A COMPACTIFICATION PROBLEM OF J. DE GROOT

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Recently, De Groot's conjecture that $\text{cmp} X = \text{def} X$ holds for every separable and metrizable space $X$ has been negatively resolved by Pol. In previous efforts to resolve De Groot's conjecture various functions like $\text{cmp}$ have been introduced. A new inequality between two of these functions is established. Many examples which have been constructed so far in relation with the conjecture are obtained by attaching a locally compact space to a compact space. An upper bound for the compactness deficiency $\text{def}$ of the resulting space is given.

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compactification compactness degree
dimension compactness deficiency

Unless stated otherwise, all spaces under consideration are separable and metrizable.

0. Introduction

In 1941 J. de Groot [7] proved the following theorem:

Theorem. A space $X$ is rimcompact (i.e., every point of $X$ has arbitrarily small neighborhoods with compact boundary) if and only if $X$ has a compactification $F(X)$ such that the dimension of the remainder $F(X) \setminus X$ does not exceed 0.
Looking for natural compactifications of manifolds Freudenthal [6] obtained a similar result. By this theorem the rimcompactness of a space $X$ may be viewed as an internal characterization of the existence of a compactification $F(X)$ of $X$ the remainder $F(X) \setminus X$ of which has dimension $\leq 0$. The similarity of the definitions of rimcompactness and zero-dimensionality then naturally leads to the conjecture below.

For any subset $U$ of a space $X$ let $\partial U$ denote the topological boundary of $U$ in $X$.

The compactness degree $\text{cmp} X$ of a space $X$ is inductively defined as follows.

(i) $\text{cmp} X = -1$ if and only if $X$ is compact,
(ii) $\text{cmp} X \leq n$ if every point of $X$ has arbitrarily small neighborhoods $U$ whose topological boundary $\partial U$ has $\text{cmp} \partial U \leq n - 1$.

The compactness deficiency $\text{def} X$ of a space $X$ is defined as the minimum of the numbers $\dim Y \setminus X$ where $Y$ varies over all compactifications of $X$. As all dimension functions agree on a separable and metrizable space there is no ambiguity here.

Various examples of spaces $X$ with $\text{cmp} X = n$ or $\text{def} X = n$, $n \in \mathbb{N}$, are exhibited in [8]. See also [9]. It is a theorem that $\text{cmp} X \leq \text{def} X$ for any space $X$. The problem whether the reverse inequality holds was posed by De Groot.

Conjecture (J. de Groot [7, 8]). For every (separable and metrizable) space $X$ the equality $\text{cmp} X = \text{def} X$ holds.

Recently R. Pol resolved this conjecture in the negative.

Example (R. Pol. [13]). There exists a (separable and metrizable) space $P$ with $\text{cmp} P = 1$ and $\text{def} P = 2$.

In previous efforts to resolve De Groot’s conjecture various functions like $\text{cmp}$ were introduced. In Section 1 a new inequality between two of these functions is established (Theorem 1). Many examples which have been constructed so far in relation with De Groot’s conjecture are obtained by attaching a locally compact space to a compact space. In Theorem 2 an upper bound for the compactness deficiency of the resulting space is given. This result is of interest as the above-mentioned example of Pol as well as an example of Van Mill (see Section 1) are of this type.

1. The inequality $\text{Cmp} \leq \text{Skl}$

The large inductive compactness degree $\text{Cmp}$ is defined in a similar way as the large inductive dimension:

(i) $\text{Cmp} X = \text{cmp} X$ if $X$ is rimcompact,
(ii) for $n \geq 1$, $\text{Cmp} X \leq n$ if every non-empty closed set has arbitrarily small neighborhoods $U$ whose topological boundary $\partial U$ has $\text{Cmp} \partial U \leq n - 1$. 
It can be proved [8] that for any space $X$ we have
\[ \text{cmp} X \leq \text{Cmp} X \leq \text{def} X. \] (1)

This is a splitting of the original problem of De Groot. The space $P$ of Pol's example has $\text{Cmp} P = 2$. So the problem whether $\text{Cmp} X = \text{def} X$ for every space $X$ is still unresolved. Up to now only very little is known about this problem [8]. It should be noticed that a similar problem for completeness degree and completeness deficiency has been positively resolved [1]. See also [11] where it is shown that for the obtaining of this positive result the restriction to metrizable spaces is essential. The analogous problem for $\sigma$-compactness degree and deficiency, posed by Nagata [12], has been resolved in the negative [2].

