weak selection for X and, let $\prec$ be a wellordering on X. For every $x\in X$ we will construct closed sets L_x, $U_x\subset X$ such that

(1) $L_x\cup U_x=X$ and $L_x\cap U_x=\{x\}$,

(2) if $y\prec x$ and if $x\in L_y$ then $L_x\subset L_y\backslash\{y\}$,

(3) if $y\prec x$ and if $x\in U_y$ then $U_x\subset U_y\backslash\{y\}$,

(4) if $z\in L_x$ and if $z\notin\bigcup\{L_y|\ y\prec x\,\&\, x\in U_y\}$ then $z\in B_x$,

(5) if $z\in U_x$ and if $z\notin\bigcup\{U_y|\ y\prec x\,\&\, x\in L_y\}$ then $z\in A_x$.

(In the total ordering on X which we will construct in this proof, L_x will be the set of all points smaller than or equal to x, and U_x will be the set of all points larger than or equal to x.)

Let x_0 be the first element of X and define

$$L_{x_0}=B_{x_0} \quad \text{and} \quad U_{x_0}=A_{x_0}.$$

Assume that we have defined L_y and U_y for all $y\prec x$ satisfying (1) through (5). Let

$$E=\{y\prec x|\ x\notin L_y\} \quad \text{and} \quad F=\{y\prec x|\ x\notin U_y\}.$$

Put

$$Z=X\backslash(\bigcup_{y\in E} L_y\cup y\bigcup_{y\in F} U_y).$$

Let $k=|E|$ and for each $\xi\leqslant k$ define points $y_\xi\in E$ in the following way:

(6) $$y_0=\min(E),$$

(7) $$y_\zeta=\min\{y\in E\cup\{x\}|\ (y_\mu\prec y \text{ for all } \mu<\zeta)\,\&\,(y\notin\bigcup_{\mu<\zeta} L_{y_\mu})\}.$$

Let $\xi\leqslant k$ be the first ordinal for which $y_\xi=x$.

Claim 1. *If $\xi_0\leqslant\xi$ then* $\bigcup\{L_y|\ y\in E\,\&\, y\prec y_{\xi_0}\}=\bigcup_{\mu<\xi_0} L_{y_\mu}$.

Take $y\in\{z\in E|\ z\prec y_{\xi_0}\}\setminus\{y_\mu|\ \mu<\xi_0\}$ and let $\mu\leqslant\xi_0$ be the first ordinal for which $y\prec y_\mu$. Since $y_\varrho\prec y$ for all $\varrho<\mu$ (notice that $\mu\neq 0$) and since $y\neq y_\mu$, by (7), $y\in\bigcup_{\varrho<\mu}L_{y_\varrho}$. Choose $\varrho<\mu$ such that $y\in L_{y_\varrho}$. Since $y_\varrho\prec y$, by (2),

$$L_y\subset L_{y_\varrho}\subset\bigcup_{\delta<\xi_0}L_{y_\delta}.$$

CLAIM 2. *If $\mu_0<\mu_1<\xi$ then $L_{y_{\mu_0}}\subset L_{y_{\mu_1}}\setminus\{y_{\mu_1}\}$.*

By (7), $y_{\mu_1}\notin L_{y_{\mu_0}}$. Consequently, $y_{\mu_1}\in U_{y_{\mu_0}}$ and therefore, by (3), $U_{y_{\mu_1}}\subset U_{y_{\mu_0}}\setminus\{y_{\mu_0}\}$. Consequently, by (1), $L_{y_{\mu_0}}\subset L_{y_{\mu_1}}\setminus\{y_{\mu_1}\}$.

