DESCRIPTIVE COMPLEXITY OF FUNCTION SPACES

BY

D. LUTZER, J. VAN MILL AND R. POL

ABSTRACT. In this paper we show that \( C_\pi(X) \), the set of continuous, real-valued functions on \( X \) topologized by the pointwise convergence topology, can have arbitrarily high Borel or projective complexity in \( R^X \) even when \( X \) is a countable regular space with a unique limit point. In addition we show how to construct countable regular spaces \( X \) for which \( C_\pi(X) \) lies nowhere in the projective hierarchy of the complete separable metric space \( R^X \).

1. Introduction. Let \( C_\pi(X) \) be the set of continuous, real-valued functions on a space \( X \) and topologize \( C_\pi(X) \) as a subspace of the full product \( R^X \). In [DGLvM] it is shown that if \( X \) is completely regular, then \( C_\pi(X) \) cannot be a \( G_{\delta}^\sigma \), \( F_{\sigma}^\sigma \) or \( G_{\delta\sigma} \)-subset of \( R^X \) unless \( X \) is discrete and that for any countable metrizable space \( X \), \( C_\pi(X) \) will be an \( F_{\sigma}^\sigma \)-subset of \( R^X \). In the terminology of [KM and K], \( C_\pi(X) \) cannot have multiplicative class 1 and cannot have additive class 1 or 2, but may have multiplicative class 2.

In this paper we study the descriptive complexity of \( C_\pi(X) \) in \( R^X \) when \( X \) is countable (so that \( R^X \) is a complete separable metric space). Our main results can be summarized as follows. 

THEOREM. (a) Given any \( \alpha < \omega_1 \), there is a countable regular space \( X \) such that \( C_\pi(X) \) is a Borel subset of \( R^X \) having additive class \( \beta \), where \( \alpha \leq \beta \leq 3+\alpha+2 \) (§§2 and 3).

(b) Given any \( n \geq 1 \) there is a countable regular space \( Y \) such that \( C_\pi(Y) \in \mathcal{L}_n(R^Y) - \mathcal{L}_{n-1}(R^Y) \), where \( \mathcal{L}_n(R^Y) \) is the family of projective sets of class \( n \) in the complete separable metric space \( R^Y \) (§4).

(c) There is a countable regular space \( Z \) such that \( C_\pi(Z) \notin \bigcup\{\mathcal{L}_n(R^Z) : 0 \leq n < \omega\} \) (§§4 and 5).

The spaces \( X, Y \) and \( Z \) in the above Theorem can be obtained from a single general construction which associates with each subset \( S \subset 2^\omega \) a certain countable regular space \( \Sigma_S \) having a unique nonisolated point. The descriptive complexity of \( S \) in \( 2^\omega \) determines the complexity of \( C_\pi(\Sigma_S) \) in \( R^{2^\omega} \). To describe \( \Sigma_S \) precisely, we begin by letting \( T_n = 2^n \) be the set of functions from \( \{0, 1, \ldots, n-1\} \) into \( \{0, 1\} \), i.e., the set of ordered \( n \)-tuples of 0’s and 1’s. Let \( T = \bigcup\{T_n : n \geq 1\} \) and partially order \( T \) by function extension. A branch of \( T \) is a maximal linearly ordered subset of \( T \), i.e., a linearly ordered subset \( B \subset T \) having \( \text{card}(B \cap T_n) = 1 \) for each \( n \geq 1 \). Observe that if \( B \) and \( \bar{B} \) are distinct branches of \( T \), then \( B \cap \bar{B} \) must be a finite set.

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Given $x \in 2^\omega$, the set $B_x = \{(x(0)), (x(0), x(1)), (x(0), x(1), x(2)), \ldots\}$ is a branch of $T$. Conversely, each branch $B$ of $T$ has the form $B = B_x$ for a unique $x \in 2^\omega$. Let $\mathcal{B} = \{B | B$ is a branch of $T\}$.

Let $\mathcal{P}(T) = \{A | A \subset T\}$ and topologize $\mathcal{P}(T)$ using open sets of the form $[Y, N] = \{A \in \mathcal{P}(T) | Y \subset A \subset T - N\}$, where $Y$ and $N$ are arbitrary finite subsets of $T$. The resulting space is compact and metrizable, and is homeomorphic to the product space $2^T$ under the mapping which identifies each subset $A \in \mathcal{P}(T)$ with its characteristic function $\chi_A$. The mapping $x \to B_x$ is easily seen to be a homeomorphism of $2^\omega$ into $\mathcal{P}(T)$ whose image is exactly the set $\mathcal{B}$ defined above.

For each subset $S \subset 2^\omega$, the collection $\{T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F) | n \geq 1, x_i \in S$ and $F \subset T$ is a finite set $\}$ is a filter base. Let $p_S$ be the filter generated by that filter base. Let $\omega$ be any point not in $T$ and $2^\omega$ and let $\Sigma_T = T \cup \{\omega\}$. Topologize $\Sigma_T$ by isolating each point of $T$ and by using the family $\{P \cup \{\omega\} | P \in p_S\}$ as a neighborhood base at $\omega$. The space $\Sigma_T$ is countable, regular, and (since $p_S$ is a free filter) is $T_1$. The spaces mentioned in the above Theorem are all of the form $\Sigma_T$ for various subsets $S$ of $2^\omega$.

