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Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility

by

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Abstract. For every $k \in \{-1, 0, 1, \ldots\}$ we construct a topologically complete separable metric AR space X_k which is *not* homeomorphic to the Hilbert space l_2, but which has the following properties:

(1) X_k embeds as a linearly convex subset of l_2.

(2) every compact subset of X_k is a Z-set and homeomorphisms between compact subsets of X_k can be extended (with control),

(3) $X_k \times X_k \approx l_2$,

(4) if $A \subseteq X_k$ is σ-compact, then A is strongly negligible iff $\dim A \leqslant k$ (in particular, $X_k \not\approx X_{k'}$ if $k \neq k'$)

(5) if $A \subseteq X_k$ is any compactum of fundamental dimension at most k, then A is negligible in X_k.

1. Introduction. *All topological spaces under discussion are separable metric.*

Toruńczyk [15] has obtained the following topological characterization of the separable Hilbert space l_2:

1.1. THEOREM. *A topologically complete* AR *space X is homeomorphic to l_2 if and only if every map $f\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ of the countable free union of Hilbert cubes is strongly approximable by maps $g\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ for which the collection $\{g(Q_i)\}_{i=1}^{\infty}$ is discrete.*

This extremely useful characterization has now become the standard method for recognizing topological Hilbert spaces. The above approximation property, referred to as the *strong discrete approximation property*, can be stated in the following way:

1.2. With respect to every admissible metric d on X, for each map $f\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ and each $\varepsilon > 0$, there exists a map $g\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ such that $d(f(y), g(y)) < \varepsilon$ for each y and $\{g(Q_i)\}_{i=1}^{\infty}$ is discrete.

In Anderson, Curtis and van Mill [3] it was shown that the strong discrete approximation property cannot be relaxed by considering only positive constants

$\varepsilon > 0$ and a fixed metric d on X. Specifically, they constructed a topologically complete AR space (X, d) with the following properties:

(1) For each map $f\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ and $\varepsilon > 0$, there exists a map $g\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ such that $d(f(q), g(q)) < \varepsilon$ for each q while moreover the family $\{g(Q_i)\}_{i=1}^{\infty}$ is discrete (this is called the *weak discrete approximation property*),

(2) every compact subset of X is a Z-set,

(3) X embeds as a linearly convex subset of l_2,

(4) $X \times X \approx l_2$,

(5) X is homogeneous,

(6) every countable subset of X is strongly negligible (in particular, $X \setminus \{\text{countable set}\} \approx X$),

(7) no Cantor set is negligible in X.

Since in l_2 every σ-compact set is strongly negligible, Anderson [2], property (7) shows that $X \not\approx l_2$. The space X is a "fake topological Hilbert space" since it has many of the familiar topological properties of l_2 but yet is not homeomorphic to l_2. As an "application" we get that the properties (1) through (6) do not characterize l_2. It is useful to push this point further. Every "fake topological Hilbert space" blocks a possible generalization of Toruńczyk's Theorem.

The aim of this paper is to construct spaces that "approximate" l_2 closer than the space X above. We are interested in dimension theory and negligibility properties. As a by-product we get a characterization of dimension in terms of negligibility. Specifically, we construct for every $k \in \{-1, 0, 1, \ldots\}$ a topologically complete AR space (X_k, d) which is *not* homeomorphic to l_2, but which has the following properties:

(1) (X_k, d) has the weak discrete approximation property,

(2) let $O \subseteq X_k$ be open and let $\mathscr{U}$ be an open covering of O. If A is compact and if $F\colon A \times [0,1] \to O$ is a homotopy that is limited by $\mathscr{U}$ such that F_0 and F_1 are embeddings, then there is a homeomorphism $h\colon X_k \to X_k$ such that (a) h is supported on O, (b) $h \circ F_0 = F_1$ and (c) $h|O$ is $\mathscr{U}$-close to the identity. Moreover, each compact subset of X_k is a Z-set,

(3) X_k embeds as a linearly convex subset of l_2,

(4) $X_k \times X_k \approx l_2$,

(5) if $A \subseteq X_k$ is σ-compact, then A is strongly negligible iff $\dim A \leqslant k$ (in particular, $X_k \not\approx X_{k'}$, if $k \neq k'$).

(6) if $A \subseteq X_k$ is a compactum of fundamental dimension at most k, then A is negligible (in particular, if $B \subseteq X_k$ is an n-cell, then B is negligible and B is strongly negligible iff $n \leqslant k$).

