A uniquely homogeneous space need not be a topological group

by

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Abstract. A (separable metric) space X is called *uniquely homogeneous* provided that for al $x, y \in X$ there is a unique homeomorphism of X taking x onto y. Each uniquely homogeneous space admits a natural group structure for which all left translations are homeomorphisms. We present an example of a uniquely homogeneous space X for which this group structure is Abelian, which implies that both left and right translations are homeomorphisms, such that X does not admit the structure of a topological group.

0. Introduction. All spaces under discussion are separable metric. A space X is called *uniquely homogeneous* provided that for all $x, y \in X$ there is a unique homeomorphism of X taking x onto y. Barit and Renaud [1], using a result of Ungar [14], showed that a uniquely homogeneous space cannot be compact, or locally compact and locally connected. The author presented in [9] an example of a connected and locally connected topological group G which is Boolean (each element of the group has order at most 2), and which has the additional property that each autohomeomorphism of G is a translation. An easy consequence of these properties of G is that G is uniquely homogeneous.

Ungar [14] defines a natural group structure on a uniquely homogeneous space X for which it is easily seen that all left translations of X are homeomorphisms, and shows that if X is compact, or locally compact and locally connected, then this group structure makes X into a topological group. In addition he showed that the group of topological isomorphisms of X is trivial. Subsequently, Barit and Renaud [1] showed that a locally compact group different from $\{0\}$ or Z_2 always has a nontrivial topological isomorphism. This gave a proof of the result that a uniquely homogeneous space containing more than one point cannot be compact, or locally compact and locally connected.

In view of the above remarks and in view of the fact that all known examples of uniquely homogeneous spaces are topological groups, the question naturally arises whether every uniquely homogeneous space admits the structure of a topological group. We will answer this question in the negative by constructing an example of a uniquely homogeneous semitopological group which does not admit the structure of a topological group. This answers a question raised in [9].

1. Preliminaries. If G is an Abelian group then we let "+" denote the group operation on G. We regard $+$ to be a function of $G\times G$ onto G and for $+(x, y)$ we write $x+y$ of course. If $K\subseteq G\times G$ then the restriction $+|K$ will also be denoted by $+$.

A group is called *Boolean* if each element of the group has order at most 2. Observe that a Boolean group is Abelian. Let G be a group. If $A\subseteq G$ then $\langle\langle A\rangle\rangle$ denotes the subgroup of G generated by A. Let G be a Boolean group and let $H\subseteq G$ be a subgroup. If $x\in G$ then it is easily seen that

$$\langle\langle H\cup\{x\}\rangle\rangle = H\cup(x+H).$$

A group $(G, .)$ equipped with a topology is called a *semitopological group* if the operation $\cdot$: $G\times G\to G$ is continuous in each variable separately. It is known that if G is a semitopological group then G is a topological group if G is Baire, see Husain [5, p. 38].

A cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. $\mathfrak{c}$ denotes $2^{\aleph_0}$.

The domain and range of a function f will be denoted by $\mathrm{dom}(f)$ and $\mathrm{range}(f)$, respectively. Observe that the cardinality of the collection of all G_δ-subsets of a given space is at most $\mathfrak{c}$. This implies that if X and Y are spaces, then the collection

$$\mathscr{F} = \{f\colon \mathrm{dom}(f) \text{ is a } G_\delta\text{-subset of } X \text{ and } \mathrm{range}(f)\subseteq Y\}$$

has cardinality at most $\mathfrak{c}$.

