A DOWKER GROUP
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Abstract: We construct, in ZFC, a normal topological group, whose product with the circle group is not normal.

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0. Introduction. The purpose of this note is to give an example of a Dowker group: i.e. a normal topological group whose product with the circle group is not normal. We construct our example in ZFC alone, applying the B(X)-construction from [HavM] to a minor modification of M.E. Rudin’s Dowker space [Ru]. The paper is organized as follows: Section 1 contains some definitions and preliminaries. In Section 2 we repeat the construction of B(X) and give some generalizations of the results from [HavM] in order to be able to show that for the modified Dowker space X of Section 4 B(X) is a topological group. In Section 3 we describe the Rudin’s Dowker space R and show that under \( \text{\neg CH} \) B(R) is not a topological group.

Our construction shows once more the usefulness of Rudin’s example: In [DovM] R was used to construct an extremally disconnected Dowker space.
1. Definitions and preliminaries. For topology see [En], for set theory see [Ku].

1.0. Free Boolean groups. Recall that a Boolean group is a group in which every element has order at most 2. Such groups are always Abelian.

For a set $X$ we define the free Boolean group $B(X)$ of $X$ to be the unique (up to isomorphism) Boolean group containing $X$ such that every function from $X$ to a Boolean group extends to a unique homomorphism from $B(X)$ to that group. For example $B(X) = \{ x \in X^2 : |x(1)| \leq \omega \}$ as a subgroup of $X^2$. We shall write the elements of $B(X)$ as formal Boolean sums of elements of $X$. For every $n \in \mathbb{N}$ define $\varphi_n : X^n \to B(X)$ by $\varphi_n(x) = x_1 \ldots x_n$ and let $X_n = \varphi_n[X^n]$.

1.1. $\mathbb{P}_\kappa$-spaces. Let $X$ be a topological space. We call $X$ a $\mathbb{P}_\kappa$-space, where $\kappa$ is a cardinal, if whenever $\mathcal{U}$ is a collection of fewer than $\kappa$ open subsets of $X$, $\cap \mathcal{U}$ is open.

1.2. $k(X)$. For a space $X$ we let

$$k(X) = \min \{ \kappa \in \omega : \text{Every open cover of } X \text{ has a subcover of cardinality less than } \kappa \}.$$ 

Observe that $k(X) = \omega$ iff $X$ is compact. Thus $k(X)$ might be called the compactness number of $X$.

From now on we assume that all spaces are Hausdorff. Observe that if $X$ is a $\mathbb{P}_\omega$-space with $k(X) = \omega$ then $X$ is simply a compact space.

For regular $\kappa$, $\mathbb{P}_\kappa$-spaces with compactness number $\kappa$ behave like compact spaces.

1.3. Proposition. Let $X$ be a $\mathbb{P}_\kappa$-space with $k(X) = \kappa$, $\kappa$ regular. Then

(i) For all $n \in \mathbb{N}$ $X^n$ is a $\mathbb{P}_\kappa$-space and $k(X^n) = \kappa$. 

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(ii) If $f: X \to Y$ is continuous where $Y$ is a $\aleph$-space (and Hausdorff) then $f$ is closed.

(iii) $X$ is normal.

Proof: Imitate the proof for $\aleph = \omega$. Note that only (i) needs regularity of $\aleph$.

2. $B(X)$ revisited. We begin this section by repeating the construction of a topology for $B(X)$ given in [HavM].

2.0. Construction. Let $X$ be a topological space. We define a topology on $B(X)$ as follows:

First for each $n$ let $\tau_n$ be the quotient topology on $X^n$ determined by $X^n$ and $\sigma_n$. We then define

$$\tau = \{ U \subseteq B(X) : U \cap X_n \in \tau_n \text{ for all } n \},$$

i.e. $\tau$ is the topology on $B(X)$ determined by the spaces $\langle X_n, \tau_n \rangle$, $n \in \mathbb{N}$. Henceforth we will always assume that $B(X)$ carries this topology.

We now list some properties of $B(X)$, proved in [HavM]. Remember that all spaces are assumed to be Hausdorff.

2.1. Properties of $B(X)$.

(i) Both $E$ and $D$ are clopen in $B(X)$.

(ii) Translations are continuous, hence $B(X)$ is homogeneous.