Observe that X_k is contractible, being an AR, whence (2) implies that X_k is homogeneous. In fact, homeomorphisms between compact subsets of X_k can always be extended (with control).

Our construction is inspired by ideas in Anderson, Curtis and van Mill [3]. However, to get (2), (5) and (6), a more delicate process is necessary. We heavily rely on results obtained recently in Dijkstra [7] and [9].

2. Preliminaries. Let $Q = \prod_{i=1}^{\infty} [-1, 1]_i$ and $s = \prod_{i=1}^{\infty} (-1, 1)_i$. The space s is homeomorphic to l_2, [1], and is called the *pseudo-interior* of Q. On these product spaces we use the standard metric $d(x, y) = \max_{i \in N} \frac{1}{2^i} \cdot |x_i - y_i|$. The interval $[0, 1]$ will be denoted by I.

If X is a space, then $\mathscr{H}(X)$ denotes the group of autohomeomorphisms of X. The identity of $\mathscr{H}(X)$, which is the identity map on X, is usually denoted by 1. We say $h \in \mathscr{H}(X)$ is *supported* on $V \subseteq X$ if h restricts to the identity map on $X \setminus V$.

Let $\mathscr{U}$ be a collection of open subsets of a space X. Maps $f, g\colon Y \to X$ are *$\mathscr{U}$-close* if for each $y \in Y$ with $f(y) \neq g(y)$ there exists a $U \in \mathscr{U}$ containing both $f(y)$ and $g(y)$ (observe that we do *not* require $\mathscr{U}$ to be an open covering of X). Note that if $h \in \mathscr{H}(X)$ is $\mathscr{U}$-close to 1 then h is supported on $\bigcup \mathscr{U}$. A homotopy $H\colon Y \times I \to X$ is *limited by* $\mathscr{U}$ if for each $y \in Y$ either $H(\{y\} \times I)$ is a point or $H(\{y\} \times I) \subseteq U$ for certain $U \in \mathscr{U}$.

A collection $\mathscr{D}$ of closed subsets of a space X is *discrete* if each point of X has a neighbourhood intersecting at most one member of $\mathscr{D}$.

A closed subset $A \subseteq X$ is a Z-set in X if, for each map $f\colon Q \to X$ and open covering $\mathscr{U}$ of X, there exists a map $g\colon Q \to X \setminus A$ which is $\mathscr{U}$-close to 1. In l_2 every compact set is a Z-set. A *σ-Z-set* in X is a countable union of Z-sets.

The symbol $X \approx Y$ means that X and Y are homeomorphic spaces. A subset $K \subseteq X$ is *negligible* in X if $X \approx X \setminus K$. The subset K is *strongly negligible* if for every open $O \subseteq X$ and every open covering $\mathscr{U}$ of O there is a homeomorphism h from X onto $X \setminus (K \cap O)$ such that h is $\mathscr{U}$-close to 1.

Let X be a space. A subgroup $\Gamma \subseteq \mathscr{H}(X)$ is called *closed* if there is an admissible metric d on X such that every $f \in \mathscr{H}(X)$ which is the uniform d-limit of a sequence of elements of Γ belongs to Γ. Observe that if X is compact, then a subgroup Γ of $\mathscr{H}(X)$ is closed if and only if Γ is a closed subset of $\mathscr{H}(X)$ when given the compact-open topology.

Let X be a space and let $\mathscr{S}$ be a collection of closed subsets of X. Then $\mathscr{S}$ is called *hereditary* if every closed subset of a member of $\mathscr{S}$ belongs to $\mathscr{S}$. We let $\mathscr{S}_\sigma$ denote the collection of all countable unions of elements of $\mathscr{S}$.

In the remaining part of this section, let X be a topologically complete space, $\mathscr{S}$ a hereditary collection of (closed) subsets of X and $\Gamma \subseteq \mathscr{H}(X)$ a closed subgroup such that $\mathscr{S}$ is invariant under the action of Γ.

2.1. Definition. An element A of $\mathscr{S}_\sigma$ is called an *$(\mathscr{S}, \Gamma)$-absorber* if for every $S \in \mathscr{S}$ and every collection $\mathscr{U}$ of open subsets of X there is an $h \in \Gamma$ such that h is $\mathscr{U}$-close to 1 while moreover $h(S \cap \bigcup \mathscr{U}) \subseteq A$.

