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A METHOD FOR CONSTRUCTING ORDERED CONTINUA

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We present a method for constructing ordered continua. We illustrate our method by constructing (i) a new order-homogeneous non-reversible continuum, and (ii) an ordered continuum with a minimal set of continuous self-maps.

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0. Introduction

In recent years some types of topological spaces were constructed having only the 'necessary'' continuous self-maps (of a special kind). Examples of this type are: a topological group having no other continuous self-maps other than the translations and the constant maps [5] and an infinite-dimensional inner-product space with only trivial bounded linear operators (an operator A is trivial if for some scalar λ, $A-\lambda I$ has finite dimensional range) [6]. For older results of this type see [3, 7]. We pursue this line a little further by constructing an ordered continuum with only the necessary continuous self-maps: for an explanation of 'necessary' in this context see Section 5. Actually our main result is a general method for constructing ordered continua, of which the above-mentioned continuum is an illustration. A second example is presented in Section 4 (this example came first in time), which is an order-homogeneous non-reversible ordered continuum. The first (real) example of this type was constructed by Shelah [8]. Our example is totally different from Shelah's (see Section 4 for an explanation) and, in our opinion, somewhat simpler.

The paper is organized as follows: Section 1 contains the necessary definitions and preliminaries. Section 2 concerns special subsets of the unit interval [0, 1]. The construction presented there is very much like the one in [4]. In Section 3 we show how to construct ordered continua from families of subsets of [0, 1]. In Sections 4 and 5 we construct the continua mentioned above using the method of Section 3, with input from Section 2.

1. Definitions and preliminaries

Our notation and terminology is fairly standard, see e.g. [1, 2].

1.0. An *ordered continuum* is a compact, connected linearly ordered topological space, equivalently, a complete and densely linearly ordered set equipped with the order topology. Two ordered continua K and L are *isomorphic* if there exists an order-preserving bijection $f: K \to L$; if there exists an order-reversing bijection $f: K \to L$ then K and L are *anti-isomorphic.*

Usually the ordering relation is denoted by $\leqslant$, we are sure this will not cause confusion. There is one exception: if $\prod_{i\in\omega} X_i$ is a product of linearly ordered sets then $<|$ denotes the lexicographic order: $x <| y$ iff for some n, $x_i = y_i$ for $i \in n$ and $x_n < y_n$.

1.1. A set $A \subseteq [0, 1]$ is a *BB-set* if A and its complement intersect every Cantor set of $[0, 1]$ (BB stands for *Bi-Bernstein*).

1.2. We fix some notation:

(i) if X is a set then $X^{<\omega}$ denotes $\bigcup_{n\in\omega} X^n$,

(ii) If $x \in X^\omega$ then $x \upharpoonright n = \langle x_0, \ldots, x_{n-1}\rangle$,

(iii) If s is a finite sequence of points, say $\langle s_0, \ldots s_{n-1}\rangle$, and x is a point then $\langle s, x\rangle = \langle s_0, \ldots, s_{n-1}, x\rangle$,

(iv) $\langle\ \rangle$ denotes the empty sequence,

(v) finally, the symbol $\approx$ is used to denote both homeomorphism of topological spaces and isomorphism of ordered sets.

2. Subsets of [0, 1]

This section contains some results on the existence of some special subsets of $[0, 1]$ and their properties. We start with our principal tool for constructing various subsets of $[0, 1]$. For convenience we adopt the following conventions: if $\mathscr{G}$ is a group of autohomeomorphisms of $\mathbb{R}$ then a set A, where $A \subseteq [0, 1]$, will be called *$\mathscr{G}$-invariant* if for all $a \in A$, $\mathscr{G}(a) \cap [0, 1]$ where $\mathscr{G}(a) = \{g(a): g \in \mathscr{G}\}$. Let f be a function such that $\operatorname{dom} f$ and $\operatorname{range} f$ are subsets of $\mathbb{R}$. If $\mathscr{G}$ is a group of autohomeomorphisms of $\mathbb{R}$ define $S(f, \mathscr{G}) = \{x \in \operatorname{dom} f: f(x) \notin \mathscr{G}(x)\}$. We call f *$\mathscr{G}$-singular* if $f(S(f, \mathscr{G}))$ has cardinality 2^ω. For every $\mathscr{G}$-singular f we choose a set $C(f, \mathscr{G}) \subseteq \mathbb{R}$ such that $f \upharpoonright C(f, \mathscr{G})$ is one-to-one while moreover $f(C(f, \mathscr{G})) = f(S(f, \mathscr{G}))$. Observe that the cardinality of $C(f, \mathscr{G})$ equals 2^ω. The sets $C(f, \mathscr{G})$ remain fixed throughout the remaining part of this paper.

2.0. Theorem. *Let $\mathscr{G}$ be a countable group of autohomeomorphisms of $\mathbb{R}$, let $\mathscr{F}$ be a family of functions such that $|\mathscr{F}| \leqslant 2^\omega$ and $\operatorname{dom} f$, $\operatorname{range} f \subseteq [0, 1]$ for all $f \in \mathscr{F}$, and let B be a subset of $[0, 1]$ of cardinality less than 2^ω. Then there exists a pairwise disjoint family $\{A_\alpha\}_{\alpha\in 2^\omega}$ of $\mathscr{G}$-invariant* BB-*subsets of $[0, 1]$ which all miss B and a* BB-*set $V \subseteq [0, 1] \setminus \bigcup_{\alpha\in 2^\omega} A_\alpha$ such that if $f \in \mathscr{F}$ is $\mathscr{G}$-singular then $|C(f, \mathscr{G}) \cap A_\alpha| = 2^\omega$ and $|f(C(f, \mathscr{G}) \cap A_\alpha) \cap V| = 2^\omega$, for every $\alpha \in 2^\omega$.*

Proof. We assume of course that every $f \in \mathcal{F}$ is $\mathcal{G}$-singular. Let $\{f_{\langle\alpha,\beta\rangle}: \langle\alpha, \beta\rangle \in 2^\omega \times 2^\omega\}$ be a listing of $\mathcal{F}$ such that each function occurs 2^ω times in each row $\{f_{\langle\alpha,\beta\rangle}: \beta \in 2^\omega\}$. Let $\{K_{\langle\alpha,\beta\rangle}: \langle\alpha, \beta\rangle \in 2^\omega \times 2^\omega\}$ be a similar listing of the set of all Cantor sets in $[0, 1]$. In addition, let $B' = \bigcup_{b \in B} \mathcal{G}(b)$. Observe that $|B'| < 2^\omega$.

We shall find points $x(\alpha, \beta, 0)$, $x(\alpha, \beta, 1)$, $y(\alpha, \beta, 0)$, $y(\alpha, \beta, 1) \in [0, 1] \setminus B'$ such that:

(1) $x(\alpha, \beta, 0) \in C(f_{\langle\alpha,\beta\rangle}, \mathcal{G})$ and $y(\alpha, \beta, 0) = f_{\langle\alpha,\beta\rangle}(x(\alpha, \beta, 0))$.

(2) $x(\alpha, \beta, 1)$, $y(\alpha, \beta, 1) \in K_{\langle\alpha,\beta\rangle}$.

(3) if $(\alpha, \beta, i) \neq (\alpha', \beta', i')$, where $\alpha, \beta, \alpha', \beta' \in 2^\omega$ and $i, i' \in 2$, then the collection $\{\mathcal{G}(x(\alpha, \beta, i)), \mathcal{G}(x(\alpha', \beta', i')), \mathcal{G}(y(\alpha, \beta, i)), \mathcal{G}(y(\alpha', \beta', i'))\}$ is pairwise disjoint.