Let X be a space. A subset $U\subseteq X$ is called *regular open* provided that $U = \mathrm{int}_X\mathrm{cl}_X U$. Let $RO(X) = \{U\subseteq X\colon U \text{ is regular open}\}$. It is well-known, and easy to prove (see Sikorski [13, § 1.20]), that $RO(X)$ is a complete Boolean algebra under the following operations:

$$U\wedge V = U\cap V,$$
$$U\vee V = \mathrm{int}_X\mathrm{cl}_X(U\cup V),$$

and

$$U' = \mathrm{int}_X(X\setminus U).$$

Let $\mathscr{B}\subseteq RO(X)$ be a subalgebra. The *Stone space* of $\mathscr{B}$, denoted by $\mathrm{st}(\mathscr{B})$, has as underlying set the set of all ultrafilters in $\mathscr{B}$. If $U\in\mathscr{B}$ then $\tilde{U} = \{\beta\in\mathrm{st}(\mathscr{B})\colon U\in\beta\}$. The topology of $\mathrm{st}(\mathscr{B})$ is generated by the collection

$$\{\tilde{U}\colon U\in\mathscr{B}\}.$$

The sets $\tilde{U}$ are *clopen* ($\equiv$ closed and open) and $\mathrm{st}(\mathscr{B})$ is compact. The reader should observe that if $\mathscr{B}$ is countable then $\mathrm{st}(\mathscr{B})$ is metrizable and that then for the definition of $\mathrm{st}(\mathscr{B})$ one does not need the full strength of the axiom of choice. Since $\mathscr{B}$ is countable, if $\gamma\subseteq\mathscr{B}$ is a filter then γ can be extended to an ultrafilter $\beta\subseteq\mathscr{B}$ by a process of countably many steps.

If X is a space, then $H(X)$ denotes the group of all autohomeomorphisms of X. The identity of X will be denoted by id.

If X is a space, and if $A\subseteq X$ then $\mathrm{cl}_X A$ and $\bar{A}$ denote the closure of A in X.

2. Unique homogeneity. In this section we will show that each uniquely homogeneous space admits a very natural group structure such that all left translations are homeomorphisms. The results in this section are included for completeness sake; they are well-known and easy to prove, see Ungar [14].

2.1. LEMMA. *Let X be homogeneous. The following statements are equivalent*:
(1) *X is uniquely homogeneous,*
(2) *if $f, g\in H(X)$ and $f(x) = g(x)$ for certain $x\in X$, then $f = g$,*
(3) *if $f\in H(X)$ and f has a fixed point, then $f = $ identity.*

Proof. (1) $\Rightarrow$ (3). Suppose that $f(x) = x$ for certain $f\in H(X)$ and $x\in X$. Since the identity is a homeomorphism taking x onto x, by (1), $f = \mathrm{id}$.

(3) $\Rightarrow$ (2). Let $h = g^{-1}f$. Then $h(x) = x$ and therefore, by (3), $h = \mathrm{id}$. We conclude that $f = g$.

(2) $\Rightarrow$ (1). Take $x, y\in X$. Since X is homogeneous, there is an $f\in H(X)$ with $(x) = y$. By (2) this f is unique. ■

Let X be uniquely homogeneous. Fix $e\in X$ and for each $x\in X$ let f_x be the unique homeomorphism taking e onto x. Define a binary operation "+" and an operation "−" on X by

$$x+y = f_x(y), \quad \text{and} \quad -x = f_x^{-1}(e).$$

2.2. LEMMA. *Let $x, y, z\in X$. Then*
(1) $x+(y+z) = (x+y)+z$,
(2) $x+e = e+x = x$,
(3) $x+(-x) = (-x)+x = e$.

Proof. (1) Observe that

$$(f_xf_y)(e) = f_x(f_y(e)) = f_x(y) = x+y,$$

and

$$f_{x+y}(e) = x+y.$$

By Lemma 2.1(2) we therefore conclude that $f_xf_y = f_{x+y}$. Since

$$x+(y+z) = x+f_y(z) = f_x(f_y(z)) = (f_xf_y)(z) \quad \text{and} \quad (x+y)+z = f_{x+y}(z),$$

we conclude that $x+(y+z) = (x+y)+z$.

(2) trivial.

(3) $x+(-x) = f_x(f_x^{-1}(e)) = e$. Notice that $f_{-x}(e) = -x = f_x^{-1}(e)$. Lemma 2.1(2) therefore implies that $f_{-x} = f_x^{-1}$. Consequently, $(-x)+x = f_{-x}(x) = f_x^{-1}(x) = e$. ■

We conclude that $+$ is a very natural group operation on X.