An *isotopy* H of X is a homotopy $H\colon X \times I \to X$ such that the function $\bar{H}\colon X \times I \to X \times I$ defined by $\bar{H}(x, t) = (H(x, t), t)$ is a homeomorphism of $X \times I$.

Let $\mathcal{U}$ be a collection of open subsets of X and let $E \subseteq \mathcal{H}(X)$. A map f is a *$\mathcal{U}$-push in E* if there is an isotopy $H: X \times I \to X$ that is limited by $\mathcal{U}$ and satisfies: $H_0 = 1$, $H_1 = f$ and $H_t \in E$ $(t \in I)$.

2.2. Definition. An element A of $\mathcal{S}_\sigma$ is a *strong $(\mathcal{S}, \Gamma)$-skeletoid* if $A = \bigcup_{i=1}^{\infty} A_i$ with $A_i \in \mathcal{S}$ and $A_i \subseteq A_{i+1}$ for every $i \in N$, such that for every open covering $\mathcal{U}$ of X, every $S \in \mathcal{S}$ and every closed subset $F \subseteq X$ with $F \cap S = \emptyset$, for each $n \in N$ there exists an $m \in N$ and a $\mathcal{U}$-push h in $\{\gamma \in \Gamma: \gamma | F \cup A_n = 1\}$ such that $h(S) \subseteq A_m$.

The following results can all be derived by a similar and standard back and forth technique. For details, see e.g. Bessaga & Pełczyński [4, Chapter IV], Toruńczyk [13], West [15], Geoghegan & Summerhill [11] and Dijkstra [7], [9]. Some of the statements below are copied literally from one of these references.

2.3. Proposition. *Strong negligibility is closed hereditary and σ-additive.*

2.4. Theorem. *Let A and B be $(\mathcal{S}, \Gamma)$-absorbers. Then for every open $O \subseteq X$, every open covering $\mathcal{U}$ of O there is an $h \in \Gamma$ such that h and 1 are $\mathcal{U}$-close and $h(A \cap O) = B \cap O$.*

2.5. Proposition. *Every strong $(\mathcal{S}, \Gamma)$-skeletoid is an $(\mathcal{S}, \Gamma)$-absorber.*

2.6. Theorem. *If A is an $(\mathcal{S}, \Gamma)$-absorber in X and S is an element of $\mathcal{S}_\sigma$, then $S \setminus A$ is strongly negligible in $X \setminus A$.*

Let $\mathcal{S}_k = \{S \subseteq Q: S$ is a Z-set and $\dim S \leqslant k\}$. The following result was established in Dijkstra [7].

2.7. Theorem. *For every $k \in \{0, 1, \ldots\}$ there is a strong $(\mathcal{S}_k, \mathcal{H}(Q))$-skeletoid A_k in Q*

(for convenience, in the remaining part of this paper we let A_{-1} be the empty set). The skeletoids A_k were constructed in the pseudo-interior s of Q (since σ-Z-sets can always be pushed into s, if one proves the existence of a skeletoid of some sort which is a countable union of Z-sets, one can always assume that it is contained in s). The skeletoids A_k are fixed in the remaining part of this paper.

Let $(p_i)_{i=1}^{\infty}$ be a sequence in $(0, 1)$ with $\lim_{i \to \infty} p_i = 1$. For each $i \in N$, let

$$W_i = \{x \in Q: x_i = 1 \text{ and } \forall j \neq i,\ -p_i \leqslant x_j \leqslant p_i\}.$$

Then W_i is a "*shrunken endface*" in the ith coordinate direction, and is easily seen to be a Z-set in Q. Let $W = \bigcup_{i=1}^{\infty} W_i$ and $Y = Q \setminus W$. Note that $A_k \cap W = \emptyset$. In addition, let

$$\Gamma_W = \{h \in \mathcal{H}(Q): h(W_i) = W_i \text{ for each } i\}.$$

It is easily seen that Γ_W is a closed subgroup of $\mathcal{H}(Q)$. The example X of Anderson, Curtis and van Mill [3] mentioned in the introduction, is $Y \setminus D$, where D is any countable dense subset of Y. Our spaces X_k are

$$X_k = Y \setminus A_k,$$

where $A_k \subseteq s$ is the strong $(\mathcal{S}_k, \mathcal{H}(Q))$-skeletoid in Q, $k \in \{-1, 0, 1, \ldots\}$. In proving that the X_k's are as required, we use the following result of Dijkstra [9, 4.3.6].

