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DESCRIPTIVE COMPLEXITY OF FUNCTION SPACES

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ABSTRACT. In this paper we show that $C_\pi(X)$, the set of continuous, real-valued functions on X topologized by the pointwise convergence topology, can have arbitrarily high Borel or projective complexity in $\mathbf{R}^X$ even when X is a countable regular space with a unique limit point. In addition we show how to construct countable regular spaces X for which $C_\pi(X)$ lies nowhere in the projective hierarchy of the complete separable metric space $\mathbf{R}^X$.

1. Introduction. Let $C_\pi(X)$ be the set of continuous, real-valued functions on a space X and topologize $C_\pi(X)$ as a subspace of the full product $\mathbf{R}^X$. In [**DGLvM**] it is shown that if X is completely regular, then $C_\pi(X)$ cannot be a G_δ-, F_σ- or $G_{\delta\sigma}$-subset of $\mathbf{R}^X$ unless X is discrete and that for any countable metrizable space X, $C_\pi(X)$ will be an $F_{\sigma\delta}$-subset of $\mathbf{R}^X$. In the terminology of [**KM** and **K**], $C_\pi(X)$ cannot have multiplicative class 1 and cannot have additive class 1 or 2, but may have multiplicative class 2.

In this paper we study the descriptive complexity of $C_\pi(X)$ in $\mathbf{R}^X$ when X is countable (so that $\mathbf{R}^X$ is a complete separable metric space). Our main results can be summarized as follows.

THEOREM. (a) *Given any $\alpha < \omega_1$, there is a countable regular space X such that $C_\pi(X)$ is a Borel subset of $\mathbf{R}^X$ having additive class β, where $\alpha \le \beta \le 3+\alpha+2$* (§§2 *and* 3).

(b) *Given any $n \ge 1$ there is a countable regular space Y such that $C_\pi(Y) \in \mathcal{L}_n(\mathbf{R}^Y) - \mathcal{L}_{n-1}(\mathbf{R}^Y)$, where $\mathcal{L}_n(\mathbf{R}^Y)$ is the family of projective sets of class n in the complete separable metric space $\mathbf{R}^Y$* (§4).

(c) *There is a countable regular space Z such that $C_\pi(Z) \notin \bigcup\{\mathcal{L}_n(\mathbf{R}^Z)\colon 0 \le n < \omega\}$* (§§4 *and* 5).

The spaces X, Y and Z in the above Theorem can be obtained from a single general construction which associates with each subset $S \subset 2^\omega$ a certain countable regular space Σ_S having a unique nonisolated point. The descriptive complexity of S in 2^ω determines the complexity of $C_\pi(\Sigma_S)$ in $\mathbf{R}^{\Sigma_S}$. To describe Σ_S precisely, we begin by letting $T_n = 2^n$ be the set of functions from $\{0, 1, \ldots, n-1\}$ into $\{0, 1\}$, i.e., the set of ordered n-tuples of 0's and 1's. Let $T = \bigcup\{T_n | n \ge 1\}$ and partially order T by function extension. A *branch* of T is a maximal linearly ordered subset of T, i.e., a linearly ordered subset $B \subset T$ having $\operatorname{card}(B \cap T_n) = 1$ for each $n \ge 1$. Observe that if B and $\hat{B}$ are distinct branches of T, then $B \cap \hat{B}$ must be a finite set.

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Given $x \in 2^\omega$, the set $B_x = \{\langle x(0)\rangle, \langle x(0), x(1)\rangle, \langle x(0), x(1), x(2)\rangle, \ldots\}$ is a branch of T. Conversely, each branch B of T has the form $B = B_x$ for a unique $x \in 2^\omega$. Let $\mathcal{B} = \{B | B \text{ is a branch of } T\}$.

Let $\mathcal{P}(T) = \{A | A \subset T\}$ and topologize $\mathcal{P}(T)$ using open sets of the form $[Y, N] = \{A \in \mathcal{P}(T) | Y \subset A \subset T - N\}$, where Y and N are arbitrary finite subsets of T. The resulting space is compact and metrizable, and is homeomorphic to the product space 2^T under the mapping which identifies each subset $A \in \mathcal{P}(T)$ with its characteristic function χ_A. The mapping $x \to B_x$ is easily seen to be a homeomorphism of 2^ω into $\mathcal{P}(T)$ whose image is exactly the set $\mathcal{B}$ defined above.

For each subset $S \subset 2^\omega$, the collection $\{T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F) | n \geq 1,\ x_i \in S$ and $F \subset T$ is a finite set$\}$ is a filter base. Let p_S be the filter generated by that filter base. Let ∞ be any point not in $T \cup 2^\omega$ and let $\Sigma_S = T \cup \{\infty\}$. Topologize Σ_S by isolating each point of T and by using the family $\{P \cup \{\infty\} | P \in p_S\}$ as a neighborhood base at ∞. The space Σ_S is countable, regular and (since p_S is a free filter) is T_1. The spaces mentioned in the above Theorem are all of the form Σ_S for various subsets S of 2^ω.

