A uniquely homogeneous space need not be a topological group

by

Jan van Mill (Amsterdam)

Abstract. A (separable metric) space $X$ is called uniquely homogeneous provided that for all $x, y \in X$ there is a unique homeomorphism of $X$ taking $x$ onto $y$. Each uniquely homogeneous space admits a natural group structure for which all left translations are homeomorphisms. We present an example of a uniquely homogeneous space $X$ for which this group structure is Abelian, which implies that both left and right translations are homeomorphisms, such that $X$ does not admit the structure of a topological group.

6. Introduction. All spaces under discussion are separable metric. A space $X$ is called uniquely homogeneous provided that for all $x, y \in X$ there is a unique homeomorphism of $X$ taking $x$ onto $y$. Barit and Renaud [1], using a result of Ungar [14], showed that a uniquely homogeneous space cannot be compact, or locally compact and locally connected. The author presented in [9] an example of a connected and locally connected topological group $G$ which is Boolean (each element of the group has order at most 2), and which has the additional property that each automorphism of $G$ is a translation. An easy consequence of these properties of $G$ is that $G$ is uniquely homogeneous.

Ungar [14] defines a natural group structure on a uniquely homogeneous space $X$ for which it is easily seen that all left translations of $X$ are homeomorphisms, and shows that if $X$ is compact, or locally compact and locally connected, then this group structure makes $X$ into a topological group. In addition he showed that the group of topological isomorphisms of $X$ is trivial. Subsequently, Barit and Renaud [1] showed that a locally compact group different from $\mathbb{R}$ or $\mathbb{Z}_2$ always has a nontrivial topological isomorphism. This gave a proof of the result that a uniquely homogeneous space containing more than one point cannot be compact, or locally compact and locally connected.

In view of the above remarks and in view of the fact that all known examples of uniquely homogeneous spaces are topological groups, the question naturally arises whether every uniquely homogeneous space admits the structure of a topological group. We will answer this question in the negative by constructing an example of a uniquely homogeneous semitopological group which does not admit the structure of a topological group. This answers a question raised in [9].
1. Preliminaries. If $G$ is an Abelian group then we let $\cdot +$ denote the group operation on $G$. We regard $\cdot +$ to be a function of $G \times G$ onto $G$ and for $(x, y)$ we write $x + y$ of course. If $K \subseteq G \times G$ then the restriction $\cdot +|_K$ will also be denoted by $\cdot +$.

A group is called Boolean if each element of the group has order at most 2. Observe that a Boolean group is Abelian. Let $G$ be a group. If $A \subseteq G$ then $\langle A \rangle$ denotes the subgroup of $G$ generated by $A$. Let $G$ be a Boolean group and let $H \subseteq G$ be a subgroup. If $x \in G$ then it is easily seen that

$$\langle x \rangle = H \cup (x + H).$$

A group $(G, \cdot)$ equipped with a topology is called a semitopological group if the operation $\cdot : G \times G \rightarrow G$ is continuous in each variable separately. It is known that if $G$ is a semitopological group then $G$ is a topological group if $G$ is Baire, see Husain [5, p. 38].

A cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. $\kappa$ denotes $2^{\aleph_0}$.

The domain and range of a function $f$ will be denoted by $\text{dom}(f)$ and $\text{range}(f)$, respectively. Observe that the cardinality of the collection of all $G_\tau$-subsets of a given space is at most $\kappa$. This implies that if $X$ and $Y$ are spaces, then the collection

$$\mathcal{P} = \{f : \text{dom}(f) \to G_\tau \text{is a $G_\tau$-subset of $X$ and range}$$(f) \subseteq Y\}$$

has cardinality at most $\kappa$.