2.8. Theorem. *Let $\mathcal{U}$ be a collection of open subsets in Q, A a compact space and $F: A \times I \to Q$ a homotopy that is limited by $\mathcal{U}$. If F_0 and F_1 are embeddings of A in Y then there is a $\mathcal{U}$-push h in Γ_W with $h \circ F_0 = F_1$.*

2.9. Corollary. *Let $\mathcal{S}$ be a hereditary collection of Z-sets in Q which is invariant under the action of $\mathcal{H}(Q)$. If A is a strong $(\mathcal{S}, \mathcal{H}(Q))$-skeletoid with $A \cap W = \emptyset$, then A is a strong $(\mathcal{S}_W, \Gamma_W)$-skeletoid, where $\mathcal{S}_W = \{S \in \mathcal{S}: S \cap W = \emptyset\}$.*

Proof. Apply Theorem 2.8. ∎

3. A generalization of Sierpiński's Theorem. The aim of this section is to prove a generalization of Sierpinski's Theorem that no continuum can be partioned into countably many pairwise disjoint nonempty closed subsets, see [12]. We need this generalization in Section 5 to prove that the spaces X_k are as required. Since we feel that the results of this section are of independent interest, we have collected them in a separate section. A better result than ours has recently been obtained by Dijkstra [8].

As usual, S^n denotes the n-sphere, $n \in \{0\} \cup N$.

3.1. Lemma. *Let n be an element of $\{0\} \cup N$. Suppose that X is a compact space such that $S^n \subseteq X$, $X \setminus S^n = \bigcup_{i=1}^{\infty} F_i$, where the F_i's are compacta with the property: for every distinct pair of natural numbers i and j, $\dim(F_i \cap F_j) < n$. Then S^n is a retract of X.*

Proof. First consider the case $n = 0$. Then $S^n = \{-1, 1\}$ and $X = \{-1, 1\} \cup \bigcup_{i=1}^{\infty} F_i$, where the F_i's are pairwise disjoint. Let C be the component of -1 in X. Then C is a countable union of disjoint compacta, since $C = \{-1\} \cup (C \cap \{1\}) \cup \bigcup_{i=1}^{\infty} (F_i \cap C)$. Sierpiński's Theorem [12] implies that $C = \{-1\}$. Because X is compact there is a clopen set O in X that separates -1 and 1. Consequently, $\{-1, 1\}$ is a retract of X.

Assume that the theorem holds for n. Let $O = X \setminus S^{n+1} = \bigcup_{i=1}^{\infty} F_i$ with each F_i compact and $\dim(F_i \cap F_j) \leqslant n$ for $i \neq j$. Let A be a countable dense subset of O and define

$$Z = A \cup \bigcup \{F_i \cap F_j: j \neq i\} \cup S^{n+1}.$$

Since $Z \setminus S^{n+1}$ is a countable union of compacta with dimension at most n, $\dim(Z \setminus S^{n+1}) \leqslant n$. Let H_1 and H_2 be two hemispheres of S^{n+1} such that $S^n = H_1 \cap H_2$. According to Lemma 1.9.1. in [10] there are closed subsets Z_1 and Z_2 of Z such that

$$Z_1 \cup Z_2 = Z, \quad S^{n+1} \cap Z_i = H_i$$

and

$$\dim((Z_1 \cap Z_2)\setminus S^n) < n.$$

Let $\tilde{X}$ be the compact subspace $\mathrm{Cl}_X(Z_1) \cap \mathrm{Cl}_X(Z_2)$. Then $\tilde{X} = S^n \cup \bigcup_{i=1}^{\infty} (F_i \cap \tilde{X})$ where for $i \neq j$, $\dim(F_i \cap F_j \cap \tilde{X}) \leqslant \dim((Z_1 \cap Z_2)\setminus S^n) < n$. The induction hypothesis implies the existence of a retraction $f\colon \tilde{X} \to S^n$. Since H_1 and H_2 are $n+1$-cells there are extensions $f_i\colon \mathrm{Cl}_X(Z_i) \to H_i$ of f with $f_i|H_i = 1$ $(i = 1, 2)$. Let $g = f_1 \cup f_2$. Since f_1 and f_2 coincide on $\tilde{X}$, the function g is a retraction from $\mathrm{Cl}_X(Z)$ onto S^{n+1}. Since Z is dense in X, g is a retraction from X onto S^{n+1}. ■

We now come to the main result in this section.