2.3. LEMMA. *All left translations of X are homeomorphisms.*

Proof. Fix $a\in X$ and define $l_a\colon X\to X$ by $l_a(x) = a+x$. Then $l_a(x) = f_a(x)$ for all $x\in X$. We conclude that $l_a = f_a$. ■

3. A theorem. The aim of this section is to prove Theorem 3.1 below, which is the key in the construction of our example. The proof of this result is similar to the proof of Theorem 3.1 in [9] except for some minor changes. Since these changes are not always obvious, we will give the proof in full detail.

3.1. THEOREM. *Let X be a Boolean semitopological group with the following properties*

(a) $X = \bigcup_{n=1}^{\infty} X_n$, *where each X_n is a topologically complete subgroup of X, and $X_n \subseteq X_{n+1}$ for all $n \in N$,*

(b) *if $x, y \in X$ then there exist an $n \in N$ and a disc $D \subseteq X_n$ containing both x and y.*

If $E \subseteq X$ is a countable subgroup, then there is a dense subgroup $H \subseteq X$ containing E such that the autohomeomorphisms of H are precisely its translations. Consequently, H is uniquely homogeneous.

Proof. Let αX be a compactification of X. Let $\mathscr{F} = \{f\colon \mathrm{dom}(f)$ and range (f) are G_δ-subsets of $\alpha X\}$. Since $|\mathscr{F}| \leqslant \mathfrak{c}$, we can enumerate $\mathscr{F}$ by $\{f_\alpha\colon \alpha < \mathfrak{c},\ \alpha \text{ even}\}$. Let $\{K_\alpha\colon \alpha < \mathfrak{c},\ \alpha \text{ odd}\}$ enumerate all Cantor sets in X. By transfinite induction, for every $\alpha < \mathfrak{c}$ we will construct subgroups $H_\alpha \subseteq X$ and subsets $V_\alpha \subseteq \alpha X$ such that

(1) if $\beta < \alpha$ then $E \subseteq H_\beta \subseteq H_\alpha$, $V_\beta \subseteq V_\alpha$ and $H_\alpha \cap V_\alpha = \emptyset$,

(2) $|H_\alpha| \leqslant |\alpha| \cdot \aleph_0$ and $|V_\alpha| \leqslant |\alpha| \cdot \aleph_0$,

(3) if α is odd then $H_\alpha \cap K_\alpha \neq \emptyset$,

(4) if α is even and if $\left|\{x \in \mathrm{dom}(f_\alpha) \cap X\colon f_\alpha(x) \notin «\bigcup_{\beta<\alpha} H_\beta \cup \{x\}»\}\right| = \mathfrak{c}$, then there is a point $x \in \mathrm{dom}(f_\alpha) \cap H_\alpha$ such that $f_\alpha(x) \in V_\alpha$.

Suppose that we have completed the construction for all $\beta < \alpha$, where $\alpha < \mathfrak{c}$. For convenience, put $H^\alpha = \bigcup_{\beta<\alpha} H_\beta$ and $V^\alpha = \bigcup_{\beta<\alpha} V_\beta \cap X$. Observe that $|H^\alpha| \leqslant |\alpha| \cdot \aleph_0 < \mathfrak{c}$, and, similarly, that $|V^\alpha| \leqslant |\alpha| \cdot \aleph_0 < \mathfrak{c}$.

Case 1. α is odd. Since $|K_\alpha| = \mathfrak{c}$, we can pick a point

$$x \in K_\alpha \setminus (H^\alpha + V^\alpha).$$

Define $H_\alpha = «H_\alpha \cup \{x\}»$ and $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. An easy check shows that H_α and V_α are as required.

