Set Theory and Topology [1]

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1. Introduction

Set theoretic topology is that part of topology that uses results and techniques from set theory in order to solve its problems. In this field set theory and topology walk hand in hand. Set theoretic topology has been quite vital for the last 20 years and has solved major problems. Mathematicians involved in this area are primarily set theorists who enjoy doing 'applied' set theory and general topologists who solve their problems by working in the well-known models of set theory where axioms such as $V=L$, $MA + not - CH$, CH or $\lozenge$ hold. Right now there is a growing number of general topologists however that try to construct their own models instead of making the set theorists prove the consistency results they need. The beautiful book *Set Theory: An Introduction to Independence Proofs* by KUNEN [7] is an important tool now for set theoretic topologists.

What are the problems considered in set theoretic topology? This question is unfortunately undecidable: one never knows whether one will run into a set theoretical problem and many famous mathematicians working in various parts of mathematics posed problems that looked sensible in their field but turned out to be set theoretical ones. We shall present several well-known examples of such problems.

Who is safe from set theory? Well, WHITEHEAD, ALEXANDROFF, WILDER and CHOQUET were not. Maybe you are. But then be wise, never pose a mathematical problem. Set theory is watching you and is ready to attack.

The reader of these notes should be aware of the fact that I am not an

1. These notes form a write-up of a lecture given at the *Topologiedag CWI,* Amsterdam, September 28, 1984.

expert in set theory. I am merely an interested amateur who is fascinated by some of the results in set theoretic topology. The problems I discuss are mostly well-known and only on one occasion do I take the liberty of discussing a problem I helped to solve.

2. Alexandroff's problem

Most people are only interested in metrizable spaces. However, there are also mathematically important spaces that are not always metrizable, for example manifolds, CW-complexes, and topological vector spaces. A *manifold* is a locally Euclidean Hausdorff space. Manifolds are certainly mathematically important and the manifolds (with or without differential or algebraic structure) being mostly studied in topology are metrizable. Let M be a manifold (not necessarily assumed to be metrizable). If $A \subseteq M$ is closed then one certainly wants to be able to extend every continuous function $f: A \rightarrow \mathbb{R}$ (the reals) to a continuous function $\bar{f}: M \rightarrow \mathbb{R}$. By the Tietze-Urysohn Theorem, see [5, 2.1.8] for details, this property of M is equivalent to the following one: every two disjoint closed subsets of M can be separated by disjoint open sets. General topologists say that M is a *normal* topological space. In the process of constructing new continuous functions from old ones it is also extremely pleasant if M has the following property: for every closed subset A of M there is a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of M such that $A = \bigcap_{n=1}^{\infty} U_n$. General topologists say that a space with this property is *perfect.* A space which is both perfect and normal is called *perfectly normal.* If X is a perfectly normal space then X has the following important property: for every closed subset A of X there is a continuous function $f: X \rightarrow \mathbb{R}$ with $f^{-1}(0) = A$. Clearly, every metric space is perfectly normal. If one wants to generalize some of the existing theory on metrizable manifolds to nonmetrizable ones, it becomes clear quite quickly that in many instances it is inevitable to restrict oneself to perfectly normal manifolds. The question then naturally arises whether there is a perfectly normal manifold which is not metrizable, i.e. whether the extension of the theory is worth while. This question was asked by ALEXANDROFF [1] and later also by WILDER [18].

As usual, $\mathfrak{c}$ denotes the cardinal number of the reals. The *Continuum Hypothesis* (abbreviated *CH*) is the statement that if X is any subset of $\mathbb{R}$ then either X is countable or the cardinality of X is $\mathfrak{c}$.

It is well-known that GÖDEL [6] proved that *CH* is consistent with the usual axioms of set theory. In addition, COHEN [2] showed that *not* $-$*CH* is consistent too. Consequently, the Continuum Hypothesis is undecidable.

It seems unlikely that *CH* has anything to do with manifolds, let alone with Alexandroff's problem. In [13] however, RUDIN and ZENOR, assuming *CH*, constructed an example of a perfectly normal nonmetrizable manifold. Later, KOZLOWSKI and ZENOR [9] even constructed such a manifold that is analytic. These contributions to the solution of Alexandroff's problem very strongly suggested a positive answer.

Let X be a compact Hausdorff space. We say that X satisfies the *countable* *chain condition* (abbreviated *ccc*) if every pairwise disjoint family of open sets in X is at most countable. *Martin's Axiom* (abbreviated *MA*) states that no compact Hausdorff *ccc* space is the union of fewer than $\mathfrak{c}$ nowhere dense sets. So if one assumes *CH* then 'fewer than $\mathfrak{c}$' means countable and hence *MA* is true by the classical Baire Category Theorem. In [16], SOLOVAY and TENNENBAUM proved that the statement $MA + not - CH$ is consistent with the usual axioms of set theory, thereby showing that *MA* is strictly weaker than *CH*.