Let $X$ be a space. A subset $U \subseteq X$ is called regular open provided that $U = \text{int} \text{cl}_U(U \cap Y)$, and

$$U = \text{int}(X \setminus U).$$

Let $\mathcal{B} \subseteq RO(X)$ be a subalgebra. The Stone space of $\mathcal{B}$, denoted by $\text{st}(\mathcal{B})$, has as underlying set the set of all ultrafilters in $\mathcal{B}$. If $U \in \mathcal{B}$ then $U = \{\beta \in \text{st}(\mathcal{B}) : U \in \beta\}$. The topology of $\text{st}(\mathcal{B})$ is generated by the collection

$$\mathcal{U} = \{U : U \in \mathcal{B}\}.$$

The sets $\mathcal{U}$ are clopen (= closed and open) and $\text{st}(\mathcal{B})$ is compact. The reader should observe that if $\mathcal{B}$ is countable then $\text{st}(\mathcal{B})$ is metrizable and that then for the definition of $\text{st}(\mathcal{B})$ one does not need the full strength of the axiom of choice. Since $\mathcal{B}$ is countable, if $\gamma \in \mathcal{B}$ is a filter then $\gamma$ can be extended to an ultrafilter $\beta \in \mathcal{B}$ by a process of countably many steps.

If $X$ is a space, then $H(X)$ denotes the group of all autohomeomorphisms of $X$. The identity of $X$ will be denoted by $\text{id}$.

If $X$ is a space, and if $A \subseteq X$ then $\text{cl}_A$ and $\overline{A}$ denote the closure of $A$ in $X$.

2. Unique homogeneity. In this section we will show that each uniquely homogeneous space admits a very natural group structure such that all left translations are homeomorphisms. The results in this section are included for completeness sake; they are well-known and easy to prove, see Ungar [14].

2.1. Lemma. Let $X$ be homogeneous. The following statements are equivalent:

(1) $X$ is uniquely homogeneous,

(2) if $f, g \in H(X)$ and $f(x) = g(x)$ for certain $x \in X$, then $f = g$,

(3) if $f \in H(X)$ and $f$ has a fixed point, then $f = \text{id}$.\n
Proof. (1) $\Rightarrow$ (2). Since the identity is a homeomorphism taking $x$ onto $x$, by $(1)$, $f = \text{id}$.

(2) $\Rightarrow$ (3). Suppose that $f(x) = x$ for certain $f \in H(X)$ and $x \in X$. Since $f(x) = x$ for certain $f \in H(X)$ and $x \in X$. Since the identity is a homeomorphism taking $x$ onto $x$, by $(1)$, $f = \text{id}$.

(3) $\Rightarrow$ (2). Let $h = g^{-1}f$. Then $h(x) = x$ and therefore, by $(3)$, $h = \text{id}$. We conclude that $f = g$.

(2) $\Rightarrow$ (1). Take $x, y \in X$. Since $X$ is homogeneous, there is an $f \in H(X)$ with $(x) = y$. By $(2)$ this $f$ is unique.\n
Let $X$ be uniquely homogeneous. Fix $x \in X$ and for each $x \in X$ let $f_x$ be the unique homeomorphism taking $e$ onto $x$. Define a binary operation $+_{-}$ and an operation $\cdot_{-}$ on $X$ by

$$x +_{-} y = f_x(y), \quad \text{and} \quad -x = f_x^{-1}(e).$$

2.2. Lemma. Let $x, y, z \in X$. Then

(1) $x + (y + z) = (x + y) + z$,

(2) $x + e = x = x + e$,

(3) $x + (-x) = (-x) + x = e$.

Proof. (1) Observe that

$$(f_x f_y)(e) = f_y(f_x(e)) = f_y(x) = x + y,$$

and

$$f_{x + y}(e) = x + y.$$ By Lemma 2.1(2) we therefore conclude that $f_x f_y = f_{x + y}$. Since

$$x + (y + z) = x + f_x(z) = f_x(f_y(z)) = f_x(f_y(x))$$
and
$$(x + y) + z = f_{x + y}(z),$$

we conclude that $x + (y + z) = (x + y) + z$.\n
(2) trivial.

(3) $x + (-x) = f_x(f_x^{-1}(e)) = e$. Notice that $f_{-x}(e) = -x = f_x^{-1}(e)$. Lemma 2.1(2) therefore implies that $f_{-x} = f_x^{-1}$. Consequently, $(-x) + x = f_{-x}(x) = f_x^{-1}(x) = e$.\n
We conclude that $+$ is a very natural group operation on $X$.