Another splitting of De Groot's problem is due to Sklyarenko [14]. A space $X$ is said to have $\text{Sk1} X \approx n$ if $X$ has a base $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ for the open sets such that for any $n + 1$ different indices $i_0, \ldots, i_n$ the intersection $\partial B_{i_0} \cap \cdots \cap \partial B_{i_n}$ is compact. It can be shown that for any space $X$ we have
\[ \text{cmp} X \leq \text{Sk1} X \leq \text{def} X. \] (2)

It should be noticed that slight modifications of the definition of $\text{Sk1} X$ can be found in the literature (e.g. [9, 14]). At this stage of the development there seems to be little point in making a definite choice.

The inequalities (1) and (2) are interrelated as follows:

**Theorem 1.** Let $X$ be a space. Then $\text{Cmp} X \leq \text{Sk1} X$.

The proof of Theorem 1 is by induction on $\text{Sk1} X$. It is shown that, for every $n \geq 1$, if $\text{Sk1} X \approx n$, then $\text{Cmp} X \approx n$. The inductive step of the proof is provided by the following lemma:

**Lemma.** Let $X$ be a space with $\text{Sk1} X \approx n$, $n \geq 1$. For any closed set $F$ and any open set $U$ with $F \subseteq U$ there exists an open set $W$ such that $F \subseteq W \subseteq \text{cl} W \subseteq U$ and $\text{Sk1} \partial W \approx n - 1$.

**Proof.** Let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable base for the open subsets of $X$ such that the intersection $\partial B_{i_0} \cap \cdots \cap \partial B_{i_n}$ is compact for any $n + 1$ different indices $i_0, \ldots, i_n \in \mathbb{N}$.

Consider the countable collection $\mathcal{D} = \{(C_i, D_i) : i \in \mathbb{N}\}$ of all pairs of elements of $\mathcal{B}$ such that $\text{cl} C_i \subseteq D_i$ and
\[ \text{cl} D_i \subseteq U \quad \text{or} \quad D_i \cap F = \emptyset. \] (3)

Observe that $\{C_i : i \in \mathbb{N}\}$ is an open cover of the space $X$. Define $V_i = D_i \setminus \bigcup \{\text{cl} C_j : j = 0, \ldots, i - 1\}, i \in \mathbb{N}$. As is easily seen, $\mathcal{V} = \{V_i : i \in \mathbb{N}\}$ is a locally finite open cover of $X$ and for each $i \in \mathbb{N}$ we have
\[ \partial V_i \subseteq \partial C_0 \cup \cdots \cup \partial C_{i-1} \cup \partial D_i. \] (4)
Now let \( W = \bigcup \{ V_i : \text{cl } V_i \subset U \} \). Obviously \( W \) is an open set and in view of (3) we have \( F \subset W \). Because \( \mathcal{V} \) is locally finite, \( \text{cl } W \subset U \) and

\[
\partial W = \bigcup \{ \partial V_i : \text{cl } V_i \subset U \}. \tag{5}
\]

As in view of (4) each \( \partial V_i \) is the union of a finite collection of closed subsets of boundaries of elements of \( \mathcal{B} \), from (5) it follows that \( \partial W \), for some subset \( S \) of \( \mathbb{N} \), is the union of the locally finite collection \( \{ E_j : j \in S \} \), where each \( E_j \) is a non-empty and closed subset of \( \partial B_j \). Observe that here and also in the sequel of the proof the original indexing of \( \mathcal{B} \) is used.