We then let

$$A_\alpha = \bigcup \{\mathcal{G}(x(\alpha, \beta, i)): \beta \in 2^\omega, i \in 2\} \cap [0, 1], \quad \alpha \in 2^\omega,$$

and

$$V = \{y(\alpha, \beta, i): \langle\alpha, \beta\rangle \in 2^\omega \times 2^\omega, i \in 2\}.$$

By definition the A_α are $\mathcal{G}$-invariant, by (2) every A_α meets every Cantor set of $[0, 1]$, hence the properties of $A_{\alpha+1}$ show that A_α is a BB-set, for by (3)

$$\alpha \neq \gamma \to A_\alpha \cap A_\gamma = \emptyset.$$

By (3) it also follows that V intersects every Cantor set in $[0, 1]$ and misses every A_α, whence V is also a BB-set. Let $f \in \mathcal{F}$ and $\alpha \in 2^\omega$. Let $J(f, \alpha) = \{\beta: f = f_{\langle\alpha,\beta\rangle}\}$. Then $C(f, \mathcal{G}) \cap A_\alpha \supseteq \{x(\alpha, \beta, 0): \beta \in J(f, \alpha)\}$ and this last set has cardinality 2^ω, so $|C(f, \mathcal{G}) \cap A_\alpha| = 2^\omega$. Furthermore, $f(C(f, \mathcal{G}) \cap A_\alpha) \cap V \supseteq \{y(\alpha, \beta, 0): \beta \in J(f, \alpha)\}$ and so $|f(C(f, \mathcal{G}) \cap A_\alpha) \cap V| = 2^\omega$. So $\{A_\alpha\}_{\alpha \in 2^\omega}$ is as required.

Let us construct the points $x(\alpha, \beta, i)$, $y(\alpha, \beta, i)$ for α, $\beta \in 2^\omega$ and $i \in 2$. Fix a well-ordering $<\!|$ of $2^\omega \times 2^\omega$ in type 2^ω. Assume that $\langle\alpha, \beta\rangle \in 2^\omega \times 2^\omega$ and that $x(\gamma, \delta, i)$ and $y(\gamma, \delta, i)$ are found for $\langle\gamma, \delta\rangle <\!| \langle\alpha, \beta\rangle$ and $i \in 2$, such that (1), (2) and (3) are fulfilled for all $\langle\gamma, \delta\rangle <\!| \langle\alpha, \beta\rangle$. Let

$$H = \bigcup \{\mathcal{G}(x(\gamma, \delta, i)): \langle\gamma, \delta\rangle <\!| \langle\alpha, \beta\rangle, i \in 2\} \cup \bigcup \{\mathcal{G}(y(\gamma, \delta, i)): \langle\gamma, \delta\rangle <\!| \langle\alpha, \beta\rangle\}.$$

Then $|H| < 2^\omega$. Put $f = f_{\langle\alpha,\beta\rangle}$. Now $|C(f, \mathcal{G})| = 2^\omega$ and $f \restriction C(f, \mathcal{G})$ is one-to-one so $|x \in C(f, \mathcal{G}): x, f(x) \notin H \cup B'\}| = 2^\omega$. Pick $x(\alpha, \beta, 0)$ from this set and let $y(\alpha, \beta, 0) = f(x(\alpha, \beta, 0))$. Furthermore, $|K_{\langle\alpha,\beta\rangle}| = 2^\omega$ and the cardinality of $H \cup B' \cup \mathcal{G}(x(\alpha, \beta, 0)) \cup \mathcal{G}(y(\alpha, \beta, 0))$ is less than 2^ω, so we can pick $x(\alpha, \beta, 1) \in K_{\langle\alpha,\beta\rangle} \setminus (H \cup B' \cup \mathcal{G}(x(\alpha, \beta, 0)) \cup \mathcal{G}(y(\alpha, \beta, 0)))$ and $y(\alpha, \beta, 1) \in K_{\langle\alpha,\beta\rangle} \setminus (H \cup B' \cup \mathcal{G}(x(\alpha, \beta, 0) \cup \mathcal{G}(x(\alpha, \beta, 1) \cup \mathcal{G}(y(\alpha, \beta, 0)))$.

Now for $z = x(\alpha, \beta, 0)$, $x(\alpha, \beta, 1)$, $y(\alpha, \beta, 0)$, or $y(\alpha, \beta, 1)$ we have $z \notin H \cup B'$ so since $\mathcal{G}$ is a group, $\mathcal{G}(z) \cap (H \cup B') = \emptyset$. It is also easily seen that the collection $\{\mathcal{G}(x(\alpha, \beta, 0)), \mathcal{G}(x(\alpha, \beta, 1)), \mathcal{G}(y(\alpha, \beta, 0)), \mathcal{G}(y(\alpha, \beta, 1))\}$ is pairwise disjoint. This completes the construction. $\square$

2.1. Corollary. *Let $\mathscr{G}$ be a countable group of autohomeomorphisms of $\mathbb{R}$, let $\mathscr{F}$ be a family of functions such that $|\mathscr{F}| \leq 2^\omega$ and* $\operatorname{dom} f$, $\operatorname{range} f \subseteq [0, 1]$ *for all $f \in \mathscr{F}$. Then there exists a family $\{A_\alpha\}_{\alpha \in 2^\omega}$ of $\mathscr{G}$-invariant* BB-*subsets of* $[0, 1]$ *with the property that for distinct α, $\beta \in 2^\omega$ we have that $A_\alpha \cap A_\beta = \mathscr{G}(0) \cup \mathscr{G}(1)$, and a* BB-*set* $V \subseteq [0, 1] \setminus \bigcup_{\alpha \in 2^\omega} A_\alpha$ *such that if $f \in \mathscr{F}$ is $\mathscr{G}$-singular then $|C(f, \mathscr{G}) \cap A_\alpha| = 2^\omega$ and $|f(C)f, \mathscr{G}) \cap A_\alpha) \cap V| = 2^\omega$, for every $\alpha \in 2^\omega$.*

Proof. Apply Theorem 2.0 with $B = \mathscr{G}(0) \cup \mathscr{G}(1)$ to get V and pairwise disjoint A_α's. Then $\{A_\alpha \cup (\mathscr{G}(0) \cup \mathscr{G}(1)): \alpha \in 2^\omega\}$ and $V \setminus (\mathscr{G}(0) \cup \mathscr{G}(1))$ are as required. □

2.2. Remark. Let $\{A_\alpha\}_{\alpha \in 2^\omega}$ and V be as in Theorem 2.0 or Corollary 2.1. In addition, let $J \subseteq 2^\omega$ be non-empty and put $A_J = \bigcup_{\alpha \in J} A_\alpha$. Then the properties of V show that A_J is a BB-set, and also if $f \in \mathscr{F}$ is $\mathscr{G}$-singular then $|f(A_J) \cap V| = 2^\omega$. Simply pick $\alpha \in J$ then $f(A_\alpha) \cap V \subseteq f(A_J) \cap V$.

We now give some applications of Theorem 2.0 to get the sets we need to build our continua.

2.3. Example. *There exists a* BB-*set* $A \subseteq [0, 1]$ *containing* 0 *and* 1 *such that* $[0, 1] \setminus A$ *is isomorphic to each interval $(x, y) \setminus A$ and such that if $f: A \to [0, 1]$ is monotonically non-increasing and* $|\operatorname{range} f| = 2^\omega$ *then* $|f(A) \setminus A| = 2^\omega$.

Proof. Let $\mathscr{G}$ be the group of all homeomorphisms of $\mathbb{R}$ of the form $x \to px + q$, with p, $q \in Q$ and $p > 0$ (as usual, Q denotes the set of rational numbers). Let

$$\mathscr{F} = \{f: [0, 1] \to [0, 1]: f \text{ is monotonically non-increasing}\}$$

Observe that $\mathscr{G}$ is a countable group and that $|\mathscr{F}| \leq 2^\omega$ since each $f \in \mathscr{F}$ has only countably many points of discontinuity. Let $\{A_\alpha\}_{\alpha \in 2^\omega}$ and V be as in Corollary 2.1, and put $A = A_0$. Observe that 0, $1 \in A$.