However, even though the function spaces $C_\pi(\Sigma_S)$ for $S \subset 2^\omega$ provide enough pathology to prove our Theorem, they are all well behaved in some senses. In §5 we prove that each $C_\pi(\Sigma_S)$ is a Baire Property subset of $\mathbf{R}^{\Sigma_S}$ and is meagre in $\mathbf{R}^{\Sigma_S}$ (equivalently, $C_\pi(\Sigma_S)$ is not a Baire space) and we exhibit a countable regular space X with a unique nonisolated point such that $C_\pi(X)$ is a second category subset of $\mathbf{R}^X$ (equivalently, $C_\pi(X)$ is a Baire space), is not a Baire Property subset of $\mathbf{R}^X$, and is not a Borel, analytic or co-analytic subset of $\mathbf{R}^X$ (see Example 5.5).

The standard references for descriptive theory in complete separable metric spaces are [**K** and **KM**]. Our topological terminology is consistent with [**E**] and [**Ox₂**] is a good source for properties of Baire spaces. The authors wish to thank Jean Calbrix and Fons van Engelen for their comments on an earlier version of this paper.

2. A lower bound for the complexity of $C_\pi(\Sigma_S)$.

2.1 THEOREM. *Let $S \subset 2^\omega$ and let $\Sigma = \Sigma_S$. Then $C_\pi(\Sigma)$ contains a relatively closed subset which is homeomorphic to S.*

PROOF. Recall that in Σ_S, the point ∞ has a neighborhood base consisting of all sets of the form $\{\infty\} \cup (T - (B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_n} \cup F))$, where $x_i \in S$ and $F \subset T$ is finite. For each $x \in 2^\omega$ define a function $f_x \colon \Sigma \to \mathbf{R}$ by $f_x(\infty) = 0$, $f_x(t) = 0$ if $t \in T - B_x$ and $f_x(t) = 1$ if $t \in B_x$. Define $\lambda \colon 2^\omega \to \mathbf{R}^\Sigma$ by $\lambda(x) = f_x$. Clearly λ is 1-1 and continuous, so that λ embeds 2^ω as a closed subspace of $\mathbf{R}^\Sigma$. Furthermore $\lambda(x) \in C_\pi(\Sigma)$ whenever $x \in S$ because for such an x, the function f_x is constant on the neighborhood $\{\infty\} \cup (T - B_x)$ of ∞. Conversely, if $f_x \in C_\pi(\Sigma)$ for some $x \in 2^\omega$, then $f_x^{-1}[(-\frac{1}{2}, \frac{1}{2})]$ must be a neighborhood of ∞ so that for some $x_1, \ldots, x_n \in S$ and some finite F, the set $f_x^{-1}[(-\frac{1}{2}, \frac{1}{2})] = \{\infty\} \cup (T - B_x)$ must contain the basic neighborhood $\{\infty\} \cup (T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F))$. But then $B_x \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$ so that $B_{x_i} \cap B_x$ is infinite for some i and hence $B_x = B_{x_i}$, i.e., $x = x_i \in S$. Therefore $\lambda[S] = C_\pi(\Sigma) \cap \lambda[2^\omega]$ showing that $\lambda[S]$ is a relatively closed subset of $C_\pi(\Sigma)$. □

2.2 COROLLARY. *If S is not a Borel subset of 2^ω (resp., if S is not a projective subset of 2^ω), then $C_\pi(\Sigma_S)$ is not a Borel subset (resp. a projective subset) of $\mathbf{R}^{\Sigma_S}$.*

PROOF. Write $\Sigma = \Sigma_S$. In the complete separable metric space $\mathbf{R}^\Sigma$, a relatively closed subset of a Borel (resp., projective) set is again a Borel (resp., projective) set in $\mathbf{R}^\Sigma$ and it is known that homeomorphisms preserve Borel (resp., projective) sets [**K**, Chapter 3, §35, IV, Corollary 1 and Chapter 3, §38, VII, Theorem 1] contrary to our assumption that S is not Borel (resp., projective) in 2^ω. $\square$

2.3 COROLLARY. *There is a countable regular space X such that $C_\pi(X)$ is not a Borel subset of $\mathbf{R}^X$.*

PROOF. Let S be a non-Borel subset of 2^ω and let $X = \Sigma_S$. Now apply 2.2. $\square$

3. An upper bound for the Borel complexity of $C_\pi(\Sigma_S)$. In §2 we proved that $C_\pi(\Sigma_S)$ always contains a closed subspace homeomorphic to S so that if S is *not* a Borel set, then neither is $C_\pi(\Sigma_S)$. In this section we study the situation where S is a Borel subset of 2^ω and we prove

3.1 THEOREM. *Let S be a Borel subset of 2^ω having additive class $\alpha \geq 1$ and let $\Sigma = \Sigma_S$. Then $C_\pi(\Sigma)$ is a Borel subset of $\mathbf{R}^\Sigma$ of class β, where $\alpha \leq \beta \leq 3+\alpha+2$.*

PROOF. Following the notation of §1, we let $p = p_S$ be the filter on T generated by all sets of the form $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$, where $x_j \in S$ for $1 \leq j \leq n$ and F is any finite subset of T.