3.2. Theorem. *Let n be a nonnegative integer and X a compact space. Suppose that R is a closed subset of X such that $X\setminus R = \bigcup_{i=1}^{\infty} F_i$, where the F_i's are compacta with the property: $\dim(F_i \cap F_j) < n$ if $i \neq j$. Then every continuous $f\colon R \to S^n$ can be extended over X.*

Proof. Put $Z = X \cup_f S^n$, the adjunction space with decomposition map $\pi\colon X \to Z$. Observe that Z is a compact (metric) space and that $\pi|X\setminus R$ is a homeomorphism. By Lemma 3.1 there is a retraction $r\colon Z \to S^n$. Define an extension $\bar{f}$ of f by $\bar{f} = r \circ \pi$. ■

4. Some topological properties of the spaces X_k. In this section we shall prove all required topological properties of the spaces X_k, except those on negligibility. The proofs of those results can be found in Section 5.

4.1. Theorem. *Let $k, k' \in \{-1, 0, 1, \ldots\}$. Then*

(1) *X_k is a topologically complete* AR *space,*

(2) *X_k embeds as a linearly convex set in l_2,*

(3) *X_k has the weak discrete approximation property,*

(4) *every compact subset of X_k is a Z-set,*

(5) *$X_k \times X_{k'} \approx l_2$.*

Proof. Since $Q\setminus X_k$ is a σ-Z-set, (1) and (2) follow from [3], Theorem 3.1. Since Q admits "arbitrarily small" maps into W, which is contained in the complement of X_k, (3) follows from [3], Theorem 3.2. That every compact subset of X_k is a Z-set follows from (3) and [3], Theorem 3.3. For (5) apply a similar technique as in [3], Theorem 3.5. ■

We now turn to homogeneity properties of X_k. Let

$$\mathscr{S}_{kW} = \{S \subseteq Y\colon S \text{ is compact and } \dim S \leqslant k\}.$$

Since every compact subset of Y is a Z-set in Q (since Q admits arbitrarily small maps in $W = Q\setminus Y$) it follows that

$$\mathscr{S}_{kW} = \{S \in \mathscr{S}_k\colon S \cap W = \emptyset\}.$$

By Corollary 2.9, A_k is a strong $(\mathscr{S}_{kW}, \Gamma_W)$-skeletoid in Q. This easily implies that for every $f \in \Gamma_W$ the set $f(A_k)$ is also a strong $(\mathscr{S}_{kW}, \Gamma_W)$-skeletoid in Q (the reader

is encouraged to check this). This observation will be used in the proof of Lemma 4.2. In addition, it is also straightforward to prove that A_k is a strong $(\mathscr{S}_{kW}, \mathscr{H}(Y))$-skeletoid in Y.

4.2. Lemma. *Let $\mathscr{U}$ be a collection of open subsets in Q, A a compact space and $F\colon A \times I \to Q$ a homotopy that is limited by $\mathscr{U}$. If F_0 and F_1 are embeddings of A in X_k then there is an $h \in \Gamma_W$ that is $\mathscr{U}$-close to 1 and has the properties $h \circ F_0 = F_1$ and $h(A_k) = A_k$.*

Proof. Let $O = \bigcup \mathscr{U}$. According to Theorem 2.8 there is an $f \in \Gamma_W$ such that $f \circ F_0 = F_1$ while moreover f and 1 are $\mathscr{U}$-close. Define $\mathscr{V} = \{U \cap f^{-1}(U)\colon U \in \mathscr{U}\}$. It is easy to verify that $\bigcup \mathscr{V} = O$. Since $f \in \Gamma_W$, it follows that $f^{-1}(A_k)$ is a strong $(\mathscr{S}_{kW}, \Gamma_W)$-skeletoid. Since $(F_0(A) \cup F_1(A)) \cap A_k = \emptyset$ we have

$$(f^{-1}(A_k) \cup A_k) \cap F_0(A) = \emptyset.$$

According to Proposition 2.5 followed by Theorem 2.4, there exists a $g \in \Gamma_W$ such that g and 1 are $\mathscr{V}'$-close, where $\mathscr{V}' = \{V\setminus F_0(A)\colon V \in \mathscr{V}\}$, while moreover $g(A_k \cap O) = f^{-1}(A_k) \cap O$. Let $h = f \circ g$. Observe that $h \in \Gamma_W$. We now have that

$$h \circ F_0 = F_1 \qquad h(A_k \cap O) = A_k \cap O \quad \text{and} \quad h|Q\setminus O = 1.$$

If $x \in O$ then there is a $U \in \mathscr{U}$ such that $\{x, g(x)\} \subseteq U \cap f^{-1}(U)$ and hence $\{x, f \circ g(x)\} \subseteq U$. We conclude that h is $\mathscr{U}$-close to 1. ■

We now come to the main result in this section.