CLAIM 3. *If $\mu_0<\mu_1<\xi$ then $L_{y_{\mu_1}}\setminus L_{y_{\mu_0}}\subset A_{y_{\mu_0}}$.*

Take $t\in L_{y_{\mu_1}}\setminus L_{y_{\mu_0}}$. Since $t\in U_{y_{\mu_0}}$ and, by (7),

$$U_{y_{\mu_0}}\subset\bigcup\{U_y|\ y\prec y_{\mu_0}\ \&\ y_{\mu_0}\in L_y\}\cup A_{y_{\mu_0}},$$

we may assume, without loss of generality that $t\in U_z$ for certain $z\prec y_{\mu_0}$ with $y_{\mu_0}\in L_z$. Assume that $y_{\mu_1}\in L_z$. We will derive a contradiction. Since $y_{\mu_0}\prec y_{\mu_1}$ and since $z\prec y_{\mu_0}$ this implies by (2), that $L_{y_{\mu_1}}\subset L_z\setminus\{z\}$. Consequently, $t\in L_z\setminus\{z\}$ and $t\in U_z$, contradicting (1). This shows that $y_{\mu_1}\notin L_z$ which implies that $y_{\mu_1}\in U_z$. Since $z\prec y_{\mu_1}$, by (3), $U_{y_{\mu_1}}\subset U_z$ and therefore $x\in U_z$. If also $x\in L_z$ then $x=z$ which is impossible since $z\prec x$. We conclude that $x\notin L_z$ or equivalently, $z\in E$. Let $\varepsilon\leqslant\mu_0$ be the smallest ordinal such that $z\preccurlyeq y_\varepsilon$. Since $y_\delta\prec z$ for every $\delta<\varepsilon$ by (7), either $z=y_\varepsilon$ or $z\in L_{y_\delta}$ for certain $\delta<\varepsilon$. If $z=y_\varepsilon$ then $y_{\mu_0}\in L_{y_\varepsilon}$ which contradicts $z\prec y_{\mu_0}$ (Claim 2). Therefore, $z\in L_{y_\delta}$ for certain $\delta<\varepsilon$. Then $z\in L_{y_\delta}\subset L_{y_{\mu_0}}\setminus\{y_{\mu_0}\}$. Since $z\prec y_{\mu_0}$ and since $y_{\mu_0}\in L_z$, by (2), we also have that

$$L_{y_{\mu_0}}\subset L_z\setminus\{z\},$$

which implies that $z\in L_{y_{\mu_0}}\subset L_z\setminus\{z\}$, a contradiction.

CLAIM 4. *If $t\in\mathrm{Cl}_X\langle\bigcup_{y\in E}L_y\rangle\setminus\bigcup_{y\in E}L_y$ then t is a cluster point of the net $\{y_\mu|\ \mu<\xi\}$.*

Suppose not. Let U be a neighborhood of t which misses $\{y_\mu|\ \mu<\xi\}$. For each neighborhood V of t we choose a point

$$x(V)\in(U\cap V)\cap\bigcup_{y\in E}L_y$$

and we let $\mu(V)$ be the smallest ordinal such that $x(V)\in L_{y_{\mu(V)}}$ (such ordinal exists by Claim 1). We choose a clusterpoint p of

$$\{y_{\mu(V)}|\ V\text{ is a neighborhood of }t\}$$

in βX and by Theorem 2.2 we may extend s to a weak selection $s_p\colon(X\cup\{p\})^2\to X\cup\{p\}$ (if we have chosen the point p in X, which will happen e.g. if ξ is finite, then the argument below still works if we simply ignore the index p everywhere we write s_p). We will first prove that each $x(V)$ is

a point of $B_{y_{\mu(V)}}$. If this is not the case, then by (4) there is a $y\prec y_{\mu(V)}$ such that $x(V)$ in L_y and $y_{\mu(V)}\in U_y$. Since $y\prec y_{\mu(V)}$ and $y_{\mu(V)}$ in U_y, by (3),

$$U_{y_{\mu(V)}}\subset U_y\setminus\{y\}$$

which implies that $L_y\subset L_{y_{\mu(V)}}$. Consequently, $x\notin L_y$ or equivalently, $y\in E$. By Claim 1 we can find a $\delta<\mu(V)$ such that $x(V)\in L_{y_\delta}$ which contradicts the minimality of $\mu(V)$.