2.3. Lemma. All left translations of $X$ are homeomorphisms.

Proof. Fix $x \in X$ and define $l_x : X \to X$ by $l_x(y) = x + y$. Then $l_x(e) = f_x(e) = f_x^{-1}(e)$ for all $x \in X$. We conclude that $l_x = f_x$.
3. A theorem. The aim of this section is to prove Theorem 3.1 below, which is the key in the construction of our example. The proof of this result is similar to the proof of Theorem 3.1 in [9] except for some minor changes. Since these changes are not always obvious, we will give the proof in full detail.

3.1. Theorem. Let $X$ be a Boolean semitopological group with the following properties

(a) $X = \bigcup_{n=1}^{\infty} X_n$, where each $X_n$ is a topologically complete subgroup of $X$, and $X_n \leq X_{n+1}$, for all $n \in \mathbb{N}$.

(b) if $x, y \in X$ then there exist $n \in \mathbb{N}$ and a disc $D \subseteq X_n$ containing both $x$ and $y$.

If $E \subseteq X$ is a countable subgroup, then there is a dense subgroup $H \subseteq X$ such that the automorphisms of $H$ are precisely its translations. Consequently, $H$ is uniquely homogeneous.

Proof. Let $\mathcal{A}^X$ be a compactification of $X$. Let $\mathcal{G} = \{ f : \text{dom}(f) \text{ and range}(f) \}$ be the $\mathcal{G}_\delta$-subsets of $\mathcal{A}^X$. Since $|\mathcal{G}| \leq \mathfrak{c}$, we can enumerate $\mathcal{G}$ by $\{ f_\alpha : \alpha < \mathfrak{c}, \alpha \text{ even} \}$. Let $\{ K_\alpha : \alpha < \mathfrak{c}, \alpha \text{ even} \}$ enumerate all Cantor sets in $X$. By transfinite induction, for every $\alpha < \mathfrak{c}$ we will construct subgroups $H_\alpha \subseteq X$ and subsets $V_\alpha \subseteq \mathcal{A}^X$ such that

1. $\beta < \alpha$ then $E \subseteq H_\beta \subseteq H_\alpha$, $V_\beta \subseteq V_\alpha$, and $H_\alpha \cap V_\alpha = \emptyset$.
2. $|H_\alpha| \leq \mathfrak{c}$ and $|V_\alpha| \leq \mathfrak{c}$.
3. if $\alpha$ is odd then $H_{\alpha} \cap K_\alpha = \emptyset$.
4. if $\alpha$ is even and if $\{ x \in \text{dom}(f_\alpha) \setminus X : f_\alpha(x) \notin \bigcup_{\beta < \alpha} H_{\beta} \cup \{ x \} \} = \emptyset$ then there is a point $x \in \text{dom}(f_\alpha) \setminus H_\alpha$ such that $f_\alpha(x) \in V_\alpha$.

Suppose that we have completed the construction for all $\beta < \alpha$, where $\alpha < \mathfrak{c}$. For convenience, put $H^* = \bigcup_{\beta < \alpha} H_{\beta}$ and $V_* = \bigcup_{\beta < \alpha} V_{\beta} \cap X$. Observe that $|H^*| < |\mathcal{A}^X|$. Suppose that $\mathcal{A}^X$ does not exist.

Case 1. $\alpha$ is odd. Since $|K_\alpha| = \mathfrak{c}$, we can pick a point $x \in K_\alpha \setminus (H^* + V_*)$.

Define $H_\alpha = \langle H_{\alpha} \cup \{ x \} \rangle$ and $V_\alpha = \bigcup_{\beta < \alpha} V_{\beta}$. An easy check shows that $H_\alpha$ and $V_\alpha$ are as required.

Case 2. $\alpha$ is even and $|S| < \mathfrak{c}$ where $S = \{ x \in \text{dom}(f_\alpha) \setminus X : f_\alpha(x) \notin \mathcal{A}^X \cup \{ x \} \}$. Define $H_\alpha = H^*$ and $V_\alpha = \bigcup_{\beta < \alpha} V_{\beta}$.

Case 3. $\alpha$ is even and $|S| = \mathfrak{c}$. By the same argument as in Case 1, we can find a point $x \in S \setminus (H^* + V^*)$.

Define $H_\alpha = \langle H_{\alpha} \cup \{ x \} \rangle$ and $V_\alpha = \bigcup_{\beta < \alpha} V_{\beta}$. Since $H_\alpha \cap V_\alpha = \emptyset$, we see that $H_\alpha$ and $V_\alpha$ are as required.