For each $m \geq 1$, define $\psi_m \colon \mathbf{R}^\Sigma \to \mathcal{P}(T)$ by $\psi_m(f) = \{t \in T \mid |f(\infty) - f(t)| \geq 1/m\}$. In Lemma 3.2 we show that ψ_m is a Borel mapping of class 1. Next, define a set $\mathcal{D} \subset \mathcal{P}(T)$ by $\mathcal{D} = \{A \in \mathcal{P}(T) | A \cap P = \varnothing \text{ for some } P \in p\}$. In Lemma 3.6 we prove that $\mathcal{D}$ is a Borel subset of $\mathcal{P}(T)$ of additive class $\leq 2+\alpha$ so that $\psi_m^{-1}[\mathcal{D}]$ is a Borel set of additive class $\leq 3+\alpha$. Because a function $f \in \mathbf{R}^\Sigma$ is continuous if and only if $\{t \in T \mid |f(\infty) - f(t)| < 1/m\}$ belongs to p for each m, we have $C_\pi(\Sigma) = \bigcap\{\psi_m^{-1}[\mathcal{D}] | m \geq 1\}$ showing that $C_\pi(\Sigma)$ is a Borel set of additive class $\beta \leq (3+\alpha+2)$.

From §2, a closed subspace of $C_\pi(\Sigma)$ is homeomorphic to S, so the additive class of $C_\pi(\Sigma)$ cannot be smaller than the additive class of S and we obtain $\alpha \leq \beta$. $\square$

All that remains is to prove some lemmas.

3.2 LEMMA. *Each ψ_m is a Borel map of class* 1.

PROOF. It is enough to show that $\psi_m^{-1}[[Y,N]]$ is an F_σ-subset of $\mathbf{R}^\Sigma$ for each basic open set $[Y,N]$ in $\mathcal{P}(T)$. Now

$$\begin{aligned}\psi_m^{-1}[[Y,N]] = \{f \in \mathbf{R}^\Sigma | Y \subset \{t \in T \mid |f(t) - f(\infty)| \geq 1/m\}\} \\ \cap \{f \in \mathbf{R}^\Sigma | \{t \in T \mid |f(t) - f(\infty)| \geq 1/m\} \subset T - N\}.\end{aligned}$$

The first of those two sets is closed and, since N is finite, the second is open. Hence their intersection is an F_σ-set, as claimed. $\square$

3.3 LEMMA. *The set $\mathcal{A} = \{A \in \mathcal{P}(T)|$ for some $B_1, \ldots, B_n \in \mathcal{B}$ and some finite $F \subset T$, $A \subset B_1 \cup \cdots \cup B_n \cup F\}$ is a σ-compact subset of $\mathcal{P}(T)$.*

PROOF. For a fixed finite $F \subset T$ and a fixed n, let $\mathcal{A}(F,n) = \{(A, B_1, \ldots, B_n) | B_i \in \mathcal{B} \text{ and } A \subset B_1 \cup \cdots \cup B_n \cup F\}$. Then $\mathcal{A}(F,n)$ is a closed subset of the compact space $\mathcal{P}(T) \times \mathcal{B}^n$. Let $\pi_n \colon \mathcal{P}(T) \times \mathcal{B}^n \to \mathcal{P}(T)$ denote first coordinate projection. Then $\mathcal{A} = \bigcup\{\pi_n[\mathcal{A}(F,n)] \colon n \geq 1 \text{ and } F \subset T \text{ is finite}\}$ so that $\mathcal{A}$ is a σ-compact subspace of $\mathcal{P}(T)$ as claimed. $\square$

3.4 NOTATION. Recall that $\mathcal{B}$ is the set of all branches of T, topologized as a subspace of the compact metric space $\mathcal{P}(T)$. Being the continuous image of 2^ω under the map $\mu(x) = B_x$, $\mathcal{B}$ is compact. For $n \geq 1$, let $\Phi_n = \{\mathcal{K} \subset \mathcal{B} | \text{card}(\mathcal{K}) = n\}$ and let $\Phi = \bigcup\{\Phi_n | n \geq 0\}$. Topologize Φ with the *Vietoris topology*, i.e., by using all subsets of Φ of the forms $\{\mathcal{K} \in \Phi | \mathcal{K} \subset \mathcal{U}\}$ and $\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{V} \neq \varnothing\}$ as a subbase where $\mathcal{U}$ and $\mathcal{V}$ are arbitrary open subsets of $\mathcal{B}$. Then Φ is a σ-compact metrizable space [**KM**, p. 392]. Recall that each branch of T is of the form B_x for some $x \in 2^\omega$ and let $\Phi_S = \{\mathcal{K} \in \Phi | \mathcal{K} \subset \{B_x | x \in S\}\} = \{\mathcal{K} | \mathcal{K}$ is a finite subset of $\{B_x | x \in S\}\}$.