(1) Let x, $y \in [0, 1]$ with $x < y$. If x, $y \in Q$ then $\varphi: t \to (y - x)t + x$ maps $[0, 1] \setminus A$ isomorphically onto $(x, y) \setminus A$ ($\mathscr{G}$-invariance of $[0, 1] \setminus A$). Otherwise let $\langle q_n \rangle_{n \in \mathbb{Z}}$ be a sequence in $(x, y) \cap Q$ such that $q_n \to x$ if $n \to -\infty$ and $q_n \to y$ if $n \to \infty$. All intervals with rational endpoints are isomorphic so we can map $(q_0, q_1) \setminus A$ isomorphically onto $(1/2, 2/3) \setminus A$, $(q_{-1}, q_0) \setminus A$ onto $(1/3, 1/2) \setminus A$, etc. The combination of those mappings yields an isomorphism between $(x, y) \setminus A$ and $[0, 1] \setminus A$.

(2) Let $f: A \to [0, 1]$ be non-decreasing and extend f to $\bar{f}: [0, 1] \to [0, 1]$ which is non-increasing, e.g., $\bar{f}(x) = \sup\{f(a): a \in A, a \leq x\}$. Then $\bar{f} \in \mathscr{F}$. Now if $|\operatorname{range} f| = 2^\omega$ then $|\operatorname{range} \bar{f}| = 2^\omega$ so we have a set $C \subseteq [0, 1]$ of cardinality 2^ω on which f is one-to-one. The set $\{x \in [0, 1]: \bar{f}(x) \in \mathscr{G}(x)\}$ is countable: every $g \in \mathscr{G}$ is strictly increasing so for each $g \in \mathscr{G}$, $g(x) = \bar{f}(x)$ for at most one $x \in [0, 1]$. So without loss of generality $\bar{f}(x) \notin \mathscr{G}(x)$ for every $x \in C$. Since $f \restriction C$ is one-to-one, we conclude that $\bar{f}$ is $\mathscr{G}$-singular. But then $|C(\bar{f}, \mathscr{G}) \cap A| = 2^\omega$ and $|\bar{f}(A) \setminus A| = 2^\omega$, since $\bar{f}(A) \cap V \subseteq \bar{f}(A) \setminus A$. Since $\bar{f}$ extends f, we find that $|f(A) \setminus A| = 2^\omega$.

For later use we mention the following. If $f:(x, y)\cap A\to[0,1]$ is non-increasing and $|\text{range } f|=2^\omega$ then $|f((x,y)\cap A)\setminus A|=2^\omega$. This follows from the observation that $(x,y)\cap A$ is isomorphic to $A\setminus\{0,1\}$ (same proof as for $[0,1]\setminus A$).

We now exhibit BB-sets which have practically no non-trivial monotone self-maps.

2.4. Example. *There exists a family* $\{A_\alpha\}_{\alpha\in 2^\omega}$ of BB-sets of $[0,1]$ and a set $V\subseteq[0,1]\setminus\bigcup_{\alpha\in 2^\omega}A_\alpha$ such that:

(1) $\alpha\neq\beta\to A_\alpha\cap A_\beta=\{0,1\}$.

(2) *If* $f:(x,y)\cap A_\alpha\to[0,1]$ *is non-decreasing or non-increasing and if for some* $C\subseteq(x,y)\cap A, |C|=2^\omega, f\restriction C$ *is one-to-one and* $\forall x\in C, f(x)\neq x$, *then* $|f(A_\alpha)\cap V|=2^\omega$.

Proof. Apply Corollary 2.1 with $\mathscr{G}=\{\text{id}\}$ and

$$\mathscr{F}=\{f: f \text{ is non-decreasing or non-increasing, dom } f \text{ is a closed subinterval of } [0,1] \text{ and range } f\subseteq[0,1]\}.$$

Corollary 2.1 gives us $\{A_\alpha\}_{\alpha\in 2^\omega}$ and V such that the A_α's pairwise intersect in $\mathscr{G}(0)\cup\mathscr{G}(1)=\{0,1\}$. The rest follows as in Example 2.3. □

We conclude with a proposition on monotone maps from BB-sets.

2.5. Lemma *Let* $A\subseteq[0,1]$ *be a* BB-*set and* $f: A\to X$ *a monotone map where* X *is any linearly ordered set. Then* $|f(A)|\leqslant\omega$ *or* $|f(A)|=2^\omega$.

Proof. Let $B=\{x: f^{-1}(x)$ contains more than one point$\}=\{x: f^{-1}(x)$ is a non-degenerate interval of $A\}$. Then $|B|\leqslant\omega$ since A is separable. Let $G=[0,1]\setminus\bigcup_{x\in B}\overline{f^{-1}(x)}$ (closure in $[0,1]$). Then G is a G_δ-subset of $[0,1]$ and f is one-to-one on $G\cap A$ and

$$f(A)=B\cup f(G\cap A).$$

Now either $|G|\leqslant\omega$ in which case $|f(A)|\leqslant\omega$, or $|G|=2^\omega$ in which case $|G\cap A|=2^\omega$ (since A is a BB-set) so $|f(A)|=2^\omega$. □

2.6. Proposition. *Let* $A\subseteq[0,1]$ *be a* BB-*set and let*

$$X_A=\{\langle x,i\rangle\in[0,1]\times\{0,1\}: x\in A\to i=0\}$$

ordered lexicographically, i.e. X_A *is the compact* LOTS *obtained by splitting the points of* $[0,1]\setminus A$. *Let* $f: X_A\to X$ *be a monotone map where* X *is any linearly ordered set. Then* $|f(X_A)|\leqslant\omega$ *or* $|f(X_A)|=2^\omega$.

Proof. By Lemma 2.5, $|f(A)|\leqslant\omega$ or $|f(A)|=2^\omega$. We show that $|f(X_A)|\leqslant\omega$ if $|f(A)|\leqslant\omega$. Let $F=\bigcup_{z\in f(A)}\overline{f^{-1}(z)\cap A}$ (closure in $[0,1]$). In addition, let $B=\{x\in[0,1]$: x is a boundary point of some $f^{-1}(z)\cap A$, $z\in f(A)\}$ and $C=[0,1]\setminus F$. Then B and C are countable: B is countable since $f(A)$ is countable and each $f^{-1}(z)\cap A$ can have at most two boundary points and C is a G_δ-set disjoint from A. Let $x \in [0,1]\setminus A$ and suppose $f(\langle x, 0\rangle) \notin f(A)$ or $f(\langle x, 1\rangle) \notin f(A)$. Then $x \notin f^{-1}(z) \cap A$ for all $z \in f(A)$, since otherwise by monotonicity of f, we have that $f(\langle x, 0\rangle) = f(\langle x, 1\rangle) = z$. So $x \in B \cup C$. We conclude that

$$f(X_A) = f(A) \cup \bigcup\{\{f(\langle x, 0\rangle), f(\langle x, 1\rangle)\}: x \in B \cup C\}$$

is countable. □

2.7. Remark. For later use we note that also $\{x \in [0,1]\setminus A: f(\langle x, 0\rangle) \neq f(\langle x, 1\rangle)\}$ is contained in $B \cup C$ and that this set is countable too.

3. Continua from subsets of [0, 1]

In this section we associate with each collection $\mathscr{A}$ of subsets of $[0,1]$ having the property that $0, 1 \in A$ for every $A \in \mathscr{A}$, an ordered continuum $L_{\mathscr{A}}$. For later use we will identify some special subspaces of these continua.

The idea is as follows: start with $[0,1]$ and let $A \subseteq [0,1]$ contain 0 and 1, replace each point of $[0,1]\setminus A$ by $[0,1]$, inside of each of these copies of $[0,1]$ take a set containing 0, 1 and replace the points in the complement by $[0,1]$, etc. This gives an inverse sequence of ordered continua with monotone bonding maps (collapsing the inserted intervals) whose limit will be the ordered continuum $L_{\mathscr{A}}$ ($\mathscr{A}$ is the set of chosen subsets of $[0,1]$). For notational purposes we shall give a different description of $L_{\mathscr{A}}$ as a subset of the lexicographically ordered Hilbert cube $[0,1]^\omega$.