Now put $H = \bigcup_{\alpha < \mathfrak{c}} H_\alpha$. We claim that $H$ is as required. Let $f : H \to H$ be a homeomorphism. Our task is to find a point $h \in H$ such that $f(x) = x + h$ for all $x \in H$.

By the well-known Laurentian theorem ([6]; see also [3, Thm. 4.5.3]), there are $G_\delta$-subsets $S$ and $T$ of $\mathcal{A}^X$ such that $f$ can be extended to a homeomorphism $f^* : S \to T$. Let $\alpha < \mathfrak{c}$ be such that $\alpha = f_*^\prime f_*$. Then, there is a point $x \in S$ such that $f_\alpha(x) \notin f_*^\prime H_*^\alpha$. Since $f_\alpha$ extends $f_*$, this is impossible, by the assumption $\mathcal{A}^X < \mathfrak{c}$.

Case 2. Not Case 1. Let $H^* = \bigcup_{\beta < \alpha} H_{\beta}$ and $U = \{ x \in S \setminus X : f_\alpha(x) \notin \bigcup_{\beta < \alpha} H_{\beta} \cup \{ x \} \}$. By assumption, $|U| < \mathfrak{c}$. For each $h \in H^*$ define $E_h = \{ x \in S \setminus X : f_\alpha(x) = x + h \}$.

We claim that $E_h$ is closed in $S$. For all $n \in \mathbb{N}$ take $x_n \in E_h$ and $x \in S$ such that $\lim x_n = x$. Then $f_\alpha(x_n) = \lim f_\alpha(x_n) = (x_n + h) = x + h$. (This uses of course only the fact that $X$ is semitopological). In addition the collection $\mathcal{F} = \{ E_h : h \in H^* \}$ is clearly pairwise disjoint. For each $h \in H^*$ let $F_h = E_h \cup f_\alpha^{-1} f_\alpha^{-1} H^*$. We conclude that the collection $\mathcal{F} = \{ F_h : h \in H^* \}$ is also pairwise disjoint. We claim that at least one set of the collection $\mathcal{F}$ is nonempty. To the contrary, suppose that there exist distinct points $x, y \in H^*$ and points $x \in F_h$ and $y \in F_k$. We first claim that $X \setminus S$ is countable. If not, then there is an $n \in \mathbb{N}$ such that $S \cap X_n$ is uncountable. Since $X_n \setminus S_n$ is a countable union of closed subsets of $X_n$, one of these closed sets must be uncountable and therefore contains a Cantor set since $X_n$ is topologically complete, [4] (see also [3, 4.5.3]). This is impossible since $H^\alpha \subseteq H^\beta$ intersects all Cantor sets. Find an $n \in \mathbb{N}$ and a disc $D \subseteq X_n$ containing both $x$ and $y$. Since $\{ x \} < \mathfrak{c}$ and $\{ f_\alpha^{-1} f_\alpha^{-1} H^* \} < \mathfrak{c}$, we can choose an arc joining $D$ which connects $x$ and $y$ but misses $U \cup f_\alpha^{-1} f_\alpha^{-1} H^* \setminus X \setminus S$. We conclude that $S \subseteq \bigcup_{\beta < \alpha} H_{\beta}$.

Put $K = \{ h \in (H^*) \cup J \neq \emptyset \}$. By assumption, $|K| > \mathfrak{c}$, whence $\{ J \cap E_h : h \in K \}$ is a partition of $K$ in at least $2$ closed and disjoint, nonempty sets. Notice that $E_h \cap J$ is closed in $J$ since $J \cap S$. By Sierpinski’s theorem, [11] (see also [3, Thm. 5.3.2]), $|K| > \mathfrak{c}$. Also, $|J| < |H^*| < \mathfrak{c}$ (so in case the Continuum Hypothesis holds, we have derived a contradiction). Find an $m > \mathfrak{c}$ such that $K \cap X_m$ is uncountable. Then $K \cap X_m$ is not closed in $X_m$ since $X_m$ is topologically complete and $\mathfrak{c}$.
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4.2. Lemma. If \( F \cap N \) is finite and if \( N \cap F = \emptyset \), then there is an \( n \leq k \) such that \( B \) contains two points \( n \) and \( m \) such that \( n \leq m < 2n \).