3.5 LEMMA. *With $\mathcal{A}$ as in 3.3, for each $A \in \mathcal{A}$ let $i(A) = \{B \in \mathcal{B} | B \cap A$ is infinite$\}$. Then $i\colon \mathcal{A} \to \Phi$ is a Borel mapping of class 2.*

PROOF. Fix $A \in \mathcal{A}$ and choose branches $B_1, \ldots, B_n$ and a finite set F with $A \subset B_1 \cup \cdots \cup B_n \cup F$. If B is any branch of T such that $A \cap B$ is infinite, then $B \cap B_k$ is infinite for some $k = 1, 2, \ldots, n$ so that B is one of the branches $B_1, \ldots, B_n$. Hence $i(A)$ is finite so $i(A) \in \Phi$. (If A is finite, then $i(A) = \varnothing \in \Phi$.)

(a) Fix an open subset $\mathcal{U}$ of $\mathcal{B}$ and consider $i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \subset \mathcal{U}\}] = \{A \in \mathcal{A} | i(A) \subset \mathcal{U}\}$. Because $\mathcal{U}$ is an open subset of the compact metric space $\mathcal{B}$, $\mathcal{U}$ is σ-compact. According to 3.3, so is $\mathcal{A}$, and we conclude that the product space $\mathcal{A} \times \mathcal{U}^n$ is σ-compact for each $n \geq 1$, where $\mathcal{U}^n$ is the product of n copies of $\mathcal{U}$. Fix $n \geq 1$ and fix a finite set $F \subset T$. Then the set $\mathcal{C}(n, F) = \{(A, B_1, \ldots, B_n) \in \mathcal{A} \times \mathcal{U}^n | A \subset B_1 \cup \cdots \cup B_n \cup F\}$ is closed in $\mathcal{A} \times \mathcal{U}^n$, so $\mathcal{C}(n, F)$ is σ-compact. Let $\pi_n\colon \mathcal{A} \times \mathcal{U}^n \to \mathcal{A}$ be first coordinate projection. Then $i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \subset \mathcal{U}\}] = \bigcup\{\pi_n[\mathcal{C}(n, F)] | n \geq 1$ and $F \subset T$ is finite$\}$ so $i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \subset \mathcal{U}\}]$ is a σ-compact subset of $\mathcal{A}$ (and therefore a $G_{\delta\sigma}$-subset of $\mathcal{A}$).

(b) Next consider $i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{V} \neq \varnothing\}]$, where $\mathcal{V}$ is a compact, open subset of $\mathcal{B}$. Then $\mathcal{B} - \mathcal{V}$ is open and $\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{V} \neq \varnothing\} = \Phi - \{\mathcal{K} \in \Phi | \mathcal{K} \subset \mathcal{B} - \mathcal{V}\}$. Hence $i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{V} \neq \varnothing\}] = \mathcal{A} - i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \subset \mathcal{B} - \mathcal{V}\}]$ which is a G_δ-subset in light of (a).

(c) Finally, consider $i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{U} \neq \varnothing\}]$, where $\mathcal{U}$ is an arbitrary open subset of $\mathcal{B}$. There is a sequence $\langle \mathcal{V}_n \rangle$ of compact, open subsets of $\mathcal{B}$ having $\mathcal{U} = \bigcup\{\mathcal{V}_n | n \geq 1\}$ so that $i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{U} \neq \varnothing\}] = \bigcup\{i^{-1}[\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{V}_n \neq \varnothing\}] | n \geq 1\}$ which is a $G_{\delta\sigma}$-set in $\mathcal{A}$ because of (b).

(d) Since sets of the form $\{\mathcal{K} \in \Phi | \mathcal{K} \subset \mathcal{U}\}$ and $\{\mathcal{K} \in \Phi | \mathcal{K} \cap \mathcal{U} \neq \varnothing\}$ form a subbase for the separable metric space Φ, it follows that i is a Borel mapping of class 2. $\square$

3.6 LEMMA. *With Φ_S as defined in 3.4, Φ_S is a Borel subset of Φ whose additive class is α (= the additive class of S).*

PROOF. For $n \geq 1$, define $\theta_n\colon (2^\omega)^n \to \Phi$ by $\theta_n(x_1, x_2, \ldots, x_n) = \{B_{x_1}, B_{x_2}, \ldots, B_{x_n}\}$. Then θ_n is continuous. Let $G_n = \{(x_1, \ldots, x_n) \in (2^\omega)^n | x_j \neq x_k$ whenever $1 \leq j < k \leq n\}$. Then G_n is open in $(2^\omega)^n$ and given $(x_1, \ldots, x_n) \in G_n$ there is an open neighborhood N of $(x_1, \ldots, x_n)$ in G_n and an open neighborhood Φ' of $\theta_n(x_1, \ldots, x_n)$ in Φ such that θ_n maps N homeomorphically onto $\Phi' \cap \Phi_n$. (We say that θ_n is a local homeomorphism from G_n onto Φ_n.)