4.3. Theorem. *Let $\mathscr{U}$ be a collection of open subsets in X_k, A a compact space and $F\colon A \times I \to X_k$ a homotopy that is limited by $\mathscr{U}$. If F_0 and F_1 are embeddings then there is an $h \in \mathscr{H}(X_k)$ that is $\mathscr{U}$-close to 1 and has the property $h \circ F_0 = F_1$.*

Proof. Let $\mathscr{U}'$ be a collection of open subsets of Q such that

$$\mathscr{U} = \{U \cap X_k\colon U \in \mathscr{U}'\}.$$

It is clear that F is limited by $\mathscr{U}'$. According to Lemma 4.2 there is an $\tilde{h} \in \Gamma_W$ that is $\mathscr{U}'$-close to 1 and has the properties $\tilde{h} \circ F_0 = F_1$ and $\tilde{h}(A_k) = A_k$. It is clear that $h = \tilde{h}|X_k \in \mathscr{H}(X_k)$ is as required. ■

4.4. Remark. In view of Theorem 2.8 it is natural to ask whether the homeomorphism of Theorem 4.3 can be chosen in such a way that it is isotopic to the identity of X_k. This is not the case for $k = 0$. We believe that for $k > 0$ the spaces X_k also behave "badly" in this respect, but we have no proof of this assertion.

Consider an isotopy $H\colon X_0 \times I \to X_0$ such that $H_0 = 1$, and define $\tilde{H}\colon X_0 \times \times I \to X_0 \times I$ by $\tilde{H}(x, t) = (H(x, t), t)$. We shall show that $H_t = 1$ for every $t \in I$. Pick an arbitrary point x in A_0 and let $(x_n)_n$ be a sequence in X_0 that converges in Q to x. There is a copy J of $[0, 1)$ in X_0 such that $\{x_n\colon n \in N\} \subseteq J$ and $J \cup \{x\} \approx I$. If we put $B = \tilde{H}(J \times I)$ then B is a closed subset of $X_0 \times I$ which is homeomorphic to $[0, 1) \times I$. Let $K = \mathrm{Cl}_{Q \times I}(B)\setminus B$ and let $\tilde{K}$ be the projection of K onto the first factor of the product $Q \times I$. Then K and $\tilde{K}$ are continua which are contained in $(W \cup A_0) \times I$ and $W \cup A_0$, respectively. Since $A_0 \cup W$ can be written as a disjoint union of countably many compacta and since $x \in \tilde{K} \cap A_0$, Sierpiński's Theorem gives that $\tilde{K} \subseteq A_0$. Now, A_0 is totally disconnected and hence $\tilde{K} = \{x\}$. This implies that $\lim_{i\to\infty} H_t(x_i) = x$ for every $t \in I$, and hence H_t can be extended over A_0 such that the extension is the identity on A_0. Since A_0 is dense in Y, we have that $H_t = 1$ for every $t \in I$.

So we may conclude that if f and g are isotopic members of $\mathscr{H}(X_0)$ then $f = g$.

4.5. Corollary. *Let A be compact and $f\colon A \to X_k$ continuous. If A' is closed in A such that $f|A'$ is an embedding and if $\mathscr{U}$ is an open covering of X_k, then there is an embedding g of A in X_k such that g and f are $\mathscr{U}$-close and $g|A' = f|A'$.*

Proof. Since the complement of X_k in Q is a σ-Z-set there is a subset R of X_k that is homeomorphic to s, see e.g. Bessaga & Pełczyński [4], Chapter V. Let C be a subset of R that is homeomorphic to $f(A)$. Since X_k is an AR, Theorem 4.1 (1), both embeddings of $f(A)$ are homotopic, and we can apply Theorem 4.3 to find an $h \in \mathscr{H}(X_k)$ such that $h \circ f(A) \subseteq R$. Since $R \approx s$, there is an embedding g of A in R such that g and $h \circ f$ are $h(\mathscr{U})$-close and $g|A' = h \circ f|A'$ (Chapman [6], 8.1). If $\tilde{g} = h^{-1} \circ g$ then $\tilde{g}$ and f are $\mathscr{U}$-close and $\tilde{g}|A' = f|A'$. ■

5. Negligibility properties of X_k. In this section we shall verify the negligibility properties of X_k which were announced in the introduction. Let $k \in \{-1, 0, 1, \ldots\}$ be fixed.