However, even though the function spaces $C_\pi(\Sigma_T)$ for $S \subset 2^\omega$ provide enough pathology to prove our Theorem, they are all well behaved in some senses. In §5 we prove that each $C_\pi(\Sigma_T)$ is a Baire Property subset of $R^{\Sigma_T}$ and is meagre in $R^{\Sigma_T}$ (equivalently, $C_\pi(\Sigma_T)$ is not a Baire space) and we exhibit a countable regular space $X$ with a unique nonisolated point such that $C_\pi(X)$ is a second category subset of $R^X$ (equivalently, $C_\pi(X)$ is a Baire space), is not a Baire Property subset of $R^X$, and is not a Borel, analytic or co-analytic subset of $R^X$ (see Example 5.5).

The standard references for descriptive theory in complete separable metric spaces are [K and KM]. Our topological terminology is consistent with [E] and [Ox2] is a good source for properties of Baire spaces. The authors wish to thank Jean Calbrix and Fons van Engelen for their comments on an earlier version of this paper.

2. A lower bound for the complexity of $C_\pi(\Sigma_T)$.

2.1 Theorem. Let $S \subset 2^\omega$ and let $\Sigma = \Sigma_T$. Then $C_\pi(\Sigma)$ contains a relatively closed subset which is homeomorphic to $S$.

Proof. Recall that in $\Sigma_T$, the point $\omega$ has a neighborhood base consisting of all sets of the form $\{\omega\} \cup (T - (B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_n} \cup F))$, where $x_i \in S$ and $F \subset T$ is finite. For each $x \in 2^\omega$ define a function $f_x: \Sigma \to R$ by $f_x(\omega) = 0$, $f_x(t) = 0$ if $t \in T - B_x$ and $f_x(t) = 1$ if $t \in B_x$. Define $\lambda: 2^\omega \to R^\Sigma$ by $\lambda(x) = f_x$. Clearly $\lambda$ is 1-1 and continuous, so that $\lambda$ embeds $2^\omega$ as a closed subspace of $R^\Sigma$. Furthermore $\lambda(x) \in C_\pi(\Sigma)$ whenever $x \in S$ because for such an $x$, the function $f_x$ is constant on the neighborhood $\{\omega\} \cup (T - B_x)$ of $\omega$. Conversely, if $f_x \in C_\pi(\Sigma)$ for some $x \in 2^\omega$, then $f_x^{-1}[(-\frac{1}{2}, \frac{1}{2})]$ must be a neighborhood of $\omega$ so that for some $x_1, \ldots, x_n \in S$ and some finite $F$, the set $f_x^{-1}[(-\frac{1}{2}, \frac{1}{2})] = \{\omega\} \cup (T - B_x)$ must contain the basic neighborhood $\{\omega\} \cup (T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F))$. But then $B_x \subset B_{x_1} \cup \cdots \cup B_{x_n} \subset F$ so that $B_x \cap B_z$ is infinite for some $i$ and hence $B_x = B_{x_i}$, i.e., $x = x_i \in S$. Therefore $\lambda[S] = C_\pi(\Sigma) \cap \lambda[2^\omega]$ showing that $\lambda[S]$ is a relatively closed subset of $C_\pi(\Sigma)$.

2.2 Corollary. If $S$ is not a Borel subset of $2^\omega$ (resp., if $S$ is not a projective subset of $2^\omega$), then $C_\pi(\Sigma_T)$ is not a Borel subset (resp. a projective subset) of $R^{\Sigma_T}$.
PROOF. Write $\Sigma = \Sigma_S$. In the complete separable metric space $\mathbb{R}^\Sigma$, a relatively closed subset of a Borel (resp., projective) set is again a Borel (resp., projective) set in $\mathbb{R}^\Sigma$ and it is known that homeomorphisms preserve Borel (resp., projective) sets [K, Chapter 3, §35, IV, Corollary 1 and Chapter 3, §38, VII, Theorem 1] contrary to our assumption that $S$ is not Borel (resp., projective) in $2^\omega$. □

2.3 COROLLARY. There is a countable regular space $X$ such that $C_\pi(X)$ is not a Borel subset of $\mathbb{R}^X$.

PROOF. Let $S$ be a non-Borel subset of $2^\omega$ and let $X = \Sigma_S$. Now apply 2.2. □

3. An upper bound for the Borel complexity of $C_\pi(\Sigma_S)$. In §2 we proved that $C_\pi(\Sigma_S)$ always contains a closed subspace homeomorphic to $S$ so that if $S$ is not a Borel set, then neither is $C_\pi(\Sigma_S)$. In this section we study the situation where $S$ is a Borel subset of $2^\omega$ and we prove

3.1 THEOREM. Let $S$ be a Borel subset of $2^\omega$ having additive class $\alpha \geq 1$ and let $\Sigma = \Sigma_S$. Then $C_\pi(\Sigma)$ is a Borel subset of $\mathbb{R}^\Sigma$ of class $\beta$, where $\alpha \leq \beta \leq 3 + \alpha + 2$.

PROOF. Following the notation of §1, we let $p = p_S$ be the filter on $T$ generated by all sets of the form $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$, where $x_j \in S$ for $1 \leq j \leq n$ and $F$ is any finite subset of $T$.