Case 2. α is even and $|S| < \mathfrak{c}$ where $S = \{x \in \mathrm{dom}(f_\alpha) \cap X\colon f_\alpha(x) \notin «H^\alpha \cup \{x\}»\}$. Define $H_\alpha = H^\alpha$ and $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

Case 3. α is even and $|S| = \mathfrak{c}$. By the same argument as in Case 1, we can find a point

$$x \in S \setminus (H^\alpha + V^\alpha).$$

Define $H_\alpha = «H^\alpha \cup \{x\}»$ and $V_\alpha = \bigcup_{\beta<\alpha} V_\beta \cup \{f_\alpha(x)\}$. Since

$$H_\alpha \cap V_\alpha = (H^\alpha \cup (x + H^\alpha)) \cap \Big(\bigcup_{\beta<\alpha} V_\beta \cup \{f_\alpha(x)\}\Big) = \emptyset,$$

we see that H_α and V_α are as required.

Now put $H = \bigcup_{\beta<\mathfrak{c}} H_\alpha$. We claim that H is as required. Let $f\colon H \to H$ be a homeomorphism. Our task is to find a point $h \in H$ such that $f(x) = x + h$ for all $x \in H$.

By the well-known Lavrentieff Theorem ([6]; see also [3, Thm. 4.3.21]), there are G_δ-subsets S and T of αX such that f can be extended to a homeomorphism $\bar{f}\colon S \to T$. Let $\alpha < \mathfrak{c}$ be such that $\bar{f} = f_\alpha$.

Case 1. $\left|\{x \in S \cap X\colon f_\alpha(x) \notin «\bigcup_{\beta<\alpha} H_\beta \cup \{x\}»\}\right| = \mathfrak{c}$. Then by (4), there is a point $x \in S \cap H$ such that $f_\alpha(x) \notin H$. Since f_α extends f, this is impossible.

Case 2. Not Case 1. Let

$$H^\alpha = \bigcup_{\beta<\alpha} H_\beta \quad \text{and} \quad U = \{x \in S \cap X\colon f_\alpha(x) \notin «\bigcup_{\beta<\alpha} H_\beta \cup \{x\}»\}.$$

By assumption, $|U| < \mathfrak{c}$. For each $h \in H^\alpha$ define

$$E_h = \{x \in S \cap X\colon f_\alpha(x) = x + h\}.$$

We claim that E_h is closed in S. For all $n \in N$ take $x_n \in E_h$ and $x \in S$ such that $\lim_{n\to\infty} x_n = x$. Then

$$f_\alpha(x) = \lim_{n\to\infty} f_\alpha(x_n) = \lim_{n\to\infty} (x_n + h) = x + h$$

(this uses of course only the fact that X is semitopological). In addition the collection $\mathscr{E} = \{E_h\colon h \in H^\alpha\}$ is clearly pairwise disjoint. For each $h \in H^\alpha$ let $F_h = E_h \setminus f_\alpha^{-1}(H^\alpha)$. We conclude that the collection $\mathscr{G} = \{F_h\colon h \in H^\alpha\}$ is also pairwise disjoint. We claim that at most one set of the collection $\mathscr{G}$ can be nonempty. To the contrary, suppose that there exist distinct points $s, t \in H^\alpha$ and points $x \in F_s$ and $y \in F_t$. We first claim that $X \setminus S$ is countable. If not, then there is an $n \in N$ such that $X_n \setminus S$ is uncountable. Since $X_n \setminus S$ is a countable union of closed subsets of X_n, one of these closed sets must be uncountable and therefore contains a Cantor set since X_n is topologically complete, [4] (see also [3, 4.5.5]). This is impossible since $H \subseteq S$ intersects all Cantor sets. Find an $n \in N$ and a disc $D \subseteq X_n$ containing both x and y. Since $|U| < \mathfrak{c}$ and since $|f^{-1}(H^\alpha)| < \mathfrak{c}$, we can choose an arc $J \subseteq D$ which connects x and y but misses $U \cup f^{-1}(H^\alpha) \cup X \setminus S$. We conclude that $J \subseteq \bigcup \mathscr{G}$. Put $K = \{h \in H^\alpha\colon F_h \cap J \neq \emptyset\}$. By assumption, $|K| \geqslant 2$, whence $\{J \cap E_h\colon h \in K\}$ is a partition of J in at least 2 closed and disjoint, nonempty sets. Notice that $E_h \cap J$ is closed in J since $J \subseteq S$. By Sierpiński's Theorem, [11] (see also [3, Thm. 5.3.2]), $|K| > \aleph_0$. Also, $|K| \leqslant |H^\alpha| < \mathfrak{c}$ (so in case the Continuum Hypothesis holds, we have derived a contradiction). Find an $m \geqslant n$ such that $K \cap X_m$ is uncountable. Then $K \cap X_m$ is not closed in X_m since X_m is topologically complete and $\aleph_0$