The point (t,p) is a cluster point of

$$\{(x(V),y_{\mu(V)})|\ V\text{ is a neighborhood of }x\}$$

in $(X\cup\{p\})^2$. Since each $x(V)\in B_{y_{\mu(V)}}$, by continuity of s_p we find that $s_p(t,p)=t$.

Fix $\varrho<\xi$ and consider the sequence

$$\{(x(V),y_\varrho)|\ V\text{ is a neighborhood of }t\text{ and }\varrho<\mu(V)\}.$$

By Claim 3 and the definition of $\mu(V)$ we find that

$$s\big((x(V),y_\varrho)\big)=y_\varrho$$

for each element of the sequence. By continuity of s this implies that $s(t,y_\varrho)=y_\varrho$. Since s_p is continuous and extends s we can now conclude that $s_p(t,p)=p$. Since $t\neq p$ we have derived a contradiction.

CLAIM 5. *If both t and u are cluster points of the net $\{y_\mu|\ \mu<\xi\}$ then $t=u$.*

Let C and D be closed and disjoint neighborhoods of, respectively, t and u. Define:

$$E=\{(y_\varrho,y_\mu)|\ y_\varrho\in D,\ y_\mu\in C\text{ and }\varrho<\mu\},$$

and

$$F=\{(y_\delta,y_\varepsilon)|\ y_\delta\in C,\ y_\varepsilon\in D\text{ and }\delta<\varepsilon\},$$

respectively. It is clear that (t,u) is a cluster point of E as well as F. If $(y_\varrho,y_\mu)\in E$ then, by Claim 3, $s(y_\varrho,y_\mu)=y_\varrho$, whence, by continuity of s, $s(u,t)=u$. In the same way, if $(y_\delta,y_\varepsilon)\in F$ then $s(y_\delta,y_\varepsilon)=y_\delta$ and consequently $s(t,u)=t$. This contradiction proves the claim.

CLAIM 6. *$\bigcup_{y\in E}L_y$ has at most one boundary point.*

Follows immediately from Claims 4 and 5.

CLAIM 7. *If $t\in Z$ and $\mu<\xi$ then $t\in A_{y_\mu}$.*

Since $t\notin L_{y_\mu}$ clearly $t\in U_{y_\mu}$. Therefore by (5), if $t\notin A_{y_\mu}$ then $t\in U_y$ for certain $y\prec y_\mu$ with $y_\mu\in L_y$. If $x\in L_y$ then $x\notin U_y$ since $x\neq y$ in which case $Z\cap U_y=\emptyset$ which contradicts $t\in Z\cap U_y$. Therefore $y\in E$. By Claim 1

$$\bigcup\{L_y|\ y\in E\ \&\ y\prec y_\mu\}=\bigcup_{\delta<\mu}L_{y_\delta}.$$

Therefore $y_\mu\in L_{y_\delta}$ for certain $\delta<\mu$ which contradicts (7).

Formally we have to consider two cases, namely that ξ is a successor or that ξ is a limit ordinal. Those two cases can be treated analogously and since the case that ξ is a limit is more complicated we will assume from now on that ξ is a limit.

Since $L_{y_\mu}\setminus\{y_\mu\}$ is open for each $\mu < \xi$ by Claims 1 and 2, $\bigcup_{y\in E} L_y$ can have at most one limit point, say a. By using precisely the same technique as above and again restricting our attention to the limit case we can find a limit ordinal η and for each $\mu<\eta$ a point $z_\mu \in F$ such that

(8) if $\mu < \delta$ then $U_{z_\mu} \subset U_{z_\delta}$,

(9) $\bigcup_{\mu<\eta} U_{z_\mu} = \bigcup_{y\in F} U_y$,

and

(10) if $t\in Z$ and $\mu<\eta$ then $t\in B_{z_\mu}$.

Again we find that $\bigcup_{y\in F} U_y$ has at most one boundary point, say b, and that this point is a cluster point of the net $\{z_\mu|\ \mu<\eta\}$.