For each \( j \in S \), a point \( p_j \in E_j \) is selected. Let \( P = \{ p_j : j \in S \} \). Observe that \( P \) is closed. Now let \( \mathcal{B}_1 = \{ B_i : i \in N_1 \} \), where \( N_1 \subset \mathbb{N} \), be the set of all \( B \in \mathcal{B} \) with \( \partial B \cap P = \emptyset \). It is easily seen that \( \mathcal{B}_1 \) is a base for the open subsets of \( X \) and that for each \( B \in \mathcal{B} \), the boundary \( \partial B \) is distinct from any boundary \( \partial B_j, j \in S \). So, in particular, \( S \cap N_1 = \emptyset \).

Let \( \mathcal{U} \) be a point-finite open cover of \( X \) such that each element of \( \mathcal{U} \) meets at most finitely many members of \( \{ E_j : j \in S \} \). Finally let \( \mathcal{B}_2 = \{ B_i : i \in N_2 \} \), where \( N_2 \subset N_1 \), be the set of all \( B \in \mathcal{B}_1 \) such that \( \text{cl } B \subset U \) for some \( U \in \mathcal{U} \) and let \( \mathcal{C} = \{ B_i \cap \partial W : i \in N_2 \} \). Obviously \( \mathcal{C} \) is a base for the open sets of \( \partial W \). We claim that \( \mathcal{C} \) witnesses the fact that \( \text{Skl } \partial W = n - 1 \). To this end, choose \( n \) different indices \( i_0, \ldots, i_{n-1} \) from \( N_2 \) and consider the intersection \( T = \partial_{\partial W} (B_{i_0} \cap \partial W) \cap \cdots \cap \partial_{\partial W} (B_{i_{n-1}} \cap \partial W) \). We shall show that \( T \) is compact. It is not hard to see that \( T \) is a closed subset of the set \( R = \partial_{\partial W} (B_{i_0} \cap \partial W) \cap \cdots \cap \partial_{\partial W} (B_{i_{n-1}} \cap \partial W) \). As each \( \text{cl } B_{i_j} \subset U \) for some \( U \in \mathcal{U}, j = 0, \ldots, n - 1 \), and \( \mathcal{U} \) is point-finite and each member of \( \mathcal{U} \) meets at most finitely many \( E_p \) where \( j \in S \), it follows that \( R = \bigcup \{ R \cap E_j : j \in F \} \) for some finite subset \( F \) of \( S \). For each \( j \in F \), the set \( R \cap E_j \) is a closed subset of \( \partial B_j \). Now \( R \cap \partial B_j \) is compact, because the \( n + 1 \) indices \( j, i_0, \ldots, i_{n-1} \) are distinct. It follows that \( R \) and, consequently, \( T \) are compact. \( \square \)

2. An upper bound for \( \text{def} \)

In the following theorem a situation is discussed which occurs in many examples which have been constructed in relation with De Groot's problem (e.g. [8, 11, 13]).

**Theorem 2.** Let \( X = A \cup B \), where \( A \) is closed and \( B \) is locally compact. Then \( \text{def } X \leq \dim A + 1 \).

**Remark.** Under the conditions stated in the theorem one also has \( \text{cmp } X \leq \dim A + 1 \). This follows from a much more general result [8, Theorem 3.3.2]: If \( X = A \cup B \), then \( \text{cmp } X \leq \dim A + \text{cmp } B + 1 \). It also follows from Theorem 2 of course since \( \text{cmp } X \leq \text{def } X \).

**Preliminary observation.** Before embarking upon the proof of Theorem 2 we first make the following observation. It was shown by Bothe [3] that each \( n \)-dimensional
compactum can be embedded in an \((n+1)\)-dimensional compact absolute retract. In Krasinkiewicz [10] an elementary proof of this fact was presented. In fact, it follows from the proof in [10], that if \(A\) is an \(n\)-dimensional compactum, then there is an \((n+1)\)-dimensional compact absolute retract \(Y\), containing \(A\) as a subspace, and a homotopy \(H: Y \times I \to Y\) such that \(H_0 = \text{id}\) and \(H_t(Y) \cap A = \emptyset\) for all \(t \in (0, 1]\).

We need the following lemma:

**Lemma.** Let \(X = A \cup B\), where \(A\) is closed and \(\dim A \leq n\). Then there is a compactification \(\partial X\) of \(X\) such that the closure of \(A\) in \(\partial X\) is a compactum of dimension \(\leq n\).

