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A COMPACTIFICATION PROBLEM OF J. DE GROOT

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Recently, De Groot's conjecture that cmp $X = \text{def}\, X$ holds for every separable and metrizable space X has been negatively resolved by Pol. In previous efforts to resolve De Groot's conjecture various functions like cmp have been introduced. A new inequality between two of these functions is established. Many examples which have been constructed so far in relation with the conjecture are obtained by attaching a locally compact space to a compact space. An upper bound for the compactness deficiency def of the resulting space is given.

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compactification compactness degree
dimension compactness deficiency

Unless stated otherwise, all spaces under consideration are separable and metrizable.

0. Introduction

In 1941 J. de Groot [7] proved the following theorem:

Theorem. *A space X is rimcompact (i.e., every point of X has arbitrarily small neighborhoods with compact boundary) if and only if X has a compactification $F(X)$ such that the dimension of the remainder $F(X)\backslash X$ does not exceed* 0.

Looking for natural compactifications of manifolds Freudenthal [6] obtained a similar result. By this theorem the rimcompactness of a space X may be viewed as an internal characterization of the existence of a compactification $F(X)$ of X the remainder $F(X)\backslash X$ of which has dimension $\leqslant 0$. The similarity of the definitions of rimcompactness and zero-dimensionality then naturally leads to the conjecture below.

For any subset U of a space X let ∂U denote the topological boundary of U in X.

The *compactness degree* cmp X of a space X is inductively defined as follows.

(i) $\operatorname{cmp} X = -1$ if and only if X is compact,

(ii) $\operatorname{cmp} X \leqslant n$ if every point of X has arbitrarily small neighborhoods U whose topological boundary ∂U has $\operatorname{cmp} \partial U \leqslant n-1$.

The *compactness deficiency* def X of a space X is defined as the minimum of the numbers $\dim Y\backslash X$ where Y varies over all compactifications of X. As all dimension functions agree on a separable and metrizable space there is no ambiguity here.

Various examples of spaces X with $\operatorname{cmp} X = n$ or $\operatorname{def} X = n$, $n \in \mathbb{N}$, are exhibited in [8]. See also [9]. It is a theorem that $\operatorname{cmp} X \leqslant \operatorname{def} X$ for any space X. The problem whether the reverse inequality holds was posed by De Groot.

Conjecture (J. de Groot [7, 8]). For every (separable and metrizable) space X the equality $\operatorname{cmp} X = \operatorname{def} X$ holds.

Recently R. Pol resolved this conjecture in the negative.

Example (R. Pol. [13]). There exists a (separable and metrizable) space P with $\operatorname{cmp} P = 1$ and $\operatorname{def} P = 2$.

In previous efforts to resolve De Groot's conjecture various functions like cmp were introduced. In Section 1 a new inequality between two of these functions is established (Theorem 1). Many examples which have been constructed so far in relation with De Groot's conjecture are obtained by attaching a locally compact space to a compact space. In Theorem 2 an upper bound for the compactness deficiency of the resulting space is given. This result is of interest as the above-mentioned example of Pol as well as an example of Van Mill (see Section 1) are of this type.

1. The inequality Cmp $\leqslant$ Skl

The *large inductive compactness degree* Cmp is defined in a similar way as the large inductive dimension:

(i) $\operatorname{Cmp} X = \operatorname{cmp} X$ if X is rimcompact,

(ii) for $n \geqslant 1$, $\operatorname{Cmp} X \leqslant n$ if every non-empty closed set has arbitrarily small neighborhoods U whose topological boundary ∂U has $\operatorname{Cmp} \partial U \leqslant n-1$.

It can be proved [8] that for any space X we have

$$\operatorname{cmp} X \leqslant \operatorname{Cmp} X \leqslant \operatorname{def} X. \tag{1}$$

This is a splitting of the original problem of De Groot. The space P of Pol's example has $\operatorname{Cmp} P = 2$. So the problem whether $\operatorname{Cmp} X = \operatorname{def} X$ for every space X is still unresolved. Up to now only very little is known about this problem [8]. It should be noticed that a similar problem for completeness degree and completeness deficiency has been positively resolved [1]. See also [11] where it is shown that for the obtaining of this positive result the restriction to metrizable spaces is essential. The analogous problem for σ-compactness degree and deficiency, posed by Nagata [12], has been resolved in the negative [2].

Another splitting of De Groot's problem is due to Sklyarenko [14]. A space X is said to have $\operatorname{Skl} X \leqslant n$ if X has a base $\mathcal{B} = \{B_i: i \in \mathbb{N}\}$ for the open sets such that for any $n+1$ different indices $i_0, \ldots, i_n$ the intersection $\partial B_{i_0} \cap \cdots \cap \partial B_{i_n}$ is compact. It can be shown that for any space X we have

$$\operatorname{cmp} X \leqslant \operatorname{Skl} X \leqslant \operatorname{def} X. \tag{2}$$

It should be noticed that slight modifications of the definition of Skl X can be found in the literature (e.g. [9, 14]). At this stage of the development there seems to be little point in making a definite choice.