Now consider the subspace S of 2^ω. Clearly $\theta_n[G_n \cap S^n] = \Phi_n \cap \Phi_S$ so θ_n is a local homeomorphism from $G_n \cap S^n$ onto $\Phi_n \cap \Phi_S$. Because S is of additive class α, so is S^n [**K**, p. 346]. Hence so is $G_n \cap S^n$ as is each relatively open subset of $G_n \cap S^n$.

(Recall that since $\alpha \geq 1$, each open subset of G_n is of additive class α.) Therefore, the metric space $\Phi_S \cap \Phi_n$ admits an open cover by sets of additive class α so that $\Phi_S \cap \Phi_n$ has additive class α [**K**, p. 358]. Because $\Phi_S = \{\emptyset\} \cup (\bigcup\{\Phi_S \cap \Phi_n | n \geq 1\})$, Φ_S also has additive class α, as claimed. □

3.7 LEMMA. *Let $\mathcal{D} = \{A \in \mathcal{P}(T) |$ some $P \in p$ has $P \cap A = \emptyset\}$. Then $\mathcal{D}$ is of additive class $2 + \alpha$.*

PROOF. With i as in 3.5, we claim that $\mathcal{D} = i^{-1}[\Phi_S]$. For let $A \in \mathcal{D}$. Choose $P \in p$ with $P \cap A = \emptyset$. Then P contains some set $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$, where $x_j \in S$, so $A \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$. Hence $A \in \mathcal{A}$ so that $i(A)$ is defined. As noted in the proof of 3.5, since $A \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$, $i(A) \subset \{B_{x_1}, \ldots, B_{x_n}\}$ showing that $i(A) \in \Phi_S$. Conversely, suppose $A \in i^{-1}[\Phi_S]$. Then either there are points $x_1, \ldots, x_n \in S$ with $i(A) = \{B_{x_1}, \ldots, B_{x_n}\}$ or else $i(A) = \emptyset$ in which case A is finite. Consider the first possibility. If the set $A - (B_{x_1} \cup \cdots \cup B_{x_n})$ were infinite, some other branch of T would have an infinite intersection with A which is impossible, so the set $F = A - (B_{x_1} \cup \cdots \cup B_{x_n})$ is finite and we have $A \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$, so that A is disjoint from $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$ which belongs to the filter p so that $A \in \mathcal{D}$. The case where A is finite is easy because then the set $P_0 = T - A$ belongs to p so that $A \in \mathcal{D}$.

Because i is a Borel map of class 2 and because by 3.6 the set Φ_S has additive class α (where α is the additive class of S), $i^{-1}[\Phi_S]$ has additive class $2 + \alpha$, as claimed. □

4. The projective hierarchy. Recall the definition of the projective classes in a complete separable metric space Z [**K**, Chapter 3, §38]:

$$\mathcal{L}_0(Z) = \{A | A \text{ is a Borel subset of } Z\},$$

$$\mathcal{L}_{n+1}(Z) = \begin{cases} \{f[A] | A \in \mathcal{L}_n(Z) \text{ and } f\colon A \to Z \text{ is continuous}\} & \text{if } n \text{ is even,} \\ \{Z - A | A \in \mathcal{L}_n(Z)\} & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $\mathcal{L}_1(Z)$ is the family of analytic sets in Z, $\mathcal{L}_2(Z)$ is the family of co-analytic sets in Z, etc. The techniques of §§2 and 3 can be used to prove an analogue of 3.1 for projective sets. In our proof we will invoke theorems which are ordinarily stated for mappings into complete metric spaces [**K**, §38, III, Propositions 2 and 5, and VII, Theorem 1], applying those results to mappings into the σ-compact metric space Φ defined in 3.4. Extending the proofs given in [**K**] to cover this situation is easily done.

4.1 THEOREM. *Suppose $S \in \mathcal{L}_r(2^\omega)$ for some $r \geq 1$. Let $\Sigma = \Sigma_S$. Then $C_\pi(\Sigma) \in \mathcal{L}_r(\mathbf{R}^\Sigma)$. Furthermore, if $S \notin \mathcal{L}_{r-1}(2^\omega)$, then $C_\pi(\Sigma) \notin \mathcal{L}_{r-1}(\mathbf{R}^\Sigma)$.*

PROOF. Define $\psi_m \colon \mathbf{R}^\Sigma \to \mathcal{P}(T)$ and $\mathcal{D} \subset \mathcal{P}(T)$ as in 3.1. Suppose we know that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$. Then by [**K**, §38, III, Proposition 5], $\psi_m^{-1}[\mathcal{D}] \in \mathcal{L}_r(\mathbf{R}^\Sigma)$ for each m so that by [**K**, §38, III, Proposition 3] we would have $C_\pi(\Sigma) = \bigcap_{m=1}^\infty \psi_m^{-1}[\mathcal{D}] \in \mathcal{L}_r(\mathbf{R}^\Sigma)$ as claimed. Thus it will be enough to show that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$.