5.1. Theorem. *Every σ-compact subset of X_k with dimension at most k is strongly negligible.*

Proof. As observed in Section 4, A_k is a strong $(\mathscr{S}_{k,W}, \mathscr{H}(Y))$-skeletoid. Now apply Proposition 2.5 and Theorem 2.6. ■

We identify S^{n-1} and the (geometrical) boundary ∂I^n of I^n, for every $n \in N$. Let X be a space. A mapping $f\colon X \to I^n$ is called *essential* if $f|f^{-1}(S^{n-1})$ cannot be extended to a map $g\colon X \to S^{n-1}$.

5.2. Lemma. *Let n be a natural number with $n > k$. If A is a compact subset of X_k and $f\colon A \to I^n$ is essential, then $f^{-1}(\mathrm{Int}(I^n))$ is not negligible in X_k.*

Proof. Let $R = f^{-1}(S^{n-1})$ and $O = A \setminus R$. In view of Corollary 4.5 we may assume that $A \times I$ is a subset of X_k such that $A \times \{0\}$ coincides with A. Assume that O is a negligible subset of X_k. This implies that $Z = (A \times I) \setminus O$ can be embedded as a closed subset of X_k. Assume that Z is reembedded as a closed subset of X_k and let $\bar{Z}$ be the closure of Z in Q. Define $Z^* = \bar{Z} \setminus Z$ and note that the local compactness of $A \times (0, 1]$ implies that $Z^* \cup R$ is compact. Also, Z^* is a closed subset of $Q \setminus X_k = A_k \cup W$. Since $Z^* \cap A_k$ is σ-compact and at most $(n-1)$-dimensional, we can find a sequence F_i, $i \in N$, of compact subsets of $Z^* \cap A_k$ such that $Z^* \cap A_k = \bigcup_{i=1}^{\infty} F_i$ and $F_i \cap F_j$ is at most $(n-2)$-dimensional for all distinct $i, j \in N$. In addition, observe that $Z^* \cap W$ is a countable disjoint union of compacta and that $W \cap A_k = \emptyset$. Theorem 3.2 implies that the map $g = f|R$ can be extended to a map $\bar{g}\colon (Z^* \cup R) \to S^{n-1}$. Since S^{n-1} is an ANR, there is an open neighbourhood U of $Z^* \cup (R \times I)$ such that the map h defined by

$$h(x) = \bar{g}(x) \quad \text{if} \quad x \in Z^* \cup R,$$
$$h(x, t) = f(x) \quad \text{if} \quad (x, t) \in R \times I,$$

can be extended to a map $\bar{h}\colon U \to S^{n-1}$. Since $(Q \setminus U) \cap (A \times (0, 1])$ is compact, there is an $\varepsilon \in (0, 1]$ such that $A \times \{\varepsilon\} \subseteq U$. Define the function $\eta\colon A \to S^{n-1}$ by $\eta(a) = \bar{h}(a, \varepsilon)$, $a \in A$. Then $\eta|R = f|R$ and $\eta(A) \subseteq S^{n-1}$, which means that f is not essential. ■

5.3. Corollary. *If $n \in N$ and $n > k$ then there exist copies of R^n in X_k that are not negligible.*

Proof. I^n is embeddable in X_k, Corollary 4.5, and 1_{I^n} is essential. ■

5.4. Corollary to Corollary: *X_k is not homeomorphic to l_2.*

Proof. As remarked in the introduction, every σ-compact subset of l_2 is strongly negligible. ■

We now come to the announced characterizations of topological dimension in terms of negligibility.

5.5. Theorem *Let $k \neq -1$. For every compact space A, the following statements are equivalent*:

(1) $\dim A \leqslant k$,

(2) *there is an embedding f of A in X_k such that for every open O in A, $f(O)$ is negligible in X_k,*

(3) *every embedding f of A in X_k has the property that for every open O in A, $f(O)$ is negligible in X_k.*

Proof. (1) → (3). If $\dim A \leqslant k$ then $f(O)$ is a σ-compactum with dimension at most k. Now apply Theorem 5.1.