$<|K \cap X_m| < \mathfrak{c}$. Let $k_i \in K \cap X_m$ $(i \in N)$ be a sequence converging to a point $k \in X_m \setminus K$. For each $i \in N$, take a point $x_i \in J \cap F_{k_i}$. By compactness of J, we may assume that $\lim_{i\to\infty} x_i = p \in J$. Since X_m is a semitopological group which is topologically complete, by Montgomery [10] (see also [5, p. 38]) X_m is a topological group with group operation $+$. We conclude that $+: X_m \times X_m \to X_m$ is continuous. Consequently,

$$f_\alpha(p) = \lim_{i\to\infty} f_\alpha(x_i) = \lim_{i\to\infty}(x_i+k_i) = p+k$$

(notice that $x_i, k_i \in X_m$ for all $i \in N$). Since $p \in J$, there is an $h \in K$ such that $p \in E_h$. Therefore,

$$f_\alpha(p) = p+h = p+k,$$

which implies that $h = k$, a contradiction.

If for each $h \in H^\alpha$ it is true that $F_h = \emptyset$, then $f_\alpha(H) \subseteq f_\alpha(U) \cup H^\alpha$. Since H intersects all Cantor sets in X, the cardinality of H must be $\mathfrak{c}$. Since f_α is one-to-one, this implies that $f_\alpha(H)$ has cardinality $\mathfrak{c}$. However, since $f_\alpha(U) \cup H^\alpha$ has cardinality less than $\mathfrak{c}$, this is impossible. Consequently, there is precisely one $h \in H^\alpha$ for which $F_h \neq \emptyset$. Since H intersects all Cantor sets in X, each nonempty open subset of H has clearly cardinality $\mathfrak{c}$. Since E_h is closed in S, and since $S \setminus E_h \subseteq U$ has cardinality less than $\mathfrak{c}$, we may conclude that $E_h = S$. This proves that f_α is a translation. ■

3.2. Remark. In the proof of the above result we used a technique originally due to Kuratowski [7]. Some specific ideas needed for the proof can be found in [8], and also in [9]. For more references and information concerning Kuratowski's technique, see [9].

4. A Boolean group structure on Q. Let Q denote the space of rational numbers. Among other things, we will show that there is a Boolean group structure $*$ on Q such that all translations are (topological) homeomorphisms while moreover the function $*: Q \times Q \to Q$ is nowhere continuous ([1]).

Let $K = \{0, 1\}^N$ and let G be a countable dense subgroup of K. Observe that G is Boolean. For all $n \in N$, define

$$A_n = \{x \in K\colon\ x_i = 0 \text{ if } i \in \{1, 2, \ldots, n-1, n+1, \ldots, 2n-1\} \text{ and } x_n = 1\}.$$

It is clear that A_n is clopen, that the diameter of A_n tends to 0 if n goes to infinity, and that each neighborhood of the identity of K, which we will denote by O_K, contains all but finitely many of the A_n's. Let $A = \bigcup_{n=1}^{\infty} A_n$. Observe that $A \in RO(K)$.

4.1. Lemma. *If $n<m<2n$ then $(A_n+A_m) \cap \bar{A} = \emptyset$.*

Proof. If $x \in A_n$ and $y \in A_m$ then $(x+y)_i = 0$ if $i \leqslant n-1$, $(x+y)_n = 1$ and there is a $j \in N$ with $n<j\leqslant 2n-1$ such that $(x+y)_j = 1$. This easily implies that $x+y \notin \bar{A}$. ■

([1]) It is well-known of course that a semitopological group need not be a topological group. In our construction we need a Boolean group structure as above. I do not know whether this has been done before.