For each $m \geq 1$, define $\psi_m : \mathbb{R}^\Sigma \to \mathcal{P}(T)$ by $\psi_m(f) = \{t \in T | |f(\infty) - f(t)| \geq 1/m\}$. In Lemma 3.2 we show that $\psi_m$ is a Borel mapping of class 1. Next, define a set $\mathcal{D} \subset \mathcal{P}(T)$ by $\mathcal{D} = \{A \in \mathcal{P}(T) | A \cap P = \emptyset \text{ for some } P \in p\}$. In Lemma 3.6 we prove that $\mathcal{D}$ is a Borel subset of $\mathcal{P}(T)$ of additive class $\leq 2 + \alpha$ so that $\psi_m^{-1}(\mathcal{D})$ is a Borel set of additive class $\leq 3 + \alpha$. Because a function $f \in \mathbb{R}^\Sigma$ is continuous if and only if $\{t \in T | |f(\infty) - f(t)| < 1/m\}$ belongs to $p$ for each $m$, we have $C_\pi(\Sigma) = \bigcap \{\psi_m^{-1}(\mathcal{D}) | m \geq 1\}$ showing that $C_\pi(\Sigma)$ is a Borel set of additive class $\beta \leq (3 + \alpha + 2)$.

From §2, a closed subspace of $C_\pi(\Sigma)$ is homeomorphic to $S$, so the additive class of $C_\pi(\Sigma)$ cannot be smaller than the additive class of $S$ and we obtain $\alpha \leq \beta$. □

All that remains is to prove some lemmas.

3.2 LEMMA. Each $\psi_m$ is a Borel map of class 1.

PROOF. It is enough to show that $\psi_m^{-1}([Y, N])$ is an $F_\sigma$-subset of $\mathbb{R}^\Sigma$ for each basic open set $[Y, N]$ in $\mathcal{P}(T)$. Now

$$\psi_m^{-1}([Y, N]) = \{f \in \mathbb{R}^\Sigma | Y \subset \{t \in T | |f(t) - f(\infty)| \geq 1/m\} \cap \{f \in \mathbb{R}^\Sigma | t \in [T, T - N] \}.$$

The first of those two sets is closed and, since $N$ is finite, the second is open. Hence their intersection is an $F_\sigma$-set, as claimed. □

3.3 LEMMA. The set $\mathcal{A} = \{A \in \mathcal{P}(T) | \text{ for some } B_1, \ldots, B_n \in \mathcal{B} \text{ and some finite } F \subset C \subset B_1 \cup \cdots \cup B_n \cup F \}$ is a $\sigma$-compact subset of $\mathcal{P}(T)$.

PROOF. For a fixed finite $F \subset T$ and a fixed $n$, let $\mathcal{A}(F, n) = \{(A, B_1, \ldots, B_n) | B_i \in \mathcal{B} \text{ and } A \subset B_1 \cup \cdots \cup B_n \cup F\}$. Then $\mathcal{A}(F, n)$ is a closed subset of the compact space $\mathcal{P}(T) \times \mathcal{B}^n$. Let $\pi_n : \mathcal{P}(T) \times \mathcal{B}^n \to \mathcal{P}(T)$ denote first coordinate projection. Then $\mathcal{A} = \bigcup \{\pi_n[\mathcal{A}(F, n)] | n \geq 1 \text{ and } F \subset T \text{ is finite}\}$ so that $\mathcal{A}$ is a $\sigma$-compact subspace of $\mathcal{P}(T)$ as claimed. □
3.4 Notation. Recall that $\mathcal{B}$ is the set of all branches of $T$, topologized as a subspace of the compact metric space $\mathcal{P}(T)$. Being the continuous image of $2^\omega$ under the map $\mu(x) = B_x$, $\mathcal{B}$ is compact. For $n \geq 1$, let $\Phi_n = \{K \subset \mathcal{B} | \text{card}(K) = n\}$ and let $\Phi = \bigcup\{\Phi_n | n \geq 0\}$. Topologize $\Phi$ with the Vietoris topology, i.e., by using all subsets of $\Phi$ of the forms $\{K \in \Phi | K \subset \mathcal{U}\}$ and $\{K \in \Phi | K \cap \mathcal{V} \neq \emptyset\}$ as a subbase where $\mathcal{U}$ and $\mathcal{V}$ are arbitrary open subsets of $\mathcal{B}$. Then $\Phi$ is a $\sigma$-compact metrizable space [KM, p. 392]. Recall that each branch of $T$ is of the form $B_x$ for some $x \in 2^\omega$ and let $\Phi_S = \{K \in \Phi | K \subset \{B_x | x \in S\}\} = \{K | K$ is a finite subset of $\{B_x | x \in S\}\}$.

3.5 Lemma. With $\mathcal{A}$ as in 3.3, for each $A \in \mathcal{A}$ let $i(A) = \{B \in \mathcal{B} | B \cap A$ is infinite\}. Then $i: \mathcal{A} \to \Phi$ is a Borel mapping of class $2$.