(3) → (2). By Corollary 4.5, this is a triviality.

(2) → (1). Assume that A satisfies (2). According to Lemma 5.2, no function from A into I^{k+1} is essential. This means that $\dim A \leqslant k$, [10, 1.9.A]. ■

5.6. Theorem. *Let A be a σ-compact space. The following statements are equivalent*:

(1) $\dim A \leqslant k$,

(2) *there is an embedding f of A in X_k such that $f(A)$ is strongly negligible in X_k,*

(3) *every subset of X_k that is homeomorphic to A is strongly negligible.*

Proof. (1) → (3). Apply Theorem 5.1.

(3) → (2). Embed with Corollary 4.5 some compactification of A in X_k.

(2) → (1). We first consider the case $k \neq -1$. Let $A = \bigcup_{i \in N} C_i$, where C_i is compact for every $i \in N$. According to Proposition 2.3 every $f(C_i)$ is strongly negligible. Observe that this implies that for every open subset O of C_i, $f(O)$ is negligible. Applying Theorem 5.5 we obtain that $\dim C_i \leqslant k$. So we may conclude with the Countable Sum Theorem ([10], 3.1.8) that $\dim A \leqslant k$.

Now let $k = -1$. Assume that A is a non-empty space that satisfies (2). If $p \in A$ then $\{f(p)\}$ is strongly negligible in X_k (Proposition 2.3). Since X_{-1} is homogeneous, Theorem 4.3, and strong negligibility is σ-additive, Proposition 2.3, every countable dense subset of $X_{-1} = Y$ is negligible. This contradicts [3], 6.2 and hence $A = \emptyset$. Note that we did not use here that A is σ-compact: Y has no strongly negligible subsets other than the empty set. ■

It is natural to ask whether in Theorem 5.6 "strong negligibility" can be replaced by "negligibility". In the remaining part of this section we shall show that this is not the case.

For information concerning shape theory, see Borsuk [5]. Recall that if X is compact then the *fundamental dimension* $\mathrm{Fd}X$ of X is defined as follows:

$$\mathrm{Fd}X = \min\{n \in N\colon \exists \text{ compact } Y \text{ with } \mathrm{Sh}\,Y = \mathrm{Sh}\,X \text{ and } \dim Y \leqslant n\}.$$

5.7. THEOREM. *Let $A \subseteq X_k$ be compact such that $\mathrm{Fd}A \leqslant k$. Then A is negligible in X_k.*

Proof (sketch). By Corollary 4.5 we can choose a compact $B \subseteq X_k$ such that $\mathrm{Sh}A = \mathrm{Sh}B$ and $\dim B \leqslant k$. Precisely as in Chapman [6], 25.2, we can construct a sequence $\{h_n\}_{n=1}^{\infty}$ of homeomorphisms of Q such that

(1) $h = \lim_{n\to\infty} h_n \circ \ldots \circ h_1 | Q\setminus A$ is a homeomorphism of $Q\setminus A$ onto $Q\setminus B$,

(2) if $x \in Q\setminus A$ then the sequence $\{h_n \circ \ldots \circ h_1(x)\}_{n=1}^{\infty}$ is eventually constant,

(3) $h_n(Q\setminus X_k) = Q\setminus X_k$ for all $n \in N$.

The difference with Chapman [6], 25.2, is (3). However, this can easily be taken care of by using Lemma 4.2 instead of Chapman [6], 19.4. By (1), (2) and (3), $h(X_k\setminus A) = X_k\setminus B$. Since $X_k\setminus B \approx X_k$ by Theorem 5.1, we conclude that $X_k\setminus A \approx X_k$. ■

Observe that Theorems 5.6 and 5.7 imply that if $k \geqslant 0$ and $B \subseteq X_k$ is an n-cell, then B is negligible and B is strongly negligible iff $n \leqslant k$. It can be shown that every n-cell is also negligible in $X_{-1} = Y$. This is due to R. D. Anderson; for details see Dijkstra [9]. We believe that the converse of Theorem 5.7 is also true if $k \geqslant 0$, i.e. that if $A \subseteq X_k$ is compact and negligible then $\mathrm{Fd}A \leqslant k$. We have only been able to show this for $k = 0$. For details see Dijkstra [9].

5.8. CONJECTURE. *Let $k \geqslant 0$ and let $A \subseteq X_k$ be compact. Then A is negligible iff $\mathrm{Fd}A \leqslant k$.*

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