The inequalities (1) and (2) are interrelated as follows:

Theorem 1. *Let X be a space. Then* $\operatorname{Cmp} X \leqslant \operatorname{Skl} X$.

The proof of Theorem 1 is by induction on Skl X. It is shown that, for every $n \geqslant 1$, if $\operatorname{Skl} X \leqslant n$, then $\operatorname{Cmp} X \leqslant n$. The inductive step of the proof is provided by the following lemma:

Lemma. *Let X be a space with* $\operatorname{Skl} X \leqslant n$, $n \geqslant 1$. *For any closed set F and any open set U with $F \subset U$ there exists an open set W such that $F \subset W \subset \operatorname{cl} W \subset U$ and* $\operatorname{Skl} \partial W \leqslant n-1$.

Proof. Let $\mathcal{B} = \{B_i: i \in \mathbb{N}\}$ be a countable base for the open subsets of X such that the intersection $\partial B_{i_0} \cap \cdots \cap \partial B_{i_n}$ is compact for any $n+1$ different indices $i_0, \ldots, i_n \in \mathbb{N}$.

Consider the countable collection $\mathcal{D} = \{(C_i, D_i): i \in \mathbb{N}\}$ of all pairs of elements of $\mathcal{B}$ such that $\operatorname{cl} C_i \subset D_i$ and

$$\operatorname{cl} D_i \subset U \quad \text{or} \quad D_i \cap F = \emptyset. \tag{3}$$

Observe that $\{C_i: i \in \mathbb{N}\}$ is an open cover of the space X. Define $V_i = D_i \setminus \bigcup \{\operatorname{cl} C_j: j = 0, \ldots, i-1\}$, $i \in \mathbb{N}$. As is easily seen, $\mathcal{V} = \{V_i: i \in \mathbb{N}\}$ is a locally finite open cover of X and for each $i \in \mathbb{N}$ we have

$$\partial V_i \subset \partial C_0 \cup \cdots \cup \partial C_{i-1} \cup \partial D_i. \tag{4}$$

Now let $W = \bigcup\{V_i: \text{cl } V_i \subset U\}$. Obviously W is an open set and in view of (3) we have $F \subset W$. Because $\mathscr{V}$ is locally finite, cl $W \subset U$ and

$$\partial W \subset \bigcup\{\partial V_i: \text{cl } V_i \subset U\}. \tag{5}$$

As in view of (4) each ∂V_i is the union of a finite collection of closed subsets of boundaries of elements of $\mathscr{B}$, from (5) it follows that ∂W, for some subset S of $\mathbb{N}$, is the union of the locally finite collection $\{E_j: j \in S\}$, where each E_j is a non-empty and closed subset of ∂B_j. Observe that here and also in the sequel of the proof the original indexing of $\mathscr{B}$ is used.

For each $j \in S$, a point $p_j \in E_j$ is selected. Let $P = \{p_j: j \in S\}$. Observe that P is closed. Now let $\mathscr{B}_1 = \{B_i: i \in N_1\}$, where $N_1 \subset \mathbb{N}$, be the set of all $B \in \mathscr{B}$ with $\partial B \cap P = \emptyset$. It is easily seen that $\mathscr{B}_1$ is a base for the open subsets of X and that for each $B \in \mathscr{B}_1$ the boundary ∂B is distinct from any boundary ∂B_j, $j \in S$. So, in particular, $S \cap N_1 = \emptyset$.

Let $\mathscr{U}$ be a point-finite open cover of X such that each element of $\mathscr{U}$ meets at most finitely many members of $\{E_j: j \in S\}$. Finally let $\mathscr{B}_2 = \{B_i: i \in N_2\}$, where $N_2 \subset N_1$, be the set of all $B \in \mathscr{B}_1$ such that cl $B \subset U$ for some $U \in \mathscr{U}$ and let $\mathscr{C} = \{B_i \cap \partial W: i \in N_2\}$. Obviously $\mathscr{C}$ is a base for the open sets of ∂W. We claim that $\mathscr{C}$ witnesses the fact that Skl $\partial W \leq n-1$. To this end, choose n different indices $i_0, \ldots, i_{n-1}$ from N_2 and consider the intersection $T = \partial_{\partial W}(B_{i_0} \cap \partial W) \cap \cdots \cap \partial_{\partial W}(B_{i_{n-1}} \cap \partial W)$. We shall show that T is compact. It is not hard to see that T is a closed subset of the set $R = \partial B_{i_0} \cap \cdots \cap \partial B_{i_{n-1}} \cap \partial W$. As each cl $B_{i_j} \subset U$ for some $U \in \mathscr{U}$, $j = 0, \ldots, n-1$, and $\mathscr{U}$ is point-finite and each member of $\mathscr{U}$ meets at most finitely many E_j, where $j \in S$, it follows that $R = \bigcup\{R \cap E_j: j \in F\}$ for some finite subset F of S. For each $j \in F$, the set $R \cap E_j$ is a closed subset of ∂B_j. Now $R \cap \partial B_j$ is compact, because the $n+1$ indices j, $i_0, \ldots, i_{n-1}$ are distinct. It follows that R and, consequently, T are compact. □

2. An upper bound for def

In the following theorem a situation is discussed which occurs in many examples which have been constructed in relation with De Groot's problem (e.g. [8, 11, 13]).