To prove that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$, we define the σ-compact set $\mathcal{A} \subset \mathcal{P}(T)$ as in 3.3, the σ-compact metric space Φ as in 3.4, the Borel measurable mapping $i\colon \mathcal{A} \to \Phi$ as in (3.5), and the set Φ_S as in 3.4. As in the proof of 3.7, $\mathcal{D} = \mathcal{A} \cap i^{-1}[\Phi_S]$. If we knew that $\Phi_S \in \mathcal{L}_r(\Phi)$, it would follow from [**K**, §38, III, Proposition 5] that $i^{-1}[\Phi_S] \in \mathcal{L}_r(\mathcal{A})$. Since $\mathcal{A}$ is σ-compact and hence in $\mathcal{L}_r(\mathcal{P}(T))$, it would follow that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$ [**K**, §38, III, Proposition 2]. Therefore it will be enough to show that $\Phi_S \in \mathcal{L}_r(\Phi)$. Define function $\theta_n\colon (2^\omega)^n \to \Phi$ as in 3.6. According to [**K**, §38, III, Proposition 1], $S^n \in \mathcal{L}_r((2^\omega)^n)$. Because each open subset H of $(2^\omega)^n$ also belongs to $\mathcal{L}_r((2^\omega)^n)$ we see that $H \cap G_n \cap S^n \in \mathcal{L}_r((2^\omega)^n)$ whenever H is open in $(2^\omega)^n$. But θ_n is known to be a local homeomorphism of $G_n \cap S^n$ onto the separable metric space $\Phi_S \cap \Phi_n$ so there is a sequence $H_1, H_2, \ldots$ of subsets of G_n such that for each k, θ_n maps $H_k \cap G_n \cap S^n$ homeomorphically onto a relatively open subset of $\Phi_n \cap \Phi_S$ and such that $\Phi_n \cap \Phi_S = \bigcup\{\theta_n[H_k \cap G_n \cap S^n] | k \geq 1\}$. Because $H_k \cap G_n \cap S^n \in \mathcal{L}_r((2^\omega)^n)$ for each k, it follows from [**K**, §38, VII, Theorem 1] that $\theta_n[H_k \cap G_n \cap S^n] \in \mathcal{L}_r(\Phi)$. But then $\Phi_n \cap \Phi_S$, being a countable union of members of $\mathcal{L}_r(\Phi)$, also belongs to $\mathcal{L}_r(\Phi)$. For the same reason, the set $\Phi_S = \bigcup\{\Phi_S \cap \Phi_n | n \geq 1\}$ also belongs to $\mathcal{L}_r(\Phi)$ as claimed.

Finally suppose $S \notin \mathcal{L}_{r-1}(2^\omega)$. According to 2.1, there is a (relatively) closed subspace S^* of $C_\pi(\Sigma)$ which is homeomorphic to S. Then $S^* = C_\pi(\Sigma) \cap D$, where D is some closed subset in $\mathbf{R}^\Sigma$. If $C_\pi(\Sigma) \in \mathcal{L}_{r-1}(\mathbf{R}^\Sigma)$, then $S^* = C_\pi(\Sigma) \cap D$ would also belong to $\mathcal{L}_{r-1}(\mathbf{R}^\Sigma)$. According to [**K**, §38, VII, Theorem 1], we would then have $S \in \mathcal{L}_{r-1}(2^\omega)$ because S is homeomorphic to S^*, which is impossible. □

4.2 COROLLARY. *For each $n \geq 1$ there is a countable regular space X_n such that $C_\pi(X_n) \in \mathcal{L}_n(\mathbf{R}^{X_n}) - \mathcal{L}_{n-1}(\mathbf{R}^{X_n})$ and there is a countable regular space Y such that $C_\pi(Y) \notin \bigcup\{\mathcal{L}_n(\mathbf{R}^Y) | n \geq 1\}$.*

PROOF. Fix n. By [**K**, §38, VI, Theorem 1] there is a set $S_n \subset 2^\omega$ having $S_n \in \mathcal{L}_n(2^\omega) - \mathcal{L}_{n-1}(2^\omega)$. Let $X_n = \Sigma_{S_n}$. To obtain the space Y, choose any $S \subset 2^\omega$ with $S \notin \bigcup\{\mathcal{L}_n(2^\omega) | n \geq 1\}$ [**K**, §38, VI, Remark 1] and let $Y = \Sigma_S$. Because $C_\pi(Y)$ contains a closed subset homeomorphic to S, $C_\pi(Y) \notin \bigcup\{\mathcal{L}_n(\mathbf{R}^Y) | n \geq 1\}$. □

5. Baire category and Baire Property subsets of $\mathbf{R}^X$. For any space Z, $\mathcal{BP}(Z)$ is the σ-algebra generated by the open sets and the first category subsets of Z. Members of $\mathcal{BP}(Z)$ are called Baire Property subsets of Z [**Ox₂**, p. 19]. For a space X with a unique limit point (such as the spaces Σ_S for $S \subset 2^\omega$ constructed in §1) it is easy to characterize which function spaces $C_\pi(X)$ belong to $\mathcal{BP}(\mathbf{R}^X)$.