Proof. Choose \( r \in N \) such that \( r > \max F \) and \( r > k \). There are \( x, y \in N \) with \( x < y < 2r \) and there is an \( n \leq k \) with \( \{x, y\} \subseteq B \). It is clear that \( x < y < 2k \).

4.3. Proposition. If \( \mathcal{E} \) is a finite family of subsets of \( A \) such that for each \( E \in \mathcal{E} \) we have that \( E \cup A \subseteq A \), then there are infinitely many \( n \in N \) with \( A_n \cap \emptyset \neq \emptyset \).

Proof. Suppose that \( F = \{n \in N: A_n \cap \emptyset \neq \emptyset\} \) is finite. By Lemma 4.2 there is an \( E \in \mathcal{E} \) and there are \( n, m \in N \) such that \( n < m < 2n \) and \( E \cap A \neq \emptyset \). Since by assumption \( E \cup A \subseteq A \), this contradicts Lemma 4.1.

4.4. Lemma. There is a countable subalgebra \( \mathcal{B} \subseteq RO(K) \) such that

\begin{enumerate}
\item \( C \subseteq K \) is clopen then \( C \in \mathcal{B} \),
\item \( \mathcal{A} \in \mathcal{B} \),
\item \( \mathcal{A} \in \mathcal{B} \),
\end{enumerate}

Proof. Let \( \mathcal{B} \) be the smallest subalgebra of \( RO(K) \) containing all clopen subsets of \( K \) and the set \( A \). Observe that \( \mathcal{B} \) is countable. Inductively construct countable subalgebras \( \mathcal{B}_{n+1} \subseteq RO(K) \) such that for all \( n \in N \),

\begin{enumerate}
\item \( \mathcal{B}_{n+1} \subseteq \mathcal{B}_n \),
\item \( \mathcal{A} \in \mathcal{B}_{n+1} \),
\end{enumerate}

Suppose that the subalgebras are constructed for all \( i \in \mathbb{N} \). Let \( \mathcal{B}_{i+1} \) be the smallest subalgebra of \( RO(K) \) containing the collection \( \{x \in A: x \in G \text{ and } B \in \mathcal{B}\} \). It is clear that \( \mathcal{B}_{i+1} \) is countable.

Now put \( \mathcal{B} = \bigcup \mathcal{B}_n \). An easy check shows that \( \mathcal{B} \) is as required.

For each \( x \in G \) define \( f_x: K \to K \) by \( f_x(t) = x + t \). Let \( X = st(\mathcal{A}) \). Define \( \varphi_x: X \to X \) by \( \varphi_x(\beta) = (f_x(\beta): B \in \mathcal{B}) \). By Stone duality, \( \varphi_x \) is a homeomorphism.

Consider the following subcollection of \( \mathcal{B} \):

\[ \gamma = \{B: B \in \mathcal{A}, B \subseteq A \text{ and } B + B \subseteq A\} \cup \{C \subseteq K: C \text{ is clopen and } A \cap C \subseteq \{x \in G \}, B \}

4.5. Lemma. If \( \mathcal{A} \subseteq \gamma \) is finite, then \( \bigwedge \mathcal{A} \neq \emptyset \).

Proof. Without loss of generality, \( \mathcal{A} = \{B_1, \ldots, B_n\} \cup \{A\} \cup \{C \subseteq K: C \text{ is clopen and } A \subseteq C\} \). Where for each \( i < m \), we have that \( B_i \cup (B_i + B_i) \subseteq A \), and \( C \) is a clopen neighborhood of \( A \).

By Proposition 4.3, the set \( \{n \in N: A_n \cap B_i = \emptyset\} \) is infinite. Since \( C \) contains almost all of the \( A_n \)'s, we conclude that \( \bigwedge \mathcal{A} \neq \emptyset \).

By the above Lemma we can extend the collection \( \gamma \) to an ultrafilter \( \beta \subseteq \mathcal{B} \).

Whence \( \beta \in st(\mathcal{A}) = X \). Define

\[ Y = \{\varphi_x(\beta): x \in G\} \]

We topologize \( Y \) by regarding it to be a subspace of \( X \).