Case 1. $a=b$. We then claim that $Z=\{x\}=\{a\}=\{b\}$. For assume that there exists a point $t\in Z\setminus\{a\}$. By Claim 7, $s(y_\mu, t)=y_\mu$ for all $\mu<\xi$ and consequently $s(a, t)=a$ since a is a limit point of $\{y_\mu\}_{\mu<\xi}$. On the other hand, by (10), $s(t, z_\mu)=t$ for all $\mu<\eta$. By the same argument $s(t, a) = s(t, b)=t$. Contradiction.

We therefore conclude that $a=b=x$ and that $Z=\{x\}$. Now define

$$L_x = \bigcup_{y\in E} L_y \cup \{x\} \quad \text{and} \quad U_x = \bigcup_{y\in F} U_y \cup \{x\}.$$

An easy check shows that our inductive hypotheses are satisfied.

Case 2. $x\neq a$ and $x\neq b$ if either a or b exists. Define

$$L_x = \bigcup_{y\in E} L_y \cup (Z\cap B_x) \quad \text{and} \quad U_x = \bigcup_{y\in F} U_y \cup (Z\cap A_x).$$

Observe that both L_x and U_x are closed since $a\in Z\cap B_x$ and $b\in Z\cap A_x$. Again an easy check shows that our inductive hypotheses are satisfied.

Case 3. $x=a$ and $x\neq b$ if b exists. Define

$$L_x = \bigcup_{y\in E} L_y \cup \{x\} \quad \text{and} \quad U_x = \bigcap_{\mu<\xi} U_{y_\mu}.$$

Case 4. $x=b$ and $a\neq x$ if a exists. Similar to case 3.

Now define $x\leqslant y$ iff $x\in L_y$. Then $\leqslant$ is a linear order. Moreover, for each $x\in X$ the sets $\{y\in X|\ y\leqslant x\}$ and $\{y\in X|\ x\leqslant y\}$ are closed. This means that X is weakly orderable w.r.t. the order $\leqslant$. So we only have to show that the space X has an open base consisting of convex sets w.r.t. $\leqslant$.

Suppose that $x\in X$ has a cozero-set neighborhood U such that U does not contain a convex open set containing x. Without loss of generality we may suppose that U does not contain any closed interval $[y, x]$ for a point $y\leqslant x$.

We construct a transfinite sequence $\{y_\zeta\}$ which now satisfies the conditions

(6) $y_0 = \min(E)$,

(7′) $y_\zeta = \min\{y\in E|\ (y_\varrho \prec y \text{ iff } \varrho<\zeta)\,\&\,(y\notin \bigcup_{\varrho<\zeta} L_{y_\varrho})\}$

if it exists. Let ξ be the first ordinal such that y_ξ does not exist.

We can now consider all points (t, y_ζ) in $(X\cup\{p\})^2$ with $t\leqslant y_\zeta$ and $t\in U$ such that there is no member of the sequence between t and y_ζ. From condition (4) it follows that $s(t, y_\zeta)=t$. If x is not an adherence point of the sequence y_ζ then we can consider the space $X\cup\{p\}$ for some $p\in\beta X$ which is in the closure of the sequence y_ζ. Without loss of generality we choose p such that p is in the closure of the chosen sequence y_ζ and then we may conclude that $s(x, p)=x$. If we consider all points (t, y_ζ) with $y_\zeta\leqslant t$ then we obtain that $s(x, p)=p$. (Compare with Claim 3.) We derived a contradiction in the same way as in Claim 4.

So we assume that x is in the closure of $\{y_\zeta\}$. A similar argument shows that x is the limit of the sequence $\{y_\zeta\}$.

Next we choose a point of βX in the closure of all intervals $[y, x]$ intersected with $X\setminus U$ and we derive a contradiction in the same way; but now we have that x is in the closure of the y_ζ and p in the closure of the t's.

Finally we conclude that the existence of a cozero-set neighborhood U which does not contain a right neighborhood of x leads to a contradiction, and so x has a local base consisting of convex open sets. This proves that X is a GO-space.

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