Proof. Fix $A \in \mathcal{A}$ and choose branches $B_1, \ldots, B_n$ and a finite set $F$ with $A \subset B_1 \cup \cdots \cup B_n \cup F$. If $B$ is any branch of $T$ such that $A \cap B$ is infinite, then $B \cap B_k$ is infinite for some $k = 1, 2, \ldots, n$ so that $B$ is one of the branches $B_1, \ldots, B_n$. Hence $i(A)$ is finite so $i(A) \in \Phi$. (If $A$ is infinite, then $i(A) = \emptyset \in \Phi$.)

(a) Fix an open subset $\mathcal{U}$ of $\mathcal{B}$ and consider $i^{-1}\{\{K \in \Phi | K \subset \mathcal{U}\}\} = \{A \in \mathcal{A} | i(A) \subset \mathcal{U}\}$. Because $\mathcal{U}$ is an open subset of the compact metric space $\mathcal{B}$, $\mathcal{U}$ is $\sigma$-compact. According to 3.3, so is $\mathcal{A}$, and we conclude that the product space $\mathcal{A} \times \mathcal{U}^n$ is $\sigma$-compact for each $n \geq 1$, where $\mathcal{U}^n$ is the product of $n$ copies of $\mathcal{U}$. Fix $n \geq 1$ and fix a finite set $F \subset T$. Then the set $\mathcal{C}(n, F) = \{(A, B_1, \ldots, B_n) \in \mathcal{A} \times \mathcal{U}^n | A \subset B_1 \cup \cdots \cup B_n \cup F\}$ is closed in $\mathcal{A} \times \mathcal{U}^n$, so $\mathcal{C}(n, F)$ is $\sigma$-compact. Let $\pi_n: \mathcal{A} \times \mathcal{U}^n \to \mathcal{A}$ be first coordinate projection. Then $i^{-1}\{\{K \in \Phi | K \subset \mathcal{U}\}\} = \bigcup\{\pi_n[\mathcal{C}(n, F)] | n \geq 1$ and $F \subset T$ is finite\} so $i^{-1}\{\{K \in \Phi | K \subset \mathcal{U}\}\}$ is a $\sigma$-compact subset of $\mathcal{A}$ (and therefore a $G_\delta$-subset of $\mathcal{A}$).

(b) Next consider $i^{-1}\{\{K \in \Phi | K \cap \mathcal{V} \neq \emptyset\}\}$, where $\mathcal{V}$ is a compact, open subset of $\mathcal{B}$. Then $\mathcal{B} \setminus \mathcal{V}$ is open and $\{K \in \Phi | K \cap \mathcal{V} \neq \emptyset\} = \Phi - \{K \in \Phi | K \subset \mathcal{B} \setminus \mathcal{V}\}$. Hence $i^{-1}\{\{K \in \Phi | K \cap \mathcal{V} \neq \emptyset\}\} = \mathcal{A} - i^{-1}\{\{K \in \Phi | K \subset \mathcal{B} \setminus \mathcal{V}\}\}$ which is a $G_\delta$-subset in light of (a).

(c) Finally, consider $i^{-1}\{\{K \in \Phi | K \cap \mathcal{U} \neq \emptyset\}\}$, where $\mathcal{U}$ is an arbitrary open subset of $\mathcal{B}$. There is a sequence $\langle \mathcal{V}_n \rangle$ of compact, open subsets of $\mathcal{B}$ having $\mathcal{U} = \bigcup\{\mathcal{V}_n | n \geq 1\}$ so that $i^{-1}\{\{K \in \Phi | K \cap \mathcal{U} \neq \emptyset\}\} = \bigcup\{i^{-1}\{\{K \in \Phi | K \cap \mathcal{V}_n \neq \emptyset\}\} | n \geq 1\}$ which is a $G_\delta$-set in $\mathcal{A}$ because of (b).

(d) Since sets of the form $\{K \in \Phi | K \subset \mathcal{U}\}$ and $\{K \in \Phi | K \cap \mathcal{U} \neq \emptyset\}$ form a subbase for the separable metric space $\Phi$, it follows that $i$ is a Borel mapping of class $2$. \endproof

3.6 Lemma. With $\Phi_S$ as defined in 3.4, $\Phi_S$ is a Borel subset of $\Phi$ whose additive class is $\alpha$ (= the additive class of $S$).

Proof. For $n \geq 1$, define $\theta_n: (2^\omega)^n \to \Phi$ by $\theta_n(x_1, x_2, \ldots, x_n) = \{B_{x_1}, B_{x_2}, \ldots, B_{x_n}\}$. Then $\theta_n$ is continuous. Let $G_n = \{(x_1, \ldots, x_n) \in (2^\omega)^n | x_j \neq x_k$ whenever $1 \leq j < k \leq n\}$. Then $G_n$ is open in $(2^\omega)^n$ and given $(x_1, \ldots, x_n) \in G_n$ there is an open neighborhood $N$ of $(x_1, \ldots, x_n)$ in $G_n$ and an open neighborhood $\Phi'$ of $\theta_n(x_1, \ldots, x_n)$ in $\Phi$ such that $\theta_n$ maps $N$ homeomorphically onto $\Phi' \cap \Phi_n$. (We say that $\theta_n$ is a local homeomorphism from $G_n$ onto $\Phi_n$.)