Theorem 2. *Let $X = A \cup B$, where A is closed and B is locally compact. Then* def $X \leq$ dim $A + 1$.

Remark. Under the conditions stated in the theorem one also has cmp $X \leq$ dim $A + 1$. This follows from a much more general result [8, Theorem 3.3.2]: If $X = A \cup B$, then cmp $X \leq$ dim $A +$ cmp $B + 1$. It also follows from Theorem 2 of course since cmp $X \leq$ def X.

Preliminary observation. Before embarking upon the proof of Theorem 2 we first make the following observation. It was shown by Bothe [3] that each n-dimensional compactum can be embedded in an $(n+1)$-dimensional compact absolute retract. In Krasinkiewicz [10] an elementary proof of this fact was presented. In fact, it follows from the proof in [10], that if A is an n-dimensional compactum, then there is an $(n+1)$-dimensional compact absolute retract Y, containing A as a subspace, and a homotopy $H: Y\times I\to Y$ such that $H_0=\mathrm{id}$ and $H_t(Y)\cap A=\emptyset$ for all $t\in(0,1]$.

We need the following lemma:

Lemma. *Let $X=A\cup B$, where A is closed and $\dim A\leq n$. Then there is a compactification δX of X such that the closure of A in δX is a compactum of dimension $\leq n$.*

Proof. Let γA be a compactification of A such that $\dim \gamma A=\dim A$ [5, 1.7.2]. Let σX be any compactification of X. Assume that $\gamma A\subset Q$, where Q denotes the Hilbert cube. Since Q is an absolute retract, the embedding $e: A\to\gamma A\subset Q$ can be extended to a mapping $\tilde{e}: X\to Q$. The mapping $h: X\to\sigma X\times Q$, defined by $h(x)=(x,\tilde{e}(x))$, is a topological embedding, which sends X onto the graph G of the mapping $\tilde{e}: X\to Q$ [4, 2.3.22]. Now X is identified with its topological copy G. Under this identification the action of $\tilde{e}$ corresponds to the action of the restriction $\pi_2|G$, where π_2 is the projection of $\sigma X\times Q$ onto the second coordinate space. Let $\gamma' X$ be the closure of G in $\sigma X\times Q$. In addition, let $\gamma' A$ be the closure of A in $\gamma' X$. It is clear that $f=\pi_2|\gamma' A$ maps $\gamma' A$ onto γA. Let δX be the adjunction space $\gamma' X\cup_f\gamma A$ [4, p. 127]. It is easily seen that the quotient mapping $\gamma' X\oplus\gamma A\to\delta X$ is a perfect mapping. It follows that δX is a metrizable compactum which satisfies all properties required. □

Proof of Theorem 2. Let X, A and B be as mentioned in the theorem and let δX be as in the above lemma. In addition, let A' denote the closure of A in δX. For convenience, assume that $A\cap B=\emptyset$. Since A' is a compactum of dimension at most n, we can find a compact absolute retract Y and a homotopy H such as in the preliminary observation. As B is locally compact and open in X, B is open in δX [4, 3.5.8]. Because Y is an absolute retract containing A' as a subspace, the identity mapping of A' can be extended to a continuous mapping $f: \delta X\backslash B\to Y$. Define $\tilde{f}: \delta X\backslash B\to Y$ by $\tilde{f}(x)=H_t(f(x))$, where $t=d(x,A')$. Clearly $\tilde{f}$ is continuous, whence a perfect mapping since $\delta X\backslash B$ is compact. Observe that $\tilde{f}|A'=\mathrm{id}$ and $\tilde{f}(x)\notin A'$ whenever $x\notin A'$. Now let $\gamma X=\delta X\cup_{\tilde{f}}\tilde{f}(\delta X\backslash B)$, the adjunction space [4, p. 127].

Observe that by the adjunction the set $\delta X\backslash B$ is replaced by $\tilde{f}(\delta X\backslash B)$ and that $\dim\tilde{f}(\delta X\backslash B)\leq n+1$. Because $\tilde{f}$ is closed, the quotient mapping $\delta X\oplus\tilde{f}(\delta X\backslash B)\to\gamma X$ is closed, whence perfect. It follows that γX is a (metrizable) compactification of X and $\dim(\gamma X\backslash X)\leq\dim\tilde{f}(\delta X\backslash B)\leq n+1$. □

Remarks. Let X be an n-simplex with an open face D removed. Write $X=A\cup B$, where A is the boundary of D and $B=X\backslash A$. Then X, A and B satisfy the requirements of Theorem 2, whence $\operatorname{def} X\leq\dim A+1=n$. Actually $\operatorname{def} X=n$ [9]. This shows that the upper bound for def in Theorem 2 cannot be improved.

The example in Van Mill [11] also shows that in Theorem 2 the metrizability of X is essential.

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