It seems extremely unlikely that an 'exotic' axiom such as $MA + not - CH$ has anything to do with Alexandroff's problem. However, in [12] RUDIN showed that under $MA + not - CH$ all perfectly normal manifolds are metrizable.

Consequently, Alexandroff's problem is undecidable.

3. WHITEHEAD'S PROBLEM

Let all groups be abelian and let 1_X denote the identity function on a set X. If A and B are groups then a surjective homomorphism $f: A \to B$ is said to *split* if there is a homomorphism $g: B \to A$ with $f \circ g = 1_B$. A group G is a *Whitehead group* if for every group B, every surjective homomorphism $f: B \to G$ with kernel isomorphic to $\mathbb{Z}$ (the integers) splits. It is clear that all free groups are whitehead and WHITEHEAD asked whether all Whitehead groups are free. STEIN [17] showed that all countable Whitehead groups are free.

GÖDEL defined a subclass L of the class V of all sets, the so-called *constructible sets.* The statement $V = L$ means that all sets are constructible: it was proven to be consistent in [6].

SHELAH [14], [15] showed that Whitehead's problem is undecidable by showing that under $V = L$ all Whitehead groups are free while under $MA + not - CH$ there exists a Whitehead group G which is not free. In fact, the group G can be constructed in *ZFC* alone, that is, its construction 'only' needs the usual axioms of set theory plus the *Axiom of Choice.* So if one is a friend of the Axiom of Choice, the group G 'really' exists and has the amazing property that it is free under $V = L$ but not so under $MA + not - CH$. For details, see also [4].

The reader should have noticed by now that without a warning we switched from topology to algebra. However, an application of Pontrjagin duality allows one to translate Whitehead's problem into topological language as follows: *is every compact arcwise-connected abelian topological group isomorphic to a product of circles?* Since I am a friend of the Axiom of Choice [1], Shelah's results imply that for me there 'really' exists a compact arcwise-connected abelian topological group which is nothing but a product of circles under $V = L$ but not under $MA + not - CH$. This is truly unbelievable.

The fact that Whitehead's problem can be formulated both into algebraic

1. If you want to support the work of the society 'Friends of the Axiom of Choice' (president: Dr. H.M. Mulder) please send a cheque to J. van Mill, Department of Mathematics, Free University, De Boelelaan 1081, Amsterdam, The Netherlands.

and topological language is not an exception for a problem that turns out to be dependent upon one's set theory. These problems can often successfully be translated into many mathematical languages and can therefore be attacked from several directions.

4. CHOQUET'S PROBLEM

Our last problem is not as important as the other two. However, since I took part in the solution of the problem myself, I take the liberty to mention this one too.

A *Boolean algebra* (abbreviated *BA*) will be identified with its universe. A *BA* B is called

complete/countably complete/weakly countably complete

if for any two subsets P and Q of B such that $p \wedge q = 0$ for $p \in P, q \in Q$

without further condition/with P or Q countable/ with P and Q countable

there is an $s \in B$ which separates P and Q, i.e. $p \leqslant s$ for $p \in P$ and $q \leqslant s'$ for $q \in Q$.

CHOQUET asked whether every weakly countably complete *BA* is a homomorphic image of a complete *BA*. LOUVEAU [10] proved that under *CH* the answer to Choquet's problem is in the affirmative for *BA*'s of cardinality at most $\mathfrak{c}$. The problem was shown to be undecidable by VAN DOUWEN and VAN MILL [3] who constructed under $MA + \mathfrak{c} = \kappa^+$, with κ any regular uncountable cardinal, an example of a weakly countably complete *BA* which is not a homomorphic image of any countably complete *BA*.

Again, Choquet's problem can be translated into purely topological language (by Stone duality) and in fact VAN DOUWEN and myself, being topologists, thought about it as a topological problem.

There are numerous other problems in topology that turned out to be set theoretical, for example, Souslin's problem, the normal Moore space conjecture, $\beta\mathbb{N}$-type problems, the *S*-space problem, covering-type problems, etc. For more information, see Rudin's monograph [4] and the recently published *Handbook of Set Theoretic Topology* (edited by K. KUNEN and J.E. VAUGHAN), [8]. Many papers on set theoretic topology appear in the journal *Topology and its Applications,* which I recommend.

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