4.2. Lemma. *If $F \subseteq N$ is finite and if $N \setminus F = \bigcup_{i=1}^{k} B_i$ then there is an $i \leqslant k$ such that B_i contains two points n and m such that $n<m<2n$.*

Proof. Choose $r \in N$ such that $r > \max F$ and $r > k$. There are $x, y \in N$ with $r \leqslant x < y < 2r$ and there is an $i \leqslant k$ with $\{x, y\} \subseteq B_i$. It is clear that $x<y<2x$. ■

4.3. Proposition. *If $\mathscr{E}$ is a finite family of subsets of A such that for all $E \in \mathscr{E}$ we have that $E+E \subseteq \bar{A}$, then there are infinitely many $n \in N$ with $A_n \cap \bigcup \mathscr{E} = \emptyset$.*

Proof. Suppose that $F = \{n \in N\colon\ A_n \cap \bigcup \mathscr{E} = \emptyset\}$ is finite. By Lemma 4.2 there is an $E \in \mathscr{E}$ and there are $n, m \in N \setminus F$ such that $n<m<2n$ and $E \cap A_n \neq \emptyset \neq A_m \cap E$. Since by assumption $E+E \subseteq \bar{A}$, this contradicts Lemma 4.1. ■

4.4. Lemma. *There is a countable subalgebra $\mathscr{B} \subseteq RO(K)$ such that*

(1) *if $C \subseteq K$ is clopen then $C \in \mathscr{B}$,*

(2) *$A \in \mathscr{B}$,*

(3) *if $x \in G$ and $B \in \mathscr{B}$ then $x+B \in \mathscr{B}$.*

Proof. Let $\mathscr{B}_0$ be the smallest subalgebra of $RO(K)$ containing all clopen subsets of K and the set A. Observe that $\mathscr{B}_0$ is countable. Inductively construct countable subalgebras $\mathscr{B}_n \subseteq RO(K)$ such that for all $n \in N$,

(a) $\mathscr{B}_{n-1} \subseteq \mathscr{B}_n$,

(b) if $x \in \mathscr{B}_{n-1}$ and $x \in G$ then $x+B \in \mathscr{B}_n$.

Suppose that the subalgebras are constructed for all $i \leqslant n$. Let $\mathscr{B}_{n+1}$ be the smallest subalgebra of $RO(K)$ containing the collection $\{x+B\colon\ x \in G \text{ and } B \in \mathscr{B}_n\}$. It is clear that $\mathscr{B}_{n+1}$ is countable.

Now put $\mathscr{B} = \bigcup_{n=0}^{\infty} \mathscr{B}_n$. An easy check shows that $\mathscr{B}$ is as required. ■

For each $x \in G$ define $f_x\colon K \to K$ by $f_x(t) = x+t$. Let $X = \mathrm{st}(\mathscr{B})$. Define $\varphi_x\colon X \to X$ by $\varphi_x(\beta) = \{f_x(B)\colon B \in \beta\}$. By Stone duality, φ_x is a homeomorphism. Consider the following subcollection of $\mathscr{B}$:

$$\gamma = \{B'\colon B \in \mathscr{B},\ B \subseteq A \text{ and } B+B \subseteq \bar{A}\} \cup \{C \subseteq K\colon\ C \text{ is clopen and } O_K \in C\} \cup \{A\}.$$

4.5. Lemma. *If $\mathscr{E} \subseteq \gamma$ is finite, then $\bigwedge \mathscr{E} \neq 0$.*

Proof. Without loss of generality, $\mathscr{E} = \{B'_1, \ldots, B'_m\} \cup \{A\} \cup \{C\}$, where for each $i \leqslant m$ we have that $B_i \cup (B_i+B_i) \subseteq \bar{A}$, and C is a clopen neighborhood of O_K. By Proposition 4.3, the set $\{n \in N\colon\ A_n \cap \bigcup_{i=1}^{m} B_i = \emptyset\}$ is infinite. Since C contains almost all of the A_n's, we conclude that $\bigwedge \mathscr{E} \neq \emptyset$. ■

By the above Lemma we can extend the collection γ to an ultrafilter $\beta \subseteq \mathscr{B}$. Whence $\beta \in \mathrm{st}(\mathscr{B}) = X$. Define

$$Y = \{\varphi_x(\beta)\colon\ x \in G\}.$$

We topologize Y by regarding it to be a subspace of X.