(iii) For each $n$, $\langle X_n, \tau_n \rangle$ is a closed subspace of $\langle X_{n+2}, \tau_{n+2} \rangle$, and consequently each $\langle X_n, \tau_n \rangle$ is a closed subspace of $B(X)$.

(iii) For each $n$, if $X^n$ is normal then $X_n$ is normal and consequently if each $X^n$ is normal then $B(X)$ is normal. For in the latter case $B(X)$ is dominated by a countable collection of closed normal subspaces and hence normal.

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(iv) If $X$ is compact then $B(X)$ is a topological group.

(v) If for each $n \in \mathbb{N}$ $X^n$ is normal and $\beta(X^n) = (\beta X)^n$ then $B(X)$ is a subspace of $B(\beta X)$ and hence a topological group.

We shall need some slight generalizations of 2.1 (iv), (v), in order to be able to show that for the space $X$ from Section 4, $B(X)$ is a topological group. The proofs are almost identical to the ones in [HavM], but for the readers' convenience we shall give rough sketches. First we generalize 2.1 (iv).

2.2. Theorem. Let $X$ be a $P_{\omega}$-space with $k(X) = \omega$, $\omega$ a regular cardinal. Then $B(X)$ is a topological group.

Proof. The case $\omega = \omega$ is covered by 2.1 (iv), also $B(X)$ is Boolean, so it suffices to show that the addition is continuous.

We assume that $\omega > \omega$.

As a quotient of a $P_{\omega}$-space each $X_n$ is a $P_{\omega}$-space.

From this it follows that $B(X)$ - and hence $B(X) \times B(X)$ - is a $P_{\omega}$-space, too.

Because $\omega > \omega$, the sequence $\{X_n \times X_n\}_{n \in \mathbb{N}}$ dominates the space $B(X) \times B(X)$.

Thus, it suffices to show that for every $n \in \mathbb{N}$ $*:X_n \times X_n \rightarrow X_{2n}$ is continuous.

By 1.3(iii) and 2.1(iii) $X^n$ and $X_n$ are normal, in particular $X_n$ is Hausdorff. So by 1.3(ii) $\varphi_n \times \varphi_n:X^n \times X^n \rightarrow X_n \times X_n$ is closed.

But now if $F \subseteq X_{2n}$ is closed then $\varphi_n[F] = (\varphi_n \times \varphi_n)[h:F_{2n} \rightarrow X_{2n}]$ is closed, where $h:X^n \times X^n \rightarrow X_{2n}$ is the obvious homomorphism.

Next we generalize 2.1(v).

2.3. Lemma. Let $Y$ be a dense subspace of $X$ and $n \in \mathbb{N}$. Assume that $Y_n$ is completely regular and $Y^n$ is $C^n$-embedded in $X^n$.
Then $Y_n$ is a $C^*$-embedded subspace of $X_n$.

Proof. Consider the following diagram:

\[ \begin{array}{c}
Y_n \\
\downarrow \phi_n^X \\
Y_n \\
\downarrow \phi_n^Y \\
X_n \\
\downarrow \phi^X \\
X_n \\
\end{array} \]

where $i$ and $j$ are the inclusion maps.

$\phi_n^X \circ i$ is continuous, $\phi_n^X \circ j = j \circ \phi_n^Y$ and $\phi_n^Y$ is quotient, so $j$ is continuous.

Let $f: Y_n \rightarrow [0,1]$ be continuous. We shall find a continuous $g: X_n \rightarrow [0,1]$ with $g \circ j = f$. Let $\tilde{f} = f \circ \phi_n^Y$ and let $\tilde{g}: X^n \rightarrow [0,1]$ be the (unique) extension of $\tilde{f}$.

From the fact that $\tilde{f}$ is constant on the fibers of $\phi_n^Y$, it is easy to deduce that $\tilde{g}$ is constant on the fibers of $\phi_n^X$. Thus, $\tilde{g}$ induces a function $g: X_n \rightarrow [0,1]$ with $g \circ \phi_n^X = \tilde{g}$ and $g$ is continuous because $\tilde{g}$ is continuous and $\phi_n^X$ is quotient.

These two facts plus the complete regularity of $Y_n$ establish that $Y_n$ is a $C^*$-embedded subspace of $X_n$.

2.4. Theorem. Let $Y$ be a dense subspace of $X$ such that $B(Y)$ is completely regular and $Y^n$ is $C^*$-embedded in $X^n$ for all $n \in \mathbb{N}$.