5.1 THEOREM. *Suppose X is a countable space with a unique limit point ∞ and let p be the trace on $X - \{\infty\}$ of the neighborhod filter of ∞. Then the following are equivalent:*

(a) *$C_\pi(X)$ is a first category subset of $\mathbf{R}^X$;*

(b) *$C_\pi(X) \in \mathcal{BP}(\mathbf{R}^X)$;*

(c) *there is an array*

$$\begin{matrix} A(1,1) & A(1,2) & A(1,3) & \cdots \\ A(2,1) & A(2,2) & A(2,3) & \cdots \\ A(3,1) & A(3,2) & A(3,3) & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

satisfying

(i) *each $A(m,n)$ is a finite subset of $X - \{\infty\}$;*

(ii) *each row $A(m,1), A(m,2), A(m,3,), \ldots$ is a pairwise disjoint sequence;*

(iii) *for every sequence $k(1), k(2), \ldots$ and every $U \in p$, $U \cap (\bigcup\{A(m,k(m)) | m \geq 1\}) \neq \varnothing$.*

PROOF. The equivalence of (a) and (c) follows from [**LM**, Theorems 6.3 and 5.1] and obviously (a) implies (b). We prove that (b) implies (a). Suppose $C_\pi(X) \in \mathcal{BP}(\mathbf{R}^X)$. To simplify notation, we will identify the countably many isolated points of X with elements of ω and we will write $X = \omega \cup \{\infty\}$. Define a function $\nu\colon \mathbf{R}^X \to \mathbf{R}^\omega \times \mathbf{R}$ by the rule that $\nu(f) = (f^*, f(\infty))$, where $f^* \in \mathbf{R}^\omega$ is given by $f^*(n) = f(n) - f(\infty)$. Then ν is a homeomorphism of $\mathbf{R}^X$ onto $\mathbf{R}^\omega \times \mathbf{R}$ and $\nu[C_\pi(X)] = C_0 \times \mathbf{R}$, where $C_0 = \{g \in \mathbf{R}^\omega |$ for each $\varepsilon > 0$ there is a neighborhood U of ∞ having $g[U \cap \omega] \subset]-\varepsilon, \varepsilon[\}$. Since $C_\pi(X) \in \mathcal{BP}(\mathbf{R}^X)$, $C_0 \times \mathbf{R} \in \mathcal{BP}(\mathbf{R}^\omega \times \mathbf{R})$.

It is easily seen that C_0 is a *tailset in* $\mathbf{R}^\omega$, i.e. that if $g \in C_0$ and if the equality $h(n) = g(n)$ holds except for finitely many values of n, then $h \in C_0$. We now need a slight variation of a result due to Oxtoby [**Ox₁**]; the proof is only trivially different from Oxtoby's argument.

5.2 LEMMA. *Let C be a tailset in $\mathbf{R}^\omega$ and suppose that $C \times \mathbf{R} \in \mathcal{BP}(\mathbf{R}^\omega \times \mathbf{R})$. Then either $C \times \mathbf{R}$ is a first category subset of $\mathbf{R}^\omega \times \mathbf{R}$ or else $C \times \mathbf{R}$ contains a dense G_δ-subset of $\mathbf{R}^\omega \times \mathbf{R}$.*

Given 5.2, either $C_0 \times \mathbf{R}$ is a first category subset of $\mathbf{R}^\omega \times \mathbf{R}$, in which case $C_\pi[X]$ is also a first category subset of $\mathbf{R}^X$, or else $C_0 \times \mathbf{R}$ contains a dense G_δ-subset of $\mathbf{R}^\omega \times \mathbf{R}$, in which case $C_\pi(X)$ contains a dense G_δ in $\mathbf{R}^X$. But the latter situation occurs if and only if X is a discrete space [**DGLvM**, Theorem 1] so that $C_\pi(X)$ must be a first category subset of $\mathbf{R}^X$, as claimed. □

5.3 REMARK. The reason for creating a variant of Oxtoby's theorem as in 5.2 is that one cannot deduce $C_0 \in \mathcal{BP}(\mathbf{R}^\omega)$ from $C_0 \times \mathbf{R} \in \mathcal{BP}(\mathbf{R}^\omega \times \mathbf{R})$.