4.6. Lemma. *If $\varphi_x(\beta) = \varphi_y(\beta)$ then $x = y$.*

Proof. Suppose that $x \neq y$. Find a clopen neighborhood C of O_K such that $(x+C) \cap (y+C) = \emptyset$. Since $x+C = f_x(C) \in \varphi_x(\beta)$ and $y+C = f_y(C) \in \varphi_y(\beta)$, we find that $\varphi_x(\beta) \neq \varphi_y(\beta)$, which is a contradiction. ■

Define a binary operation $*$ on Y by $\varphi_x(\beta) * \varphi_y(\beta) = \varphi_{x+y}(\beta)$. By Lemma 4.6, the operation $*$ is well defined. For notational simplicity, the point $\varphi_x(\beta)$ will be denoted by t_x from now on.

4.7. LEMMA. *$(Y, *)$ is a Boolean group with identity β. In addition, all translations of Y are (topological) homeomorphisms of Y.*

Proof. That $(Y, *)$ is a Boolean group with identity β is trivial. Take $t_x \in Y$ and consider the translation $\xi(t_y) = t_x * t_y$. Observe that

$$\xi(t_y) = t_{x+y} = \varphi_{x+y}(\beta) = \varphi_x(\varphi_y(\beta)) = \varphi_x(t_y)$$

for all $y \in Y$. Consequently, $\xi = \varphi_x | Y$. Since by construction $\varphi_x | Y$ is an autohomeomorphism of Y, we conclude that ξ is an autohomeomorphism of Y. ■

It is easily seen that X has no isolated points. Since Y is dense in X, we conclude that $Y \approx Q$, Sierpiński [12].

4.8. THEOREM. *There is a Boolean group structure $*$ on Q such that*

(1) $*: Q \times Q \to Q$ is nowhere continuous,

(2) all translations of Q are (topological) homeomorphisms.

Proof. It clearly suffices to show that $*: Y \times Y \to Y$ is not continuous in (β, β). We claim that $*^{-1}(\tilde{A})$ is not a neighborhood of (β, β) in $Y \times Y$. Indeed, to the contrary assume that there is a neighborhood $\tilde{B}$ of β in X, where $B \in \mathscr{B}$, such that

$$*((\tilde{B} \cap Y) \times (\tilde{B} \cap Y)) \subseteq \tilde{A}.$$

It is clear that without loss of generality we may assume that $B \subseteq A$. Take $x, y \in B \cap G$. Then B is a neighborhood of both x and y, so

$$(t_x, t_y) \in (\tilde{B} \cap Y) \times (\tilde{B} \cap Y)$$

and consequently, $t_x * t_y \in \tilde{A}$ or, equivalently, $\varphi_{x+y}(\beta) \in \tilde{A}$. Since $\varphi_{x+y}(\beta) = \{(x+y)+E\colon E \in \beta\}$, there is an $E \in \beta$ such that $(x+y)+E = A$. Since $O_K \in E$ this implies that $(x+y)+O_K \in \tilde{A}$. We conclude that $x+y \in \tilde{A}$. Since $B \cap G$ is dense in B, we therefore have that $B+B \subseteq \tilde{A}$. By definition of γ, $B' \in \gamma \subseteq \beta$. Since by assumption $B \in \beta$, this is a contradiction. ■