Then $B(Y)$ is a $C^*$-embedded subspace of $B(X)$.

Proof.

If $U \subseteq B(X)$ is open then for each $n \in \mathbb{N}$ $U \cap B(Y) \cap Y_n = U \cap Y_n$ is open in $Y_n$, so $U \cap B(Y)$ is open in $B(Y)$.

If $f: B(Y) \rightarrow [0,1]$ is continuous, then for each $n \in \mathbb{N}$ we obtain a (unique) extension $g_n: X_n \rightarrow [0,1]$ of $f \upharpoonright Y_n$. It is easy to check that the $g_n$'s are compatible and that $g = \bigcup_{n \in \mathbb{N}} g_n$ is a continuous extension of $f$. 

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2.5. Corollary. If X and Y are as in 2.4, then B(Y) is a topological group if B(X) is.

3. Dowker spaces. We describe Rudin's Dowker space and give some variations.

3.0. Construction. Let $\kappa_0$ be a cardinal and for $n \in \mathbb{N}$ let $\kappa_n$ be the $n$th successor of $\kappa_0$. Let $P = \prod_{n \in \mathbb{N}} \kappa_n + 1$ i.e. the box product (see e.g. [Wi]) of the ordinal spaces $\kappa_1 + 1, \kappa_2 + 1, \ldots$. Let $\mathcal{X} = \{ f \in P : \forall n \in \mathbb{N} \text{ cf}(f(n)) > \kappa_0 \}$ and $X = \{ f \in \mathcal{X} : \exists i \in \mathbb{N} \forall n \in \mathbb{N} \text{ cf}(f(n)) \not\in \kappa_i \}.$

Then X is always a Dowker space. We shall briefly indicate why and refer to [Ru] for full proofs.

3.1. X is not countably paracompact ([Ru, II]). For $n \in \mathbb{N}$ let $D_n = \{ f \in X : \exists i \geq n f(i) = \kappa_i \}$. Then $\{ D_n : n \in \mathbb{N} \}$ witnesses that X is not countably paracompact.

3.2. X is dense in $X'$. For A and B are closed and disjoint in X then their closures are disjoint in $X'$ ([Ru] Lemmas 5 and 6). Lemma 5 says that $X'$ is a $P_{\omega_1}$-space and Lemma 6 establishes that $\overline{A} \cap B_n = \emptyset$ for all $n$ where $A_n = \{ f \in A : \forall i \geq n \text{ cf}(f(i)) \not\in \kappa_n \}$ (closures in $X'$).

In Section 4 we shall reprove that $X'$ is paracompact, thereby establishing (collectionwise) normality of X.

For the rest of this section we let $\kappa_0 = \omega_0$ so that $\kappa_1 = \omega_1$ for $i \in \mathbb{N}$. Moreover we shall call this Dowker space $R$.

We shall show that if $2^{\omega_1} \not\geq \omega_2$ then $B(R)$ is not a topological group.

3.4. Let $H$ be a topological group which is also a $P_{\omega_1}$-space.
then $H$ has a local base at the identity consisting of open subgroups.

For let $U_0 \ni e$ be open. Inductively find open $U_n \ni e$ for $n \in \mathbb{N}$ such that always $U_n = U_n^{-1}$ and $U_{n+1} \subseteq U_n$. Then $\bigcap_{n \in \mathbb{N}} U_n$ is an open subgroup contained in $U_0$.

3.5. Let $G$ be an open subgroup of $B(R)$. For $x \in R$ let $G_x = \{ y : x + y \in G \}$, then $\{ G_x : x \in R \}$ is an open partition of $R$. Note that $G_x$ is the intersection of $R$ and the coset $x + G$.

3.6. Let $f \in P$ be such that for all $n \in \mathbb{N}$ $0 < f(n) < \omega_n$ and $f(n) = f(n+1)$ and $\sup_{n \in \mathbb{N}} f(n) = \omega_\omega$.

For $A \in [\mathbb{N}]^{\omega}$ let $C_A = \{ h : \forall x A \leftrightarrow h(x) \notin f(n) \}$. Then $\mathcal{C} = \{ C_A : A \in [\mathbb{N}]^{\omega} \}$ is a clopen partition of $R$ of size $2^{\omega}$.