3.0 Definition. Let $\mathscr{A}$ be a collection of subsets of $[0,1]$ with the property that $0, 1 \in A$ for every $A \in \mathscr{A}$. Assume that $\mathscr{A}$ is indexed (not necessarily faithfully) by $[0,1]^{<\omega}$. We put

$$L_{\mathscr{A}} = \{\boldsymbol{x} \in [0,1]^\omega: \text{if } x_n \in A_{\boldsymbol{x}\restriction n} \text{ for some } n \text{ then } x_i = 0 \text{ for } i > n\};$$

we order $L_{\mathscr{A}}$ lexicographically.

3.1. Lemma. *$L_{\mathscr{A}}$ is an ordered continuum.*

Proof. We have to show that $<\!|$ is a dense and complete order on $L_{\mathscr{A}}$. We first make the following remark: if $\boldsymbol{x} \in L_{\mathscr{A}}$ and $n \in \omega$ then $\boldsymbol{x}^n = \langle \boldsymbol{x} \restriction n, 0, 0, 0, \ldots\rangle \in L_{\mathscr{A}}$. For if $x_i \in A_{\boldsymbol{x}\restriction i}$ for some $i \in n$ then $\boldsymbol{x}^n = \boldsymbol{x} \in L_{\mathscr{A}}$, and otherwise, since $x_i^n = 0$ for $i \geq n$, we have $(x_i^n \in A_{\boldsymbol{x}^n\restriction i} \to x_j^n = 0)$ for $j > i \geq n$; so $\boldsymbol{x}^n \in L_{\mathscr{A}}$.

(1) $<\!|$ is dense.

Let $\boldsymbol{x} <\!| \boldsymbol{y}$, say $\boldsymbol{x} \restriction n = \boldsymbol{y} \restriction n$ and $x_n < y_n$ for some n. Since $y_n \neq 0$ we have $y_i \notin A_{\boldsymbol{y}\restriction i}$ for $i \in n$. Pick $z \in (x_n, y_n)$. Then $\boldsymbol{z} = \langle \boldsymbol{x} \restriction n, z, 0, 0, \ldots\rangle \in L_{\mathscr{A}}$, by the same argument as above and $\boldsymbol{x} <\!| \boldsymbol{z} <\!| \boldsymbol{y}$.

(2) $<\!|$ is complete.

Let $C \subseteq L_{\mathscr{A}}$. If $C = \emptyset$ then $\mathbf{0} = \langle 0, 0, 0, \ldots \rangle = \min L_{\mathscr{A}} = \sup C$. So assume $C \neq \emptyset$. We show that the usual strategy for finding a supremum in $[0, 1]^{\omega}$ also yields a supremum in $L_{\mathscr{A}}$. Let $C_0 = \{x_0 \colon \boldsymbol{x} \in C\}$ and $c_0 = \sup C_0$. Now if $c_0 \notin C_0$ then $\langle c_0, 0, 0, \ldots \rangle = \sup C$. Observe that this point belongs to $L_{\mathscr{A}}$. If $c_0 \in C_0$ let $C_1 = \{x_1 \colon \boldsymbol{x} \in C, x_0 = c_0\}$ and $c_i = \sup C_i$. If $c_1 \notin C_1$ then $\langle c_0, c_1, 0, 0, \ldots \rangle = \sup C$; observe that $\langle c_0, c_1, 0, 0, \ldots \rangle \in L$. Continue this process. If the process stops at n then $\langle c_0, c_1, \ldots, c_n, 0, 0, \ldots \rangle = \sup C$. Observe that this point belongs to $L_{\mathscr{A}}$. If the process does not stop then $\langle c_0, c_1, \ldots, c_n,, \ldots \rangle = \max C$. $\square$

In case $\mathscr{A}$ has only one element A we will write L_A instead of $L_{\{A\}}$.

We will now study some subspaces of $L_{\mathscr{A}}$ which we will encounter when checking the various properties of $L_{\mathscr{A}}$. Although some of the results are valid for general subsets of $[0, 1]$, we assume in view of our applications that every $A \in \mathscr{A}$ is a BB-set containing 0 and 1.

3.2. Definition. Let $B_{\mathscr{A}} = \{\boldsymbol{x} \in L_{\mathscr{A}} \colon \forall n \in \omega,\ x_n \notin A_{\boldsymbol{x} \restriction n}\}$.

3.3. Lemma. *As a subspace of $L_{\mathscr{A}}$, $B_{\mathscr{A}}$ is homeomorphic to the zero-dimensional Baire space of weight 2^{ω} (i.e. the product of countably many discrete spaces of cardinality 2^{ω}).*

Proof. For each $s \in [0, 1]^{<\omega}$ let $f_s \colon [0, 1] \setminus A_s \to \mathbb{R}$ be a bijection. Define $f \colon B_{\mathscr{A}} \to \mathbb{R}^{\omega}$ by

$$f(\boldsymbol{x}) = \langle f_{\langle\,\rangle}(x_0), f_{\langle x_0 \rangle}(x_1), \ldots, f_{\boldsymbol{x} \restriction n}(x_n), \ldots \rangle.$$

It is straightforward to check that f is a bijection. For $\boldsymbol{x} \in B_{\mathscr{A}}$ and $n \in \omega$ let

$$B(\boldsymbol{x}, n) = \{\boldsymbol{y} \in B_{\mathscr{A}} \colon \boldsymbol{y} \restriction n = \boldsymbol{x} \restriction n\}.$$

For $\boldsymbol{x} \in \mathbb{R}^{\omega}$ and $n \in \omega$ let

$$C(\boldsymbol{x}, n) = \{\boldsymbol{y} \in \mathbb{R}^{\omega} \colon \boldsymbol{y} \restriction n = \boldsymbol{x} \restriction n\}.$$

The collection $\{C(\boldsymbol{x}, n) \colon \boldsymbol{x} \in \mathbb{R}^{\omega}, n \in \omega\}$ generates the Baire-space topology on $\mathbb{R}^{\omega}$, furthermore $f(B(\boldsymbol{x}, n)) = C(f(\boldsymbol{x}), n)$ for all $\boldsymbol{x}$ and n, so $\{B(\boldsymbol{x}, n) \colon \boldsymbol{x} \in B_{\mathscr{A}}, n \in \omega\}$ induces this same topology on $B_{\mathscr{A}}$. On the other hand, define for $\boldsymbol{x} \in B_{\mathscr{A}}$ and $n \in \omega$ the points $\boldsymbol{x}(n)$ and $\boldsymbol{y}(n)$ by

$$\boldsymbol{x}(n) \restriction n = \boldsymbol{y}(n) \restriction n = \boldsymbol{x} \restriction n; \qquad \boldsymbol{x}(n)_n = 0, \qquad \boldsymbol{y}(n)_n = 1;$$

$$\boldsymbol{x}(n)_i = \boldsymbol{y}(n)_i = 0 \quad \text{for } i > n.$$

Then $B(\boldsymbol{x}, n) = (\boldsymbol{x}(n), \boldsymbol{y}(n)) \cap B_{\mathscr{A}}$ for all n, while moreover

$$\boldsymbol{x} = \sup_{n \in \omega} \boldsymbol{x}(n) \quad \text{and} \quad \boldsymbol{x} = \inf_{n \in \omega} \boldsymbol{y}(n).$$

So $\{B(\boldsymbol{x}, n) \colon \boldsymbol{x} \in B_{\mathscr{A}}, n \in \omega\}$ generates the subspace topology of $B_{\mathscr{A}}$. $\square$

Again $B_A = B_{\{A\}}$. The space B_A will be used in Section 4.

3.4. Definition. Let $n \in \omega$ and $s \in [0, 1]^n$ such that $s_i \notin A_{s\restriction i}$ for $i \in n$. Put

$$I_s = \{\boldsymbol{x} \in L_A \colon \boldsymbol{x} \restriction n = s\} = [\langle s, 0, 0, \ldots\rangle, \langle s, 1, 0, 0, 0, \ldots\rangle]$$

and

$$A(s) = \{\boldsymbol{x} \in I_s \colon x_n \in A_s\}.$$

3.5. Lemma. (1) *$A(s)$ and A_s are isomorphic and homeomorphic.*

(2) *$\overline{A(s)}$ is isomorphic and homeomorphic to the* LOTS *X_{A_s} defined in Remark* 2.7.