5. Spaces of measurable functions. Let X be a space. A function $f\colon [0,1] \to X$ is said to be *measurable* if $f^{-1}(U)$ is a Borel subset of $[0,1]$ for every open subset $U \subseteq X$. Measurable functions $f, g\colon [0,1] \to X$ are called *equivalent* if

$$\lambda(\{t \in [0,1]\colon f(t) \neq g(t)\}) = 0,$$

where λ denotes Lebesgue measure on $[0,1]$. Let M_X be the topological space of equivalence classes of measurable functions from $[0,1]$ into X with the topology

of convergence in measure. The topology of M_X is determined by the metric

$$\varrho(f, g) = \Big(\int_0^1 (d(f(t), g(t)))^2 dt\Big)^{1/2},$$

where d is any bounded metric compatible with the topology of X. The topology of M_X does not depend on the chosen metric d on X. Bessaga and Pełczyński [2] show that $M_X \approx l_2$, the separable Hilbert space, if and only if X is completely metrizable and contains more than one point. It is easily seen that the set of constant functions is closed in M_X and isometric to X. We identify X and this isometric copy of X in M_X.

Let $*$ be the Boolean group structure on Q given by Theorem 4.8. Define a Boolean group structure, which we will also denote by $*$, on M_Q as follows:

$$(f * g)(t) = f(t) * g(t).$$

It is trivial to verify that $*$ is indeed a Boolean group structure on M_Q and that Q is a subgroup of M_Q. It is also easily seen that all translations of M_Q are homeomorphisms, whence M_Q is a semitopological group which is not a topological group since Q is a subgroup, see Bessaga and Pełczyński [2]. Define

$$X = \{f \in M_Q\colon f([0,1]) \text{ is finite}\}.$$

5.1. LEMMA. *X is a subgroup of M_Q and $Q \subseteq X$.*

Proof. If $f, g \in X$ then $(f * g)([0,1])$ is contained in the subgroup generated by $f([0,1])$ and $g([0,1])$. This subgroup is clearly finite since Q is a Boolean group. That $Q \subseteq X$ is trivial. ■

5.2. LEMMA. *If $F \subseteq Q$ is a finite subgroup, then $B(F) = \{f \in M_Q\colon f([0,1]) \subseteq F\}$ is a subgroup of M_Q which is homeomorphic to M_F.*

Proof. Obvious. ■

We will identify $B(F)$ and M_F.

5.3. LEMMA. *$X = \bigcup_{n=1}^{\infty} X_n$, where X_n is a topologically complete subgroup of X and $X_n \subseteq X_{n+1}$ for all $n \in N$. Moreover, if $x, y \in X$ then there are an $n \in N$ and a disc $D \subseteq X_n$ containing both x and y.*

Proof. Write Q as $\bigcup_{n=1}^{\infty} F_n$, where each F_n is a finite subgroup of Q and $F_n \subseteq F_{n+1}$ for all $n \in N$. Then $X = \bigcup_{n=1}^{\infty} B(F_n)$. Since $B(F_n) \approx M_{F_n} \approx l_2$, see Bessaga and Pełczyński [2], we see that if $X_n = B(F_n)$ then X_n is as required, Lemma 5.2. ■

We conclude that the semitopological group X satisfies the conditions of Theorem 3.1. Therefore, X contains a dense subgroup H which contains Q such that all autohomeomorphisms of H are translations. Because of this, it easily follows that H is uniquely homogeneous.

5.4. THEOREM. *H is a uniquely homogeneous semitopological group but H does not admit the structure of a topological group.*

Proof. To the contrary, assume that "$\circ$" is a topological group structure on H. If $f: H \to H$ is defined by $f(x) = x^{-1}$ then f is a homeomorphism having a fixed point. We conclude that f is the identity, Lemma 2.1(3). Therefore, $(H, \circ)$ is Boolean. Without loss of generality we may assume that the identities of $(H, \circ)$ and $(H, *)$ are both equal to the same point $e \in H$. Take $x \in H$ arbitrarily. The translation $f(t) = x \circ t$ maps e onto x. However, there is only one homeomorphism mapping e onto x, namely the translation $g(t) = x * t$. We conclude that for all $t \in H$ we have that $x \circ t = x * t$. Since x was arbitrary, we find that $\circ = *$. However, $*: H \times H \to H$ is not continuous since by construction $*: Q \times Q \to Q$ is not continuous. Contradiction. ■

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