For each $A$ find $x_{A,1}, x_{A,2} \in C_A$ such that
- for some $n \in \mathbb{N}$ $\text{cf}(x_{A,1}(n)) = \omega_1$ and $x_{A,1}(n)$ is not isolated
  in \( \{ \alpha < \omega_n : \text{cf}(\alpha) > \omega_0 \} \)
- for some $n \in \mathbb{N}$ $\text{cf}(x_{A,2}(n)) = \omega_2$.

Now using $2^{\omega} \times \omega_2$ we extract from $\mathcal{C}$ a clopen partition $\mathcal{V}_\omega$:
\( \alpha \in \omega_2 \} \) of $R$ together with points $\{ x_\alpha : \alpha \in \omega_2 \}$ such that
(i) $x_\alpha \in V_\alpha$ for each $\alpha$.

(ii) If $\alpha \in \omega_1$ then there is a decreasing sequence $\{ C_{\alpha,\beta} : \beta \in \omega_2 \}$ of clopen sets with $x_\alpha \in \bigcap_{\beta \in \omega_2} C_{\alpha,\beta}$.

(iii) if $\alpha \in \omega_2 \setminus \omega_1$, a similar sequence $\{ C_{\alpha,\beta} : \beta \in \omega_1 \}$ of length $\omega_1$.

3.7. For $\alpha \in \omega_2$ define $\mathcal{D}_\alpha$ as follows:

if $\alpha \in \omega_1$ \( \mathcal{D}_\alpha = \{ V_\beta : \beta \in \omega_1 \wedge \beta \neq \alpha \} \cup \{ C_{\alpha,\tau} : \tau \in \omega_1 \wedge \tau \neq \omega_1 \wedge \tau \} \cup \{ V_\gamma \setminus C_{\alpha,\tau} : \tau \in \omega_1 \wedge \tau \} \)

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If $\alpha \in \omega_1 \setminus \omega_1$, $\Delta_\alpha = \{ V_\beta : \beta \in \omega_2 \setminus \omega_1 \cap \beta + \omega \} \cup$ 
\( \{ C_{\alpha \beta} : \gamma \in \omega_1 \} \cup \{ V_\gamma \setminus C_{\alpha \beta} : \gamma \in \omega_1 \} \).

For each $\alpha \in \omega_2$, $\Delta_\alpha \cup \{ V_\alpha \}$ is a clopen partition of $R$.

3.8. We define an open set $O \subseteq X^4$ as follows:

$$O = \bigcup_{\alpha \in \omega_3} V^4_\alpha \cup \bigcup_{\lambda \in \omega_2} \bigcup_{\gamma \in \omega_1} U_\lambda \Delta_\lambda \cup \bigcup_{\sigma \in S_4} \sigma [V^2_\alpha \setminus W]$$

($S_4$ acts on $X^4$ in the obvious way $\sigma (x_1, \ldots , x_4) = (x_{\sigma (1)}, \ldots , x_{\sigma (4)})$).

Then $O = \Theta_4 \cup \{ g_4 [0] \}$ so that $g_4 [0]$ is a neighborhood of $0$ in $X_4$ (the verification is straightforward).

3.9. Now suppose that $G$ is an open subgroup of $B(B)$ such that $G \cap X_4 \subseteq g_4 [0]$; we shall show that this gives a contradiction.

The partition $\{ G_x : x \in X \}$ has the following property:

if $\{ a, b, c, d \} \cap G_x$ has 0, 2 or 4 elements for each $x \in X$ then

$a + b + c + d \in G$.

Any partition refining $\{ G_x : x \in X \}$ also has this property, so $\mathcal{W}$, the common refinement of $\{ G_x : x \in X \}$ and $\{ V_\alpha : \alpha \in \omega_2 \}$ also has this property.

Fix for each $\alpha \in \omega_2$, $V_\alpha \in \mathcal{W}$ with $x_\alpha \in V_\alpha$, then $W_\alpha \cap V_\alpha$ of course.