Now consider the subspace $S$ of $2^\omega$. Clearly $\theta_n[G_n \cap S^n] = \Phi_n \cap \Phi_S$ so $\theta_n$ is a local homeomorphism from $G_n \cap S^n$ onto $\Phi_n \cap \Phi_S$. Because $S$ is of additive class $\alpha$, so is $S^n$ [K, p. 346]. Hence so is $G_n \cap S^n$ as is each relatively open subset of $G_n \cap S^n$. 


(Recall that since $\alpha \geq 1$, each open subset of $G_n$ is of additive class $\alpha$.) Therefore, the metric space $\Phi_S \cap \Phi_n$ admits an open cover by sets of additive class $\alpha$ so that $\Phi_S \cap \Phi_n$ has additive class $\alpha$ [K, p. 358]. Because $\Phi_S = \{\emptyset\} \cup (\bigcup \{\Phi_S \cap \Phi_n[n \geq 1]\}$, $\Phi_S$ also has additive class $\alpha$, as claimed. \qed

3.7 Lemma. Let $\mathcal{D} = \{A \in \mathcal{P}(T) | \text{some } P \in p \text{ has } P \cap A = \emptyset\}$. Then $\mathcal{D}$ is of additive class $2 + \alpha$.

Proof. With $i$ as in 3.5, we claim that $\mathcal{D} = i^{-1}[\Phi_S]$. For let $A \in \mathcal{D}$. Choose $P \in p$ with $P \cap A = \emptyset$. Then $P$ contains some set $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$, where $x_j \in S$, so $A \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$. Hence $A \in \mathcal{A}$ so that $i(A)$ is defined. As noted in the proof of 3.5, since $A \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$, $i(A) \subset \{B_{x_1}, \ldots, B_{x_n}\}$ showing that $i(A) \in \Phi_S$. Conversely, suppose $A \in i^{-1}[\Phi_S]$. Then either there are points $x_1, \ldots, x_n \in S$ with $i(A) = \{B_{x_1}, \ldots, B_{x_n}\}$ or else $i(A) = \emptyset$ in which case $A$ is finite. Consider the first possibility. If the set $A - (B_{x_1} \cup \cdots \cup B_{x_n})$ were infinite, some other branch of $T$ would have an infinite intersection with $A$ which is impossible, so the set $F = A - (B_{x_1} \cup \cdots \cup B_{x_n})$ is finite and we have $A \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$, so that $A$ is disjoint from $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$ which belongs to the filter $p$ so that $A \in \mathcal{D}$. The case where $A$ is finite is easy because then the set $P_0 = T - A$ belongs to $p$ so that $A \in \mathcal{D}$.

Because $i$ is a Borel map of class 2 and because by 3.6 the set $\Phi_S$ has additive class $\alpha$ (where $\alpha$ is the additive class of $S$), $i^{-1}[\Phi_S]$ has additive class $2 + \alpha$, as claimed. \qed

4. The projective hierarchy. Recall the definition of the projective classes in a complete separable metric space $Z$ [K, Chapter 3, §38]:

$$\mathcal{L}_0(Z) = \{A|A \text{ is a Borel subset of } Z\},$$

$$\mathcal{L}_{n+1}(Z) = \{\{f[A]|A \in \mathcal{L}_n(Z) \text{ and } f: A \to Z \text{ is continuous}\} \text{ if } n \text{ is even},$$

$$\{Z - A|A \in \mathcal{L}_n(Z)\} \text{ if } n \text{ is odd.}$$

Thus, $\mathcal{L}_1(Z)$ is the family of analytic sets in $Z$, $\mathcal{L}_2(Z)$ is the family of co-analytic sets in $Z$, etc. The techniques of §§2 and 3 can be used to prove an analogue of 3.1 for projective sets. In our proof we will invoke theorems which are ordinarily stated for mappings into complete metric spaces [K, §38, III, Propositions 2 and 5, and VII, Theorem 1], applying those results to mappings into the $\sigma$-compact metric space $\Phi$ defined in 3.4. Extending the proofs given in [K] to cover this situation is easily done.

4.1 Theorem. Suppose $S \in \mathcal{L}_r(2^\omega)$ for some $r \geq 1$. Let $\Sigma = \Sigma_S$. Then $C_\alpha(\Sigma) \in \mathcal{L}_r(\mathcal{R}^\Sigma)$. Furthermore, if $S \notin \mathcal{L}_{r-1}(2^\omega)$, then $C_\alpha(\Sigma) \notin \mathcal{L}_{r-1}(\mathcal{R}^\Sigma)$.

Proof. Define $\psi_m: \mathcal{R}^\Sigma \to \mathcal{P}(T)$ and $\mathcal{D} \subset \mathcal{P}(T)$ as in 3.1. Suppose we know that $\mathcal{D} \subset \mathcal{L}_r(\mathcal{P}(T))$. Then by [K, §38, III, Proposition 5], $\psi_m^{-1}[\mathcal{D}] \in \mathcal{L}_r(\mathcal{R}^\Sigma)$ for each $m$ so that by [K, §38, III, Proposition 3] we would have $C_\alpha(\Sigma) = \bigcap_{m=1}^{\infty} \psi_m^{-1}[\mathcal{D}] \in \mathcal{L}_r(\mathcal{R}^\Sigma)$ as claimed. Thus it will be enough to show that $\mathcal{D} \subset \mathcal{L}_r(\mathcal{P}(T))$.