Proof. (1) The map $f_s \colon A(s) \to A_s$ defined by $f_s(\boldsymbol{x}) = x_n$ is both an isomorphism and a homeomorphism.

(2) The same holds for $g_s \colon \overline{A(s)} \to X_{A_s}$ defined by $g_s(\boldsymbol{x}) = \langle x_n, x_{n+1}\rangle$, for it is easily seen that $\overline{A(s)} = \{\boldsymbol{x} \in I_s \colon (x_n \in A_s) \vee (x_n \notin A_s \wedge x_{n+1} \in \{0, 1\})\}$. $\square$

If $\mathscr{A}$ has only one element A we use A_s to denote $A(s)$. This occurs only in Section 4.

3.6. Remark. Notice that in Lemma 3.5 $\overline{A(s)} \setminus A(s)$ is homeomorphic to

$$([0, 1] \setminus A_s \times \{0\}) \cup ([0, 1] \setminus A_s \times \{1\}) \subseteq X_{A_s}.$$

As both $[0, 1] \setminus A_s \times \{0\}$ and $[0, 1] \setminus A_s \times \{1\}$ carry the Sorgenfrey topology we see that $\overline{A(s)} \setminus A(s)$ is a union of two subspaces of the Sorgenfrey-line.

We collect some results on $L_{\mathscr{A}}$ which will be of use to us in the next sections. First some notation. For $n \in \omega$ we let

$$A_n = \{\boldsymbol{x} \in L_{\mathscr{A}} \colon x_n \in A_{x\restriction n} \wedge \forall i \in n,\ x_i \notin A_{x\restriction i}\}.$$

Note that

$$A_n = \bigcup \{A(s) \colon s \in [0, 1]^n \wedge \forall i \in n,\ s_i \notin A_{s\restriction i}\},$$

so in particular, $A_0 = A(\langle\ \rangle)$. Also note that

$$\bigcup_{n\in\omega} A_n = L_{\mathscr{A}} \setminus B_{\mathscr{A}}.$$

3.7. Proposition. *Both $\bigcup_{n\in\omega} A_n$ and $B_{\mathscr{A}}$ are dense in $L_{\mathscr{A}}$.*

Proof. Let $\boldsymbol{x} <| \boldsymbol{y}$ in $L_{\mathscr{A}}$, say $\boldsymbol{x} \restriction n = \boldsymbol{y} \restriction n$ and $x_n < y_n$. Pick $a \in (x_n, y_n) \cap A_{x\restriction n}$ and $b \in (x_n, y_n) \setminus A_{x\restriction n}$. Put $\boldsymbol{a} = \langle \boldsymbol{x} \restriction n, a, 0, 0, \ldots\rangle$ and $\boldsymbol{b} = \langle \boldsymbol{x} \restriction n, b, 0, 0, \ldots\rangle$. Then $\boldsymbol{a} \in (\boldsymbol{x}, \boldsymbol{y}) \cap A_n$ and $\boldsymbol{b} \in (\boldsymbol{x}, \boldsymbol{y}) \cap B_{\mathscr{A}}$. $\square$

3.8. Proposition. *Let $s \in [0, 1]^n$ be such that $s_i \notin A_{s\restriction i}$ for $i \in n$. Let $D \subseteq [0, 1]$ and let $f \colon D \to I_s$ be continuous. Then $\{x \in [0, 1] \setminus A_s \colon f(D) \cap I_{\langle s,x\rangle} \neq \emptyset\}$ is countable.*

Proof. Let $D_1 = \{x \in [0,1]\backslash A_s : f(D) \cap I^\circ_{\langle s,x\rangle} \neq \emptyset\}$ (here $^\circ$ denotes interior). Then D_1 is countable since

$$\{f^{-1}(I^\circ_{\langle s,x\rangle}) : x \in D_1\}$$

is a pairwise disjoint family of open subsets of D. Let $D_2 = \{x \in [0,1]\backslash A_s : \langle s, x, 1, 0, 0, 0, \ldots\rangle \in f(D)\}$. By Lemma 3.5 and Remark 3.6 we have that $E_2 = \{\langle s, x, 1, 0, 0, \ldots\rangle : x \in D_2\}$ is homeomorphic to a D_2 as subspace of the Sorgenfrey-line. Let $\mathscr{B}$ be a countable basis for $[0,1]$. For each $x \in D_2$ pick $B_x \in \mathscr{B}$ such that $\langle s, x, 1, 0, 0, \ldots\rangle \in f(B_x \cap f^{-1}(E_2)) \subseteq \{\langle s, y, 1, 0, 0, \ldots\rangle : y \in D_2 \wedge y \geqslant x\}$. If $x \neq y$ then $B_x \neq B_y$, whence D_2 is countable. Similarly, $D_3 = \{x \in [0,1]\backslash A_s : \langle s, x, 0, 0, \ldots\rangle \in f(D)\}$ is countable We conclude that

$$\{x \in [0,1]\backslash A_s : f(D) \cap I_{\langle s,x\rangle} \neq \emptyset\} = D_1 \cup D_2 \cup D_3$$

is countable. $\square$

Observe that $\{x \in [0,1]\backslash A_s : f(D) \cap I_{\langle s,x\rangle} \neq \emptyset\} = \pi_n f(D) \cap ([0,1]\backslash A_s)$ (here π_n is the projection onto the nth factor of $[0,1]^\omega$ of course).

4. An order-homogeneous non-reversible continuum

Let $A \subseteq [0,1]$ be the BB-set from Example 2.3. We claim that $L_{\mathscr{A}}$ is as required. We first show that L_A is order-homogeneous.

4.0. Lemma. *B_A is isomorphic to each sum of finitely many copies of itself.*

Proof. It suffices to show that $B_A \approx B_A + B_A$ where $B_A + B_A$ denotes the ordered union of two disjoint copies of B_A. Let $a \in A$. Then $[0,1]\backslash A = (0,a)\backslash A \cup (a,1)\backslash A \approx [0,1]\backslash A + [0,1]\backslash A$. So

$$\begin{aligned} B_A &\approx ([0,1]\backslash A) \times ([0,1]\backslash A)^{\mathbb{N}} \approx ([0,1]\backslash A + [0,1]\backslash A) \times ([0,1]\backslash A)^{\mathbb{N}} \\ &\approx B_A + B_A. \quad \square \end{aligned}$$

4.1. Lemma. *B_A is isomorphic to each clopen initial and final segment of itself.*

Proof. Let C be a clopen initial segment and $D = B_A \backslash C$. Let $C_0 = \{x_0 : \boldsymbol{x} \in C\}$ and $D_0 = \{x_0 : \boldsymbol{x} \in D\}$. Assume $C_0 \cap D_0 = \emptyset$. If C_0 has no maximum then $C_0 \approx [0,1]\backslash A$ and hence $B_A \approx C_0 \times ([0,1]\backslash A)^{\mathbb{N}} = C$. If C_0 has a maximum, say a_0, then

$$\begin{aligned} C &= \{\boldsymbol{x} \in B_A : x_0 < a_0\} + \{\boldsymbol{x} \in B_A : x_0 = a_0\} \\ &= (0, a_0)\backslash A \times ([0,1]\backslash A)^{\mathbb{N}} + \{a_0\} \times ([0,1]\backslash A)^{\mathbb{N}} \\ &\approx B_A + B_A \end{aligned}$$