**Proof.** Let \(\gamma A\) be a compactification of \(A\) such that \(\dim \gamma A = \dim A\) [5, 1.7.2]. Let \(\sigma X\) be any compactification of \(X\). Assume that \(\gamma A \subseteq Q\), where \(Q\) denotes the Hilbert cube. Since \(Q\) is an absolute retract, the embedding \(e: A \to \gamma A \subseteq Q\) can be extended to a mapping \(\tilde{e}: X \to Q\). The mapping \(h: X \to \sigma X \times Q\), defined by \(h(x) = (x, \tilde{e}(x))\), is a topological embedding, which sends \(X\) onto the graph \(G\) of the mapping \(\tilde{e}: X \to Q\) [4, 2.3.22]. Now \(X\) is identified with its topological copy \(G\). Under this identification the action of \(\tilde{e}\) corresponds to the action of the restriction \(\pi_2|G\), where \(\pi_2\) is the projection of \(\sigma X \times Q\) onto the second coordinate space. Let \(\gamma' X\) be the closure of \(G\) in \(\sigma X \times Q\). In addition, let \(\gamma' A\) be the closure of \(A\) in \(\gamma' X\). It is clear that \(f = \pi_2|\gamma' A\) maps \(\gamma' A\) onto \(\gamma A\). Let \(\partial X\) be the adjunction space \(\gamma' X \cup_f \gamma A\) [4, p. 127]. It is easily seen that the quotient mapping \(\gamma' X \oplus \gamma A \to \partial X\) is a perfect mapping. It follows that \(\partial X\) is a metrizable compactum which satisfies all properties required. \(\Box\)

**Proof of Theorem 2.** Let \(X, A\) and \(B\) be as mentioned in the theorem and let \(\partial X\) be as in the above lemma. In addition, let \(A'\) denote the closure of \(A\) in \(\partial X\). For convenience, assume that \(A \cap B = \emptyset\). Since \(A'\) is a compactum of dimension at most \(n\), we can find a compact absolute retract \(Y\) and a homotopy \(H\) such as in the preliminary observation. As \(B\) is locally compact and open in \(X\), \(B\) is open in \(\partial X\) [4, 3.5.8]. Because \(Y\) is an absolute retract containing \(A'\) as a subspace, the identity mapping of \(A'\) can be extended to a continuous mapping \(f: \partial X \setminus B \to Y\). Define \(\tilde{f}: \partial X \setminus B \to Y\) by \(\tilde{f}(x) = H_t(f(x))\), where \(t = d(x, A')\). Clearly \(\tilde{f}\) is continuous, whence a perfect mapping since \(\partial X \setminus B\) is compact. Observe that \(\tilde{f}|A' = \text{id}\) and \(\tilde{f}(x) \in A'\) whenever \(x \in A'\). Now let \(\gamma X = \partial X \cup_f \tilde{f}(\partial X \setminus B)\), the adjunction space [4, p. 127].

Observe that by the adjunction the set \(\partial X \setminus B\) is replaced by \(\tilde{f}(\partial X \setminus B)\) and that \(\dim \tilde{f}(\partial X \setminus B) \leq n + 1\). Because \(\tilde{f}\) is closed, the quotient mapping \(\partial X \oplus \tilde{f}(\partial X \setminus B) \to \gamma X\) is closed, whence perfect. It follows that \(\gamma X\) is a (metrizable) compactification of \(X\) and \(\dim(\gamma X \setminus X) \leq \dim \tilde{f}(\partial X \setminus B) \leq n + 1\). \(\Box\)

**Remarks.** Let \(X\) be an \(n\)-simplex with an open face \(D\) removed. Write \(X = A \cup B\), where \(A\) is the boundary of \(D\) and \(B = X \setminus A\). Then \(X\), \(A\) and \(B\) satisfy the requirements of Theorem 2, whence \(\text{def} X \leq \dim A + 1 = n\). Actually \(\text{def} X = n\) [9]. This shows that the upper bound for \(\text{def}\) in Theorem 2 cannot be improved.
The example in Van Mill [11] also shows that in Theorem 2 the metrizability of $X$ is essential.

References