4.6. Lemma. If \( \varphi_x(\beta) = \varphi_y(\beta) \) then \( x = y \).
Proof. Suppose that \( x \neq y \). Find a clopen neighborhood \( C \) of \( O_x \) such that \((x+C) \cap (y+C) = \emptyset\). Since \( x+C = f_x(C) \in \phi_x(\beta) \) and \( y+C = f_y(C) \in \phi_y(\beta) \), we find that \( \phi_x(\beta) \neq \phi_y(\beta) \), which is a contradiction. Define a binary operation \( \ast \) on \( Y \) by \( \phi_x(\beta) \ast \phi_y(\beta) = \phi_{x+y}(\beta) \). By Lemma 4.6, the operation \( \ast \) is well defined. For notational simplicity, the point \( \phi_x(\beta) \) will be denoted by \( t_x \) from now on.

4.7. Lemma. \((Y, \ast)\) is a Boolean group with identity \( \beta \). In addition, all translations of \( Y \) are (topological) homeomorphisms of \( Y \).

Proof. That \((Y, \ast)\) is a Boolean group with identity \( \beta \) is trivial. Take \( t_x \in Y \) and consider the translation \( \xi(t_x) = t_x \ast t_y \). Observe that

\[
\xi(t_x) = t_x + t_y = \phi_{x+y}(\beta) = \phi_{\xi(t_x)}(\beta) = \phi(t_x)
\]

for all \( y \in Y \). Consequently, \( \xi = \phi(t_x) \) since by construction \( \phi(t_x) \) is an auto-homeomorphism of \( Y \). We conclude that \( \xi \) is an auto-homeomorphism of \( Y \).

It is easily seen that \( Y \) has no isolated points. Since \( Y \) is dense in \( X \), we conclude that \( Y = \mathcal{Q} \) is a homeomorphic copy of \( X \).

4.8. Theorem. There is a Boolean group structure \( \ast \) on \( \mathcal{Q} \) such that

1. \( \ast : Q \times Q \to Q \) is nowhere discontinuous,
2. all translations of \( \mathcal{Q} \) are (topological) homeomorphisms.

Proof. It clearly suffices to show that \( : Y \times Y \to Y \) is not continuous in \((\beta, \beta) \). We claim that \( : (\beta, \beta) \) is not a neighborhood of \((\beta, \beta) \) in \( Y \times Y \). Indeed, to the contrary assume that there is a neighborhood \( B \) of \( \beta \) in \( X \), where \( B \) is a neighborhood of both \( x \) and \( y \), so

\[
(t_x, t_y) \in (B \times Y) \times (B \times Y)
\]

and consequently, \( t_x + t_y \in A \), or, equivalently, \( \phi_{x+y}(\beta) \in A \). Since \( \phi_{x+y}(\beta) = (x+y) + E \in B \), there is an \( E \in \beta \) such that \((x+y) + E = A \). Since \( O_x \in E \), this implies that \( (x+y) + O_x \in A \). We conclude that \( x+y \in A \). Since \( B \) is a neighborhood of \( \beta \) and \( B \) is dense in \( B \), we therefore have that \( B + B \subseteq A \). By definition of \( \beta \), \( B' \subseteq B \). Since by assumption \( B \neq \beta \), this is a contradiction.

5. Spaces of measurable functions. Let \( X \) be a space. A function \( f : [0, 1] \to X \) is said to be measurable if \( f^{-1}(U) \) is a Borel subset of \( [0, 1] \) for every open subset \( U \subseteq X \). Measurable functions \( f, g : [0, 1] \to X \) are called equivalent if

\[
\lambda\{t \in [0, 1] : f(t) \neq g(t)\} = 0,
\]

where \( \lambda \) denotes Lebesgue measure on \([0, 1]\). Let \( M_X \) be the topological space of equivalence classes of measurable functions from \([0, 1]\) into \( X \) with the topology of convergence in measure. The topology of \( M_X \) is determined by the metric

\[
\delta(f, g) = \frac{1}{2} \int (d(f(t), g(t)))^2 dt^{1/2},
\]

where \( d \) is any bounded metric compatible with the topology of \( X \). The topology of \( M_X \) does not depend on the chosen metric \( d \) on \( X \). Besaga and Pelczyński [2] show that \( M_X \approx l_1 \), the separable Hilbert space, if and only if \( X \) is completely metrizable and contains more than one point. It is easily seen that the set of constant functions is closed in \( M_X \) and isometric to \( X \). We identify \( X \) and this isometric copy of \( X \) in \( M_X \).