(Lemma 4.0). If $C_0 \cap D_0 \neq \emptyset$., then, as is easily seen, $C_0 \cap D_0 = \{a_0\}$ for some $a_0 \in [0, 1]\backslash A$. Let $C_1 = \{x_1 : x \in C \wedge x_0 = a_0\}$ and $D_1 = \{x_1 : x \in D \wedge x_0 = a_0\}$. If $C_1 \cap D_1 = \emptyset$ and C_1 has no maximum then $C \approx ((0, a_0)\backslash A \times ([0, 1]\backslash A)^{\mathbb{N}}) + (\{a_0\} \times C_1 \times ([0, 1]\backslash A)^{\mathbb{N}}) \approx B_A + B_A \approx B_A$ (Lemma 4.0). If $C_1 \cap D_1 = \emptyset$ and C_1 has a maximum a_1 then $C \approx ((0, a_0)\backslash A \times ([0, 1]\backslash A)^{\mathbb{N}}) + (\{a_0\} \times (0, a_1)\backslash A \times ([0, 1]\backslash A)^{\mathbb{N}}) + (\{a_0\} \times \{a_1\} \times ([0, 1]\backslash A)^{\mathbb{N}}) \approx B_A + B_A + B_A \approx B_A$ (Lemma 4.0). If $C_1 \cap D_1 \neq \emptyset$, say $C_1 \cap D_1 = \{a_1\}$, continue. If the process stops we find that C is isomorphic to a sum of finitely many copies of B_A and hence to B_A; if the process does not stop we find a point $\langle a_0, a_1, \ldots \rangle \in C \cap D$ which is impossible.

We can of course show simultaneously that $D \approx B_A$. □

4.2. Lemma. *B_A is isomorphic to each interval $(x, y) \cap B_A$.*

Proof. If $x, y \in \bigcup_{n \in \omega} A_n$ then $(x, y) \cap B_A$ is clopen in B_A, and a clopen final segment of the clopen initial segment $(\mathbf{0}, y) \cap B_A$ of B_A. So applying Lemma 4.1 twice we see that $B_A \approx (x, y) \cap B_A$. If x and y are arbitrary find sequences $\langle p_n \rangle_{n \in \mathbb{Z}}$ in $(x, y)\backslash B_A$ and $\langle q_n \rangle_{n \in \mathbb{Z}}$ in $L_A \backslash B_A$ respectively, such that $p_n \downarrow x$, $q_n \downarrow \mathbf{0}$ (if $n \to -\infty$) and $p_n \uparrow y$, $q_n \uparrow \mathbf{1}$ (if $n \to \infty$). Now map $(p_n, p_{n+1}) \cap B_A$ isomorphically onto $(q_n, q_{n+1}) \cap B_A$ for each $n \in \mathbb{Z}$. The combination of these maps is an isomorphism of $(x, y) \cap B_A$ onto B_A. □

4.3. Theorem. *L_A is order homogeneous.*

Proof. Let $x <| y$ in L_A and let $f : B_A \to B_A \cap (x, y)$ be an isomorphism (Lemma 4.2). As B_A is dense in L_A (Proposition 3.7), f extends to a unique isomorphism $\bar{f} : L_A \to [x, y]$. □

Next we show that L_A is very strongly non-reversible.

4.4. Theorem. *Let $f : L_A \to L_A$ be continuous and non-increasing. Then f is constant. In particular, there cannot be an order-reversing autohomeomorphism of L_A.*

Proof. (1) $\pi_0 f(\bigcup_{n \in \omega} A_n)$ is countable.

We identify, using Lemma 3.5, A_s and A and $\bar{A}_s$ and X_A for each s. Let $s \in ([0, 1]\backslash A)^{<\omega}$. By Proposition 3.8 $|\pi_0 f(A_s)\backslash A| \leq \omega$ (we use the case $n = 0$). By Example 2.3 $|\pi_0 f(A_s)| < 2^\omega$, and so by Lemma 2.5 $|\pi_0 f(A_s)| \leq \omega$. We also conclude that by Proposition 2.6 $|\pi_0 f(\bar{A}_s)| \leq \omega$.

Furthermore, if $x \in [0, 1]\backslash A$ and $\pi_0 f$ is constant on $I_{s^\frown x}$ then $\pi_0 f(I_{s^\frown x}) \subseteq \pi_0 f(\bar{A}_s)$, and the number of $x \in [0, 1]\backslash A$ for which $\pi_0 f$ is not constant on $I_{s^\frown x}$ is by Remark 2.7 at most countable; call this set T_s. Summarizing we have that $|\pi_0 f(\bar{A}_s)| \leq \omega$ and

$$\pi_0 f\left(I_s \cap \bigcup_{n \in \omega} A_n \right) \subseteq \pi_0 f(\bar{A}_s) \cup \bigcup_{x \in T_s} \pi_0 f\left(I_{s^\frown x} \cap \bigcup_{n \in \omega} A_n \right).$$

Let $T \subseteq ([0,1]\backslash A)^{<\omega}$ be the union of the following sets:

$$T_0 = \{\langle\ \rangle\}, \qquad T_{n+1} = \bigcup_{s \in T_n} \{s\hat{\ }x \colon x \in T_s\}.$$

Then T is countable and $\pi_0 f(\bigcup_{n\in\omega} A_n) \subseteq \bigcup_{s\in T} \pi_0 f(\bar{A}_s)$ so indeed $|\pi_0 f(\bigcup_{n\in\omega} A_n)| \leqslant \omega$.

(2) $\pi_0 f(\bigcup_{n\in\omega} A_n)$ consists of one point.

Suppose not and let $p<q$ be distinct points of this set. Pick a point $r \in (p,q)\backslash(A \cup \pi_0 f(\bigcup_{n\in\omega} A_n))$. It follows that $J = (\langle r,0,0,\ldots\rangle, \langle r,1,0,0,\ldots\rangle)$ is disjoint from $f(\bigcup_{n\in\omega} A_n)$ and hence disjoint from $\overline{f(\bigcup_{n\in\omega} A_n)} \supseteq f(L_A)$. But $f(L_A)$ contains points on the left and on the right of J and must therefore be disconnected contradicting the continuity of f.

(3) Denote the point from (2) by x_0. If $x_0 \in A$ then f is constant with value $\langle x_0, 0, 0, \ldots\rangle$. If $x_0 \in [0,1]\backslash A$ then f maps L_A into I_{x_0} and we find x_1 such that $\pi_1 f(L_A) = \{x_1\}$. If $x_1 \in A$ then $f \equiv \langle x_0, x_1, 0, 0, \ldots\rangle$; if not, continue. If this process stops at n then $f \equiv \langle x_0, x_1, \ldots, x_n, 0, 0, \ldots\rangle$; otherwise we find $x \in B_A$ (with coordinates $x_0, x_1, \ldots$, etc.) such that $f \equiv x$. $\square$

Our continuum is different from Shelah's [8] for the following reason: Shelah's continuum is an Aronzajn continuum, hence it contains an uncountable subset without any uncountable subset isomorphic to a subset of $\mathbb{R}$. Our continuum has the property that every uncountable subset contains an uncountable subset isomorphic to a subset of $\mathbb{R}$. To see this, let $D \subseteq L_A$ be uncountable. For each $n \in \omega$ let

$$T_n = \{s \in ([0,1]\backslash A)^n \colon D \cap I_s \neq \emptyset\},$$

and let $T = \bigcup_{n\in\omega} T_n$. Then T is a tree if we define $s \leqslant t \Leftrightarrow s \subseteq t$.

Case 1. Some T_n is uncountable.

Let n be the first integer for which T_n is uncountable; since $T_0 = \{\langle\ \rangle\}$, $n>0$. Pick $s \in T_{n-1}$ such that $T_s = \{t \in T_n \colon s \leqslant t\}$ is uncountable. For each $t \in T_s$, pick $d_t \in I_t \cap D$. Then $\{d_t \colon t \in T_s\}$ is isomorphic to the uncountable subset $\{t_{n-1} \colon t \in D_s\}$ of $[0,1]$.

Case 2. Every T_n is countable.

Let $T' = \{s \in T \colon I_s \cap D$ is uncountable$\}$. Then T' is a subtree of T.