To prove that $\mathcal{D} \subset \mathcal{L}_r(\mathcal{P}(T))$, we define the $\sigma$-compact set $\mathcal{A} \subset \mathcal{P}(T)$ as in 3.3, the $\sigma$-compact metric space $\Phi$ as in 3.4, the Borel measurable mapping $i: \mathcal{A} \to \Phi$ as in (3.5), and the set $\Phi_S$ as in 3.4. As in the proof of 3.7, $\mathcal{D} = \mathcal{A} \cap i^{-1}[\Phi_S]$. If we knew that $\Phi_S \in \mathcal{L}_r(\Phi)$, it would follow from [K, §38, III, Proposition 5] that $i^{-1}[\Phi_S] \in \mathcal{L}_r(\mathcal{A})$. Since $\mathcal{A}$ is $\sigma$-compact and hence in $\mathcal{L}_r(\mathcal{P}(T))$, it would follow...
that \( D \in \mathcal{L}_r(\mathcal{P}(T)) \) [K, §38, III, Proposition 2]. Therefore it will be enough to show that \( \Phi_S \in \mathcal{L}_r(\Phi) \). Define function \( \theta_n : (2^\omega)^n \to \Phi \) as in 3.6. According to [K, §38, III, Proposition 1], \( S^n \in \mathcal{L}_r((2^\omega)^n) \). Because each open subset \( H \) of \( (2^\omega)^n \) also belongs to \( \mathcal{L}_r((2^\omega)^n) \) we see that \( H \cap G_n \cap S^n \in \mathcal{L}_r((2^\omega)^n) \) whenever \( H \) is open in \( (2^\omega)^n \). But \( \theta_n \) is known to be a local homeomorphism of \( G_n \cap S^n \) onto the separable metric space \( \Phi_S \cap \Phi_n \) so there is a sequence \( H_1, H_2, \ldots \) of subsets of \( G_n \) such that for each \( k \), \( \theta_n \) maps \( H_k \cap G_n \cap S^n \) homeomorphically onto a relatively open subset of \( \Phi_n \cap \Phi_S \) and such that \( \Phi_n \cap \Phi_S = \bigcup \{ \theta_n[H_k \cap G_n \cap S^n] \mid k \geq 1 \} \). Because \( H_k \cap G_n \cap S^n \in \mathcal{L}_r((2^\omega)^n) \) for each \( k \), it follows from [K, §38, VII, Theorem 1] that \( \theta_n[H_k \cap G_n \cap S^n] \in \mathcal{L}_r(\Phi) \). But then \( \Phi_n \cap \Phi_S \), being a countable union of members of \( \mathcal{L}_r(\Phi) \), also belongs to \( \mathcal{L}_r(\Phi) \). For the same reason, the set \( \Phi_S = \bigcup \{ \Phi_S \cap \Phi_n \mid n \geq 1 \} \) also belongs to \( \mathcal{L}_r(\Phi) \) as claimed.

Finally suppose \( S \notin \mathcal{L}_{r-1}(2^\omega) \). According to 2.1, there is a (relatively) closed subspace \( S^* \) of \( C_\tau(\Sigma) \) which is homeomorphic to \( S \). Then \( S^* = C_\tau(\Sigma) \cap D \), where \( D \) is some closed subset in \( \mathbb{R}^\Sigma \). If \( C_\tau(\Sigma) \in \mathcal{L}_{r-1}(\mathbb{R}^\Sigma) \), then \( S^* = C_\tau(\Sigma) \cap D \) would also belong to \( \mathcal{L}_{r-1}(\mathbb{R}^\Sigma) \). According to [K, §38, VII, Theorem 1], we would then have \( S \in \mathcal{L}_{r-1}(2^\omega) \) because \( S \) is homeomorphic to \( S^* \), which is impossible. \( \square \)

4.2 COROLLARY. For each \( n \geq 1 \) there is a countable regular space \( X_n \) such that \( C_\tau(X_n) \in \mathcal{L}_r(\mathbb{R}^{X_n}) - \mathcal{L}_{r-1}(\mathbb{R}^{X_n}) \) and there is a countable regular space \( Y \) such that \( C_\tau(Y) \notin \bigcup \{ \mathcal{L}_n(\mathbb{R}^Y) \mid n \geq 1 \} \).

PROOF. Fix \( n \). By [K, §38, VI, Theorem 1] there is a set \( S_n \subset 2^\omega \) having \( S_n \in \mathcal{L}_n(2^\omega) - \mathcal{L}_{n-1}(2^\omega) \). Let \( X_n = \Sigma S_n \). To obtain the space \( Y \), choose any \( S \subset 2^\omega \) with \( S \notin \bigcup \{ \mathcal{L}_n(2^\omega) \mid n \geq 1 \} \) [K, §38, VI, Remark 1] and let \( Y = \Sigma S \). Because \( C_\tau(Y) \) contains a closed subset homeomorphic to \( S \), \( C_\tau(Y) \notin \bigcup \{ \mathcal{L}_n(\mathbb{R}^Y) \mid n \geq 1 \} \). \( \square \)

5. Baire category and Baire Property subsets of \( \mathbb{R}^X \). For any space \( Z \), \( \mathcal{B}(Z) \) is the \( \sigma \)-algebra generated by the open sets and the first category subsets of \( Z \). Members of \( \mathcal{B}(Z) \) are called Baire Property subsets of \( Z \) [OX2, p. 19]. For a space \( X \) with a unique limit point (such as the spaces \( \Sigma S \) for \( S \subset 2^\omega \) constructed in §1) it is easy to characterize which function spaces \( C_\tau(X) \) belong to \( \mathcal{B}(\mathbb{R}^X) \).