For each $\alpha \in \omega_2$ let

$$\beta_\alpha = \min \{ \beta : \beta \in \omega_2 \setminus \omega_1 \}. $$

Find $\gamma_0 \in \omega_2 \setminus \omega_1$, $\gamma_1 \in \omega_1$ and $S \subseteq \omega_2 \setminus \omega_1$ unbounded such that

for $\alpha \in \omega_1$

$$\beta_\alpha = \gamma_0 \quad \text{and}$$

for $\alpha \in S$

$$\beta_\alpha = \gamma_1 .$$

Now pick $\sigma_2 \in S$ and $\tau_0 \in S_4$ and pick $y_1 \in W_{\sigma_2} \cap C_{\tau_0}$ and

$y_2 \in W_{\tau_0} \setminus C_{\tau_0}$.  \( \tau_0 \in \tau_1 . \)

Consider $F = \{ x_\alpha, y_1, y_2 \}$. Then $x_\alpha, y_1, y_2 \in G$ because $| F \cap W_{\alpha} | = 1$ and $W_{\alpha} = 0$ and 806
\[ F \cap W = \emptyset, W = W_{i_{1}i_{2}}. \text{ On the other hand } x_{i_{1}} + y_{1} + x_{i_{2}} + y_{2} \notin \phi \sigma_{4}(L) \text{ because } (x = (x_{i_{1}}, y_{1}, x_{i_{2}}, y_{2})): \]

- for no \( \sigma \in V_{\infty} \) so \( x \notin \bigcup_{\sigma \in V_{\infty}} V_{\infty}^{4} \)

- if \( x \in \sigma \{V_{\infty}^{2} \times V_{\infty}^{2}\} \) for some \( \sigma \in V_{\infty} \) then \( F \cap V_{\infty} = \emptyset \) so \( \alpha = \gamma_{1} \) or \( \alpha = \gamma_{2} \). If \( \alpha = \gamma_{1} \), then, since \( \langle x_{i_{1}}, y_{2} \rangle \in V_{\infty} \), either \( V = V_{i_{1}} \) or \( V = V_{i_{2}} \), but both are impossible since \( x_{i_{1}} \notin C_{\gamma_{1}}, V_{\infty} \) or \( x_{i_{2}} \notin C_{\gamma_{2}}, V_{\infty} \), thus \( \alpha = \gamma_{2} \) is impossible.

Thus, combining 3.6 and 3.9, we find that \( B(R) \) is not a topological group, assume \( 2 \omega \geq \omega_{2} \). This leaves open what will happen if \( 2 \omega = \omega_{1} \).

3.10. Question. Is \( B(R) \) a topological group under CH ?

4. A good Dowker space. In this section we let \( \omega_{0} = 2^{\omega} \) and we let \( X \) be the Dowker space constructed in 3.0. We shall show that \( B(X) \) is a topological group, and in fact a Dowker group.

To begin we quote from [Ha] the following fact

4.0. For each \( n \in \mathbb{N} \) \( X' \) is homeomorphic with \( (X')^{n} \) and the homeomorphism can be chosen to map \( X \) onto \( X^{n} \).

Furthermore we need the following

4.1. \( X' \) is paracompact and \( k(X') = \omega_{1} \)

Proof. We fix some notation: for \( f, g \in \mathcal{P} \) we say \( f \leq g \) iff \( f(n) \leq g(n) \) for all \( n \) and \( f < g \) iff \( f(n) < g(n) \) for all \( n \). For \( f, g \in \mathcal{P} \) with \( f \leq g \) we put

\[ U_{f,g} = X' \cap \bigcap_{n \in \mathbb{N}} (f(n), g(n)] = \{ h \in X' : f < h \leq g \} \]

For \( U = U_{f,g} \) put \( t_{U}(n) = \sup \{ h(n) : h \in U \} (n \in \mathbb{N}) \). Then \( U_{f,g} \cap X = U_{f,t_{U}} \cap X' \) and \( t_{U}(n) \) is always a limit ordinal.

Let \( \mathcal{U} \) be an open cover of \( X' \). We find a disjoint open refinement \( \mathcal{U} \) of \( \mathcal{U} \) of size \( \leq 2^{\omega} = \omega_{1} \). We define a sequence

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Let \( \mathcal{U}_\alpha \subseteq \mathcal{F}_\alpha \) of disjoint basic open covers of \( X' \) such that

(i) \( \alpha \in \beta \in \omega_1 \rightarrow \mathcal{U}_\beta \text{ refines } \mathcal{U}_\alpha \)

(ii) \( \alpha \in \omega_1 \rightarrow |\mathcal{U}_\alpha| \leq 2^\omega \)

(iii) \( \alpha \in \omega_1 \land U \in \mathcal{U}_\alpha \rightarrow \{ v \in \mathcal{U}_{\alpha + \beta} : \forall U \subseteq U \} = \{ U \} \text{ iff } U \subseteq \emptyset \) for some \( D \in \mathcal{D} \).