Subcase 2.1. For some $s \in T'$ we have that $D \cap A_s$ is uncountable.

Define f_s as in the proof of Lemma 3.5. Then $f_s(D \cap A_s)$ is isomorphic to $D \cap A_s$ and $f_s(D \cap A_s)$ is an uncountable subset of $[0,1]$.

Subcase 2.2. For all $s \in T'$ we have that $D \cap A_s$ is countable.

Consider $D' = D\backslash(\bigcup_{s\in T\backslash T'} I_s \cup \bigcup_{s\in T'} A_s)$. Then D' is uncountable because $\bigcup_{s\in T\backslash T'} I_s \cup \bigcup_{s\in T'} A_s$ is countable and $D' \subseteq B_A$. For every $s \in T'_n = \{t \in T' \colon t \in T_n\}$ the set $\{t \in T'_{n+1} \colon s \leqslant t\}$ is countable and ordered by $t' \leqslant t \Leftrightarrow t'_n \leqslant t_n$. Since every countable subset of $[0,1]$ is isomorphic to a subset of Q, We can embed the set of branches of T' into the lexicographic product Q^ω, which itself is embeddable into $\mathbb{R}$. As every element of D' determines a branch of T', we see that D' is embeddable into $\mathbb{R}$.

5. An ordered continuum with a minimum set of continuous self-maps

In this section we present an ordered continuum with only the necessary continuous self-maps. To see what this means, let X be an ordered continuum. Then for $x<y$ in X there exists a continuous map $f_{xy}: X \to X$ defined by

$$f_{xy}(z)=\begin{cases} x & \text{if } z \leqslant x, \\ z & \text{if } x \leqslant z \leqslant y, \\ y & \text{if } y \leqslant z. \end{cases}$$

Let us call such a map a *canonical retraction.* Thus whatever properties X may have, it will always have the canonical retractions among its continuous self-maps. The continuum which we construct in this section will have no continuous self-maps besides the canonical retractions.

Let $\mathscr{A}$ and V be as in Example 2.5. Index $\mathscr{A}$ in a one-to-one way by $[0,1]^{<\omega}$, and let $L=L_{\mathscr{A}}$. Then L is as required. The following lemma will be the key in showing this.

5.1. Lemma. *Let $\boldsymbol{p} <| \boldsymbol{q}$ in L, and let $f:[\boldsymbol{p}, \boldsymbol{q}] \to L$ be continuous and monotonically non-decreasing, such that $f([\boldsymbol{p}, \boldsymbol{q}]) \cap [\boldsymbol{p}, \boldsymbol{q}]=\emptyset$. Then f is constant.*

Proof. (1) For some $s \in [0,1]^n$, $\boldsymbol{p}=\langle s, 0, 0, \ldots\rangle$ and $\boldsymbol{q}=\langle s, 1, 0, 0, \ldots\rangle$, so $[\boldsymbol{p}, \boldsymbol{q}]=I_s$.

In this case we can use virtually the same proof as in Theorem 4.4. The only problem is to show that if t extends s and if $m \in \omega$ then $\pi_m f(A(t))$ is countable. To begin with note that $n>0$ because of the condition on f. Let $m=0$ and let t extend s. Consider $\bar{f}=\pi_0 \circ f \circ f_t^{-1}: A_t \to [0,1]$ (here f_t is defined as in the proof of Lemma 3.5). By Proposition 3.8, $\bar{f}(A_t) \backslash A_{\langle\ \rangle}$ is countable hence $\bar{f}(A_t) \cap V$ is countable. Next assume that $|\bar{f}(A_t) \cap A_{\langle\ \rangle}|=2^\omega$. Now since $A_t \cap A_{\langle\ \rangle}=\{0,1\}$ we see that for a set $C \subseteq A_t$ of cardinality 2^ω, $\bar{f} \restriction C$ is one-to-one while moreover $\bar{f}(x) \neq x$ for every $x \in C$. But then, using an extension $f^*:[0,1] \to [0,1]$ of $\bar{f}$, Example 2.4 ensures that $|\bar{f}(A_t) \cap V|=2^\omega$, which is impossible. Hence $|\bar{f}(A_t)| \leqslant |\bar{f}(A_t) \backslash A_{\langle\ \rangle}|+|\bar{f}(A_t) \cap A_{\langle\ \rangle}|<2^\omega$ and so by Lemma 2.5, $\bar{f}(A_t)=\pi_0 f(A(t))$ is countable. So $\pi_0 f$ is constant, say with value x_0, hence $f \equiv \langle x_0, 0, 0, \ldots\rangle$ or $f(I_s) \subseteq I_{\langle x_0\rangle}$. Repeat the process to find a constant value for f. At stage $i+1$, because $f(I_s) \cap I_s=\emptyset$, we know that for all t extending s we have that $t \neq \langle x_0, \ldots, x_i\rangle$, so by the above reasoning with $A_{\langle x_0, \ldots, x_i\rangle}$ in place of $A_{\langle\ \rangle}$, $\pi_{i+1} f(A(t))$ is countable.

(2) For some $s \in [0,1]^n$ and $p, q \in [0,1]$, $p=\langle s, p, 0, 0, \ldots\rangle$ and $q=\langle s, q, 0, 0, \ldots\rangle$.

Let $x \in [p, q] \backslash A_s$. Then by (1), f is constant on $I_{\langle s, x\rangle}$ say with value $\boldsymbol{r}_x$. Define $\bar{f}:[p, q] \to L$ by

$$\begin{cases} \bar{f}(a)=f(\langle s, a, 0, 0, \ldots\rangle) & \text{for } a \in [p, q] \cap A_s, \\ \bar{f}(x)=\boldsymbol{r}_x & \text{for } x \in [p, q] \backslash A_s. \end{cases}$$

Since f is continuous, $\bar{f}$ is continuous. But $\bar{f}([p, q])$ is separable and L contains no separable intervals, so $\bar{f}$ is constant. But then f is constant.

(3) $\boldsymbol{p}$ and $\boldsymbol{q}$ are arbitrary.

Find n such that $s := \boldsymbol{p} \restriction n = \boldsymbol{q} \restriction n$ and $p_n < q_n$. Then f is constant on the interval $[\langle s, p_n, 1, 0, 0, \ldots\rangle, \langle s, q_n, 0, 0, \ldots\rangle]$ by the same method as in (2). Also by (2), f is constant on the interval $[\langle s, q_n, \ldots, q_{n+i}, 0, 0, \ldots\rangle, \langle s, q_n, \ldots, q_{n+i+1}, 0, 0, \ldots\rangle]$ for each $i \geqslant 0$, and consequently, f is constant on $[\langle s, p_n, 1, 0, 0, \ldots\rangle, \boldsymbol{q}]$. We also have that f is constant on $[\langle s, p_n, \ldots, p_{n+i}, p_{n+i+1}, 0, 0, \ldots\rangle, \langle s, p_n, \ldots, p_{n+i}, 1, 0, \ldots\rangle]$ for each $i \geqslant 0$ such that $p_{n+i+1} < 1$, and consequently f is constant on the interval $[\boldsymbol{p}, \langle s, p_n, 1, 0, 0, \ldots\rangle]$. We conclude that f is constant on $[\boldsymbol{p}, \boldsymbol{q}]$. $\square$

From this lemma we now deduce:

5.1. Lemma. *Let f: $L \to L$ be a continuous monotonically non-decreasing map. If for some $\boldsymbol{a} \in L$ we have $f(\boldsymbol{a}) \mid> \boldsymbol{a}$ then $f(\boldsymbol{x}) = f(\boldsymbol{a})$ for all $\boldsymbol{x} <\mid \boldsymbol{a}$, and dually if for some $\boldsymbol{a} \in L$ we have that $f(\boldsymbol{a}) <\mid \boldsymbol{a}$ then $f(\boldsymbol{x}) = f(\boldsymbol{a})$ for all $\boldsymbol{x} \mid> \boldsymbol{a}$.*