5.1 THEOREM. Suppose \( X \) is a countable space with a unique limit point \( \infty \) and let \( p \) be the trace on \( X - \{ \infty \} \) of the neighborhood filter of \( \infty \). Then the following are equivalent:

(a) \( C_\tau(X) \) is a first category subset of \( \mathbb{R}^X \);

(b) \( C_\tau(X) \in \mathcal{B}(\mathbb{R}^X) \);

(c) there is an array

\[
\begin{array}{cccc}
A(1,1) & A(1,2) & A(1,3) & \cdots \\
A(2,1) & A(2,2) & A(2,3) & \cdots \\
A(3,1) & A(3,2) & A(3,3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]

satisfying

(i) each \( A(m,n) \) is a finite subset of \( X - \{ \infty \} \);

(ii) each row \( A(m,1), A(m,2), A(m,3), \ldots \) is a pairwise disjoint sequence;

(iii) for every sequence \( k(1), k(2), \ldots \) and every \( U \in p \), \( U \cap (\bigcup \{ A(m,k(m)) \mid m \geq 1 \}) \neq \emptyset \).
PROOF. The equivalence of (a) and (c) follows from [LM, Theorems 6.3 and 5.1] and obviously (a) implies (b). We prove that (b) implies (a). Suppose \( C_\pi(X) \in B\mathcal{P}(\mathbb{R}^X) \). To simplify notation, we will identify the countably many isolated points of \( X \) with elements of \( \omega \) and we will write \( X = \omega \cup \{\infty\} \). Define a function \( \nu: \mathbb{R}^X \to \mathbb{R}^\omega \times \mathbb{R} \) by the rule that \( \nu(f) = (f^*, f(\infty)) \), where \( f^* \in \mathbb{R}^\omega \) is given by \( f^*(n) = f(n) - f(\infty) \). Then \( \nu \) is a homeomorphism of \( \mathbb{R}^X \) onto \( \mathbb{R}^\omega \times \mathbb{R} \) and \( \nu(C_\pi(X)) = C_0 \times \mathbb{R} \), where \( C_0 = \{g \in \mathbb{R}^\omega | \text{ for each } \varepsilon > 0 \text{ there is a neighborhood } U \text{ of } \infty \text{ having } g[U \cap \omega] \subset ]-\varepsilon, \varepsilon[\} \). Since \( C_\pi(X) \in B\mathcal{P}(\mathbb{R}^X) \), \( C_0 \times \mathbb{R} \in B\mathcal{P}(\mathbb{R}^\omega \times \mathbb{R}) \).

It is easily seen that \( C_0 \) is a tailset in \( \mathbb{R}^\omega \), i.e. that if \( g \in C_0 \) and if the equality \( h(n) = g(n) \) holds except for finitely many values of \( n \), then \( h \in C_0 \). We now need a slight variation of a result due to Oxtoby [Ox1]; the proof is only trivially different from Oxtoby’s argument.

5.2 LEMMA. Let \( C \) be a tailset in \( \mathbb{R}^\omega \) and suppose that \( C \times \mathbb{R} \in B\mathcal{P}(\mathbb{R}^\omega \times \mathbb{R}) \). Then either \( C \times \mathbb{R} \) is a first category subset of \( \mathbb{R}^\omega \times \mathbb{R} \) or else \( C \times \mathbb{R} \) contains a dense \( G_\delta \)-subset of \( \mathbb{R}^\omega \times \mathbb{R} \).

Given 5.2, either \( C_0 \times \mathbb{R} \) is a first category subset of \( \mathbb{R}^\omega \times \mathbb{R} \), in which case \( C_\pi[X] \) is also a first category subset of \( \mathbb{R}^X \), or else \( C_0 \times \mathbb{R} \) contains a dense \( G_\delta \)-subset of \( \mathbb{R}^\omega \times \mathbb{R} \), in which case \( C_\pi[X] \) contains a dense \( G_\delta \) in \( \mathbb{R}^X \). But the latter situation occurs if and only if \( X \) is a discrete space [DGLvM, Theorem 1] so that \( C_\pi(X) \) must be a first category subset of \( \mathbb{R}^X \), as claimed. \( \square \)

5.3 REMARK. The reason for creating a variant of Oxtoby’s theorem as in 5.2 is that one cannot deduce \( C_0 \in B\mathcal{P}(\mathbb{R}^\omega) \) from \( C_0 \times \mathbb{R} \in B\mathcal{P}(\mathbb{R}^\omega \times \mathbb{R}) \).

5.4 COROLLARY. For each \( S < 2^\omega \), the function space \( C_\pi(\Sigma_S) \) is a first category subset of \( \mathbb{R}^{\Sigma_S} \).