Let \( \mathcal{U}_0 = \{ x \} \).

For \( x \in X' \) and \( \alpha \in \omega_1 \), \( U_{x, \alpha} \) is always the unique element of \( \mathcal{U}_\alpha \) containing \( x \). If \( \alpha \) is a limit, put \( U_{x, \alpha} = \bigcap \{ U_{x, \beta} : \beta \in \alpha \} \) and \( \mathcal{U}_\alpha = \{ U_{x, \alpha} : x \in X' \} \). If \( \alpha \) is found make \( \mathcal{U}_\alpha \) as follows.

Let \( U \in \mathcal{U}_\alpha \). If \( U \subseteq \emptyset \) for some \( D \in \mathcal{D} \), put \( S(U) = \{ U \} \). Otherwise consider two cases.

a) For some \( \mu = \text{cf } \langle t_u(n) : n \in \omega \rangle \in 2^\omega \) (i.e. \( t_u \subseteq X' \)). Let \( \langle \lambda_\xi : \xi < \mu \rangle \) be a strictly increasing, continuous and cofinal sequence in \( t_u(n) \) with \( \lambda_0 = 0 \) and \( \text{cf } (\lambda_\xi) < 2^\omega \) for all \( \xi \).

Put \( U_\xi = \{ f \in U : \lambda_\xi < f(n) < \lambda_{\xi + 1} \} \) (\( \xi < \mu \)) and let \( S(U) = \{ U_\xi : \xi < \mu \} \).

b) For all \( n \) \( \text{cf } \langle t_u(n) : n \in \omega \rangle \) \( \in 2^\omega \) (i.e. \( t_u \subseteq X' \)); pick \( 0 \in \mathcal{D} \) with \( t_u \subseteq 0 \) and \( f < t_u \) such that \( U_f, t_u \subseteq 0 \). For \( A \subseteq \mathbb{N} \) let \( U_A = \{ h \in U : n \in A \rightarrow h(n) < t_u(n) \} \) and set \( S(U) = \{ U_A : A \subseteq \mathbb{N} \} \).

Now let \( \mathcal{U}_{\alpha + 1} = \{ S(U) : U \in \mathcal{U}_\alpha \} \). It follows that always \( |S(U)| \leq 2^\omega \) and hence inductively that \( |\mathcal{U}_\alpha| \leq 2^\omega \) for \( \alpha \in \omega_1 \).

Let \( \mathcal{U} = \{ U \in \mathcal{U}_\alpha : S(U) = \{ U \} \} \). Then, as in [Ru], \( \mathcal{U} \) is a disjoint open refinement of \( \mathcal{D} \) and by construction \( |\mathcal{U}| \leq 2^\omega \).

The above argument is from [Ru] but we included it because we need to know that the refinement is not too big.

We now collect everything together in.

4.2. Theorem. \( B(X) \) is a Dowker group.
Proof. (i) $X = X_1$ is a closed subspace of $B(X)$, so $B(X)$ is not countably paracompact.

(iii) From 3.3, 4.0 and 4.1 it follows that for all $n \geq X^n$ is normal and $C^*$-embedded in $(X')^n$, hence $B(X)$ is normal by 2.1.

(iii) $X'$ is a $P_{\aleph_1}$-space and $k(X') = \aleph_1$ hence $B(X')$ is a topological group.

(iv) By 2.5 $B(X)$ is a topological group.

4.3. Remark. Actually, the method of Section 3 and this section yield the following result:

If $X$ is the space constructed in 3.0 then

(i) if $2^\omega = \aleph_0$ then $B(X)$ is a topological group,

(ii) if $2^\omega \geq \aleph_2$ then $B(X)$ is not a topological group.

This leaves open a generalization of the question 3.10:

Is $B(X)$ a topological group if $2^\omega = \aleph_1$?

If we specialize by setting $\aleph_0 = \omega_1$ then we obtain a space $X$ for which $B(X)$ is a topological group if $2^\omega = \omega_1$, not a topological group if $2^\omega \geq \omega_2$ and maybe (not) a topological group if $2^\omega = \omega_2$.

References


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