Proof. Let $\boldsymbol{x} = \inf\{\boldsymbol{y} \leqslant \boldsymbol{a}: f(\boldsymbol{y}) = f(\boldsymbol{a})\}$. Suppose $\boldsymbol{0} <\mid \boldsymbol{x}$. Then as f is continuous $f(\boldsymbol{x}) = f(\boldsymbol{a})$ and for some $z <\mid \boldsymbol{x}$, $f([z, \boldsymbol{x}]) \subseteq (\boldsymbol{a}, \boldsymbol{1}]$. Hence $f([z, \boldsymbol{x}]) \cap [z, \boldsymbol{x}] = \emptyset$ and hence f is constant on $[z, \boldsymbol{x}]$ (Lemma 5.0), but then $z <\mid \boldsymbol{x}$ and $f(z) = f(\boldsymbol{x}) = f(\boldsymbol{a})$, a contradiction. So $\boldsymbol{x} = \boldsymbol{0}$. $\square$

We can now show:

5.2. Lemma. *Let f: $L \to L$ be monotonically non-decreasing and continuous. Then f is a canonical retraction.*

Proof. Let $f(\boldsymbol{0}) = \boldsymbol{p}$ and $f(\boldsymbol{1}) = \boldsymbol{q}$. We show that $f = f_{\boldsymbol{p},\boldsymbol{q}}$. Let $\boldsymbol{x} \in (\boldsymbol{0}, \boldsymbol{p})$. Then $f(\boldsymbol{x}) \mid\geqslant \boldsymbol{p} \mid\geqslant \boldsymbol{x}$ so $f(\boldsymbol{x}) = f(\boldsymbol{0}) = \boldsymbol{p}$. Let $\boldsymbol{x} \in (\boldsymbol{q}, \boldsymbol{1})$. Then similarly, $f(\boldsymbol{x}) = \boldsymbol{q}$. Let $\boldsymbol{x} \in (\boldsymbol{p}, \boldsymbol{q})$. If $f(\boldsymbol{x}) \mid> \boldsymbol{x}$ then $f(\boldsymbol{0}) = f(\boldsymbol{x}) \mid> \boldsymbol{x} \mid> \boldsymbol{p}$, contradiction. Similarly $f(\boldsymbol{x}) <\mid \boldsymbol{x}$ is impossible. So indeed $f = f_{\boldsymbol{q},\boldsymbol{q}}$. $\square$

With each continuous function f: $L \to L$ we associate four monotone functions as follows:

$$f_1(\boldsymbol{x}) = \sup\{f(\boldsymbol{y}): \boldsymbol{y} \leqslant\mid \boldsymbol{x}\}, \qquad f_2(\boldsymbol{x}) = \inf\{f(\boldsymbol{y}): \boldsymbol{y} \leqslant\mid \boldsymbol{x}\},$$

$$f_3(\boldsymbol{x}) = \sup\{f(\boldsymbol{y}): \boldsymbol{y} \mid\geqslant \boldsymbol{x}\}. \qquad f_4(\boldsymbol{x}) = \inf\{f(\boldsymbol{y}): \boldsymbol{y} \mid\geqslant \boldsymbol{x}\}.$$

It is straightforward to check that these functions are continuous, that f_1 and f_4 are non-decreasing, that f_2 and f_3 are non-increasing and that $f_4 \leqslant\mid f \leqslant\mid f_1$ and $f_2 \leqslant\mid f \leqslant\mid f_3$. We now get:

5.3. Theorem. *If f: $L \to L$ is continuous, then f is a canonical retraction.*

Proof. Almost the same proof as in Theorem 4.4 will show that every non-increasing continuous self-map is constant. Let f: $L \to L$ be continuous. Then f_2 and f_3 are constant, and so, since $f_2(\mathbf{0}) = f(\mathbf{0})$ and $f_3(\mathbf{1}) = f(\mathbf{1})$, we conclude that for all $x \in L$,

$$f(\mathbf{0}) \leqslant\!| f(x) \leqslant\!| f(\mathbf{1}).$$

But then $f_1(\mathbf{0}) = f_4(\mathbf{0}) = f(\mathbf{0})$ and $f_1(\mathbf{1}) = f_4(\mathbf{1}) = f(\mathbf{1})$, so $f_1 = f_4 = f_{f(\mathbf{0}),f(\mathbf{1})}$ (Lemma 5.2). Hence $f = f_{f(\mathbf{0}),f(\mathbf{1})}$, since $f_4 \leqslant\!| f \leqslant\!| f_1$. □

6. Some additional remarks

In this section we collect some additional results which can be proved in virtually the same way as in Sections 4 and 5.

6.0. To begin with, let $\{A_\alpha\}_{\alpha\in 2^\omega}$ and V be as in the proof of Example 2.3. Consider the family $\{A_J : J \subseteq 2^\omega, J \neq \emptyset\}$ from Remark 2.2. Then each continuum L_{A_J} is order-homogeneous and non-reversible. It can be shown that for $J \neq J'$ we have L_{A_J} and $L_{A_{J'}}$ are non-isomorphic. By pairing of sets J_1 and J_2 for which $L_{A_{J_1}}$ is isomorphic to $L_{A_{J_2}}$ with the reverse order, we get a family of 2^{2^ω} order-homogeneous non-reversible continua such that no two continua are isomorphic or anti-isomorphic.

6.1. A similar remark applies to the example of Section 5. To get 2^{2^ω} non-isomorphic continua with only trival continuous self-maps, simply permutate the family $\{A_{\langle x\rangle} : x \in [0,1] \backslash A_{\langle\ \rangle}\}$. Different permutations yield different continua and the number of these permutations is 2^{2^ω}.

6.2. If we let $\mathscr{G} = \{\mathrm{id}, x \to 1 - x\}$ and $\mathscr{F} = \{f : \operatorname{dom} f = [x, y] \subseteq [0, 1]$, range $f \subseteq [0, 1]$ and f is monotonically non-decreasing or non-increasing$\}$ and apply Corollary 2.1 to get $\{A_\alpha\}_{\alpha\in 2^\omega}$ and V. Then we get an ordered continuum L with precisely one reversing map φ and such that whenever $f : L \to L$ is continuous then:

(1) f is a canonical retraction, or

(2) we can find $\boldsymbol{p} \leqslant\!| \boldsymbol{q} \leqslant\!| \boldsymbol{r} \leqslant\!| \boldsymbol{s}$ in L such that $\mathbf{0} \leqslant\!| \boldsymbol{x} \leqslant\!| \boldsymbol{p} \to f(\boldsymbol{x}) = \boldsymbol{p}$, $\boldsymbol{p} \leqslant\!| \boldsymbol{x} \leqslant\!| \boldsymbol{q} \to f(\boldsymbol{x}) = \boldsymbol{x}$, $\boldsymbol{q} \leqslant\!| \boldsymbol{x} \leqslant\!| \boldsymbol{r} \to f(\boldsymbol{x}) = \boldsymbol{q}$, $\boldsymbol{r} \leqslant\!| \boldsymbol{x} <\!| \boldsymbol{s} \to f(\boldsymbol{x}) = \varphi(\boldsymbol{x})$ and $\boldsymbol{s} \leqslant\!| \boldsymbol{x} \leqslant\!| \mathbf{1} \to f(\boldsymbol{x}) = \varphi(\boldsymbol{s})$, or

(3) we can find a $g : L \to L$ satisfying (1) or (2) such that $f = \varphi \circ g$.

Use a two-to-one indexing such that for $s \in [0, 1]^{<\omega}$, $A_s = A_{s'}$ (where $s'_i = 1 - s_i$). Then $\varphi : L \to L$ defined by

$$\varphi(\boldsymbol{x}) = \begin{cases} \langle 1 - x_0, 1 - x_1, \ldots\rangle & \text{if } \boldsymbol{x} \in B_{\mathscr{A}}, \\ \langle 1 - x_0, 1 - x_n, 0, 0, \ldots\rangle & \text{if } \boldsymbol{x} \in A_n (n \in \omega), \end{cases}$$

is the reversing map.

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