Let \( * \) be the Boolean group structure on \( \mathcal{Q} \) given by Theorem 4.8. Define a Boolean group structure, which we will also denote by \( * \), on \( M_X \) as follows:

\[
(f \ast g)(t) = f(t) \ast g(t).
\]

It is trivial to verify that \( * \) is indeed a Boolean group structure on \( M_X \) and that \( \mathcal{Q} \) is a subgroup of \( M_X \). It is also easily seen that all translations of \( M_X \) are homeomorphisms, whence \( M_X \) is a semi-topological group which is not a topological group since \( \mathcal{Q} \) is a subgroup, see Besaga and Pelczyński [2]. Define

\[
X = \{ f \in M_X : f([0, 1]) \text{ is finite} \},
\]

5.1. Lemma. \( X \) is a subgroup of \( M_X \) and \( \mathcal{Q} \subseteq X \).

Proof. If \( f, g \in X \) then \( (f \ast g)([0, 1]) \) is contained in the subgroup generated by \( f([0, 1]) \) and \( g([0, 1]) \). This subgroup is clearly finite since \( Q \) is a Boolean group. That \( \mathcal{Q} \subseteq X \) is trivial.

5.2. Lemma. If \( F \subseteq Q \) is a finite subgroup, then \( B(F) = \{ f \in M_X : f([0, 1]) \subseteq F \} \) is a subgroup of \( M_X \) which is homeomorphic to \( M_F \).

Proof. Obvious.

We will identify \( B(F) \) and \( M_F \).

5.3. Lemma. \( X = \bigcup_{n=1}^{\infty} X_n \), where \( X_n \) is a topologically complete subgroup of \( X \) and \( X_n \subseteq X_{n+1} \) for all \( n \in \mathbb{N} \). Moreover, if \( x, y \in X \) then there are \( n \in \mathbb{N} \) and a disc \( D \subseteq X_n \) containing both \( x \) and \( y \).

Proof. Write \( Q \) as \( \bigcup_{n=1}^{\infty} F_n \), where each \( F_n \) is a finite subgroup of \( Q \) and \( F_n \subseteq F_{n+1} \) for all \( n \in \mathbb{N} \). Then \( X = \bigcup_{n=1}^{\infty} B(F_n) \). Since \( B(F_n) \approx M_{F_n} \approx l_1 \), see Besaga and Pelczyński [2], we see that if \( X_n = B(F_n) \) then \( X_n \) is as required, Lemma 5.2.

We conclude that the semi-topological group \( X \) satisfies the conditions of Theorem 3.1. Therefore, \( X \) contains a dense subgroup \( H \) which contains \( \mathcal{Q} \) such that all auto-homeomorphisms of \( H \) are translations. Because of this, it easily follows that \( H \) is uniquely homogeneous.
5.4. THEOREM. \( H \) is a uniquely homogeneous semitopological group but \( H \) does not admit the structure of a topological group.

Proof. To the contrary, assume that \( \ast \) is a topological group structure on \( H \). If \( f: H \to H \) is defined by \( f(x) = x^{-1} \) then \( f \) is a homeomorphism having a fixed point. We conclude that \( f \) is the identity, Lemma 2.1(3). Therefore, \( (H, \ast) \) is Boolean. Without loss of generality we may assume that the identities of \( (H, \ast) \) and \( (H, \cdot) \) are both equal to the same point \( e \in H \). Take \( x \in H \) arbitrarily. The translation \( f(t) = x \cdot t \) maps \( e \) onto \( x \). However, there is only one homeomorphism mapping \( e \) onto \( x \), namely the translation \( g(t) = x \ast t \). We conclude that for all \( t \in H \) we have that \( x \ast t = x \ast t \). Since \( x \) was arbitrary, we find that \( \ast = \cdot \). However, \( \ast: H \times H \to H \) is not continuous since by construction \( \ast: Q \times Q \to Q \) is not continuous. Contradiction.

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