PROOF. We define an array \( A(m, n) \) as follows using the tree \( T = \bigcup_{n=1}^{\infty} T_n \):

(i) \( A(1, n) = T_n \) for \( n \geq 1 \);
(ii) \( A(2, 1) = T_1 \cup T_2 \), \( A(2, 2) = T_3 \cup T_4 \), \( A(2, 3) = T_5 \cup T_6 \), \( \ldots \);
(iii) in general, \( A(m, n) = T_{(n-1)m+1} \cup \cdots \cup T_{nm} \).

Obviously each \( A(m, n) \) is finite and because the sets \( T_1, T_2, \ldots \) are pairwise disjoint, each row \( A(m, 1), A(m, 2), \ldots \) of the array is pairwise disjoint. Suppose \( k(1), k(2), \ldots \) is a sequence of positive integers and suppose \( U = T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F) \), where \( x_i \in S \) and \( F \) is a finite subset of \( T \). If \( \varnothing = U \cap (\bigcup \{A(m, k(m)) | m \geq 1\}) \), then \( \bigcup \{A(m, k(m)) | m \geq 1\} \subset B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_n} \cup F \). Observe that for a fixed level \( T_j \) of the tree \( T \), \( \text{card}(B_{x_j} \cap T_n) = 1 \) so that \( \text{card}(T_j \cap (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)) \leq n + \text{card}(F) \). Choose \( m > n + \text{card}(F) \). Then the set \( A(m, k(m)) \) contains a level \( T_j \) of \( T \) where \( \text{card}(T_j) \geq 2^m \) so that \( T_j \cap (B_{x_1} \cup \cdots \cup B_{x_n} \cup F) \) must have cardinality greater than \( n + \text{card}(F) \), contrary to our observation above. \( \square \)

In closing let us give one more example of a countable regular space \( X \) with a unique isolated point \( \infty \) which has a "bad" function space. Unlike the examples so far, \( C_\pi(X) \) is a second category subset of \( \mathbb{R}^X \).

5.5 EXAMPLE. Let \( p \) be a free ultrafilter on \( \omega \) and topologize the set \( X = \omega \cup \{\infty\} \) by isolating all points of \( \omega \) and by using all sets of the form \( \{\infty\} \cup U \), where \( U \in p \), as neighborhoods of \( \infty \). Then \( C_\pi(X) \) is a second category subset of \( \mathbb{R}^X \) and \( C_\pi(X) \notin L_1(\mathbb{R}^X) \cup L_2(\mathbb{R}^X) \).

PROOF. That \( C_\pi(X) \) is a second category subset of \( \mathbb{R}^X \) follows from the equivalence of (a) and (c) in 5.1 (cf. [LM, 5.1 and 6.3] for details). Suppose
\( C_\pi(X) \in \mathcal{L}_n(R^X) \), where \( n \in \{1, 2\} \). Define \( j : 2^\omega \to R^X \) by the rule that if \( f \in 2^\omega \) then \( j(f) = \hat{f} \in R^X \) where \( \hat{f} \) is given by
\[
\hat{f}(x) = \begin{cases} 
  f(x) & \text{if } x \in \omega, \\
  1 & \text{if } x = \infty.
\end{cases}
\]
Then \( j \) is continuous so that by [K, §38, III, Proposition 2], \( j^{-1}[C_\pi(X)] \in \mathcal{L}_n(2^\omega) \).
Hence \( j^{-1}[C_\pi(X)] \) is a measurable subset of \( 2^\omega \) (with respect to product measure \( \mu \)) because all analytic and co-analytic subsets of \( 2^\omega \) are measurable [L, p. 243, Proposition 3.24]. But \( j^{-1}[C_\pi(X)] = \{ x \in 2^\omega \mid \text{for some } U \in p, x(n) = 1 \text{ for each } n \in U \} \) so that \( j^{-1}[C_\pi(X)] \) is seen to be a tailset in \( 2^\omega \). Hence Kolmogorov’s “0-1 law” guarantees that \( \mu[j^{-1}[C_\pi(X)]] = 0 \) or \( \mu[j^{-1}[C_\pi(X)]] = 1 \) [Ox₂, p. 84].
However, consider the function \( J : 2^\omega \to 2^\omega \) given by \( J(f) = f \oplus 1 \), where \( \oplus \) denotes coordinatewise addition modulo 2, i.e., the usual group operation of \( 2^\omega \). Since \( \mu \) is translation invariant, \( J \) is a measure preserving transformation on \( 2^\omega \). Because \( p \) is an ultrafilter, \( J[j^{-1}[C_\pi(X)]] = 2^\omega - j^{-1}[C_\pi(X)] \) so that both \( \mu[j^{-1}[C_\pi(X)]] = 0 \) and \( \mu[j^{-1}[C_\pi(X)]] = 1 \) are impossible. Therefore \( C_\pi(X) \notin \mathcal{L}_0(R^X) \cup \mathcal{L}_1(R^X) \cup \mathcal{L}_2(R^X) \), as claimed. \( \square \)

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DEPARTMENT OF MATHEMATICS, MIAMI UNIVERSITY, OXFORD, OHIO 45056

VRIJE UNIVERSITEIT, AMSTERDAM, THE NETHERLANDS

MATHEMATICS INSTITUTE, UNIVERSITY OF WARSAW, WARSAW, POLAND