5.4 COROLLARY. *For each $S \subset 2^\omega$, the function space $C_\pi(\Sigma_S)$ is a first category subset of $\mathbf{R}^{\Sigma_S}$.*

PROOF. We define an array $A(m,n)$ as follows using the tree $T = \bigcup_1^\infty T_n$;

(i) $A(1,n) = T_n$ for $n \geq 1$;

(ii) $A(2,1) = T_1 \cup T_2$, $A(2,2) = T_3 \cup T_4$, $A(2,3) = T_5 \cup T_6, \ldots$;

(iii) in general, $A(m,n) = T_{(n-1)m+1} \cup \cdots \cup T_{nm}$.

Obviously each $A(m,n)$ is finite and because the sets $T_1, T_2, \ldots$ are pairwise disjoint, each row $A(m,1), A(m,2), \ldots$ of the array is pairwise disjoint. Suppose $k(1), k(2), \ldots$ is a sequence of positive integers and suppose $U = T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$, where $x_i \in S$ and F is a finite subset of T. If $\varnothing = U \cap (\bigcup\{A(m,k(m)) | m \geq 1\})$, then $\bigcup\{A(m,k(m)) | m \geq 1\} \subset B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_n} \cup F$. Observe that for a fixed level T_j of the tree T, $\operatorname{card}(B_{x_i} \cap T_n) = 1$ so that $\operatorname{card}(T_j \cap (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)) \leq n + \operatorname{card}(F)$. Choose $m > n + \operatorname{card}(F)$. Then the set $A(m, k(m))$ contains a level T_j of T where $\operatorname{card}(T_j) \geq 2^m$ so that $T_j \cap (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$ must have cardinality greater than $n + \operatorname{card}(F)$, contrary to our observation above. □

In closing let us give one more example of a countable regular space X with a unique isolated point ∞ which has a "bad" function space. Unlike the examples so far, $C_\pi(X)$ is a second category subset of $\mathbf{R}^X$.

5.5 EXAMPLE. Let p be a free ultrafilter on ω and topologize the set $X = \omega \cup \{\infty\}$ by isolating all points of ω and by using all sets of the form $\{\infty\} \cup U$, where $U \in p$, as neighborhoods of ∞. Then $C_\pi(X)$ is a second category subset of $\mathbf{R}^X$ and $C_\pi(X) \notin \mathcal{L}_1(\mathbf{R}^X) \cup \mathcal{L}_2(\mathbf{R}^X)$.

PROOF. That $C_\pi(X)$ is a second category subset of $\mathbf{R}^X$ follows from the equivalence of (a) and (c) in 5.1 (cf. [**LM**, 5.1 and 6.3] for details). Suppose

$C_\pi(X) \in \mathcal{L}_n(\mathbf{R}^X)$, where $n \in \{1,2\}$. Define $j\colon 2^\omega \to \mathbf{R}^X$ by the rule that if $f \in 2^\omega$ then $j(f) = \hat{f} \in \mathbf{R}^X$ where $\hat{f}$ is given by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \omega, \\ 1 & \text{if } x = \infty. \end{cases}$$

Then j is continuous so that by [**K**, §38, III, Proposition 2], $j^{-1}[C_\pi(X)] \in \mathcal{L}_n(2^\omega)$. Hence $j^{-1}[C_\pi(X)]$ is a measurable subset of 2^ω (with respect to product measure μ) because all analytic and co-analytic subsets of 2^ω are measurable [**L**, p. 243, Proposition 3.24]. But $j^{-1}[C_\pi(X)] = \{x \in 2^\omega \mid \text{for some } U \in p, x(n) = 1 \text{ for each } n \in U\}$ so that $j^{-1}[C_\pi(X)]$ is seen to be a tailset in 2^ω. Hence Kolmogorov's "0-1 law" guarantees that $\mu[j^{-1}[C_\pi(X)]] = 0$ or $\mu[j^{-1}[C_\pi(X)]] = 1$ [**Ox₂**, p. 84]. However, consider the function $J\colon 2^\omega \to 2^\omega$ given by $J(f) = f \oplus \bar{1}$, where $\bar{1} \in 2^\omega$ is constantly equal to 1 and $\oplus$ denotes coordinatewise addition modulo 2, i.e., the usual group operation of 2^ω. Since μ is translation invariant, J is a measure preserving transformation on 2^ω. Because p is an ultrafilter, $J[j^{-1}[C_\pi(X)]] = 2^\omega - j^{-1}[C_\pi(X)]$ so that both $\mu[j^{-1}[C_\pi(X)]] = 0$ and $\mu[j^{-1}[C_\pi(X)]] = 1$ are impossible. Therefore $C_\pi(X) \notin \mathcal{L}_0(\mathbf{R}^X) \cup \mathcal{L}_1(\mathbf{R}^X) \cup \mathcal{L}_2(\mathbf{R}^X)$, as claimed. □

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