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FUNCTION SPACES OF LOW BOREL COMPLEXITY

J. DIJKSTRA, T. GRILLIOT, D. LUTZER[1] AND J. van MILL[2]

ABSTRACT. In this paper we investigate situations in which the space $C_\pi(X)$ of continuous, real-valued functions on X is a Borel subset of the product space $\mathbf{R}^X$. We show that for completely regular, nondiscrete spaces, $C_\pi(X)$ cannot be a G_δ, an F_σ, or a $G_{\delta\sigma}$ subset of $\mathbf{R}^X$, but it can be an $F_{\sigma\delta}$ or $G_{\delta\sigma\delta}$ subset.

I. Introduction Let $C_\pi(X)$ be the set of continuous real-valued functions on a topological space X, and topologize $C_\pi(X)$ as a subspace of the full product $\mathbf{R}^X$. Assuming that there are enough continuous functions on X to separate points, $C_\pi(X)$ is a dense subspace of $\mathbf{R}^X$. In this paper we examine situations in which $C_\pi(X)$ is a Borel subset of $\mathbf{R}^X$ and occurs very early in the Borel hierarchy of $\mathbf{R}^X$. We prove that if X is completely regular, $C_\pi(X)$ cannot be a G_δ, $G_{\delta\sigma}$ or F_σ subset of $\mathbf{R}^X$ unless X is discrete. Examples show that $C_\pi(X)$ can be a $G_{\delta\sigma\delta}$ or $F_{\sigma\delta}$ subset of $\mathbf{R}^X$. In another paper **[LMP]** it will be shown that $C_\pi(X)$ can have arbitrarily high Borel or projective class in $\mathbf{R}^X$ and that $C_\pi(X)$ may be entirely outside of the projective hierarchy even when X is countable.

In what follows, no separation axioms are assumed unless specifically stated, except that completely regular spaces are assumed to be T_1. However, this lack of separation is partially an illusion, since all of our main results (Theorems 1, 6, and 7) deal with situations where $C_\pi(X)$ is dense in $\mathbf{R}^X$, and that is known to be equivalent to the separation axiom "given distinct points x and y of X, some continuous f: $X \to \mathbf{R}$ has $f(x) \neq f(y)$." (Such a space is said to be *completely Hausdorff*.) Examples 10 and 11 show the crucial role played by this separation axiom in our paper.

For any set S, both $\overline{S}$ and $\mathrm{cl}(S)$ will be used to denote the closure of S.

II. Certain discrete spaces. Our first result was originally obtained in **[LM]** using a complicated proof. We begin by presenting an elementary proof which has the advantage (over the proof in **[LM]**) that no separation axioms are assumed.

1. THEOREM. *For any space X, if $C_\pi(X)$ contains a dense G_δ subset of $\mathbf{R}^X$ then X is discrete.*

PROOF. Suppose X is not discrete. Then there is a function $g \in \mathbf{R}^X - C_\pi(X)$. Define Φ: $\mathbf{R}^X \to \mathbf{R}^X$ by $\Phi(f) = f + g$. Then Φ is a homeomorphism of $\mathbf{R}^X$ onto itself

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and $\Phi[C_\pi(X)] \subset \mathbf{R}^X - C_\pi(X)$. Thus $\mathbf{R}^X - C_\pi(X)$ also contains a dense G_δ subset of $\mathbf{R}^X$, which is impossible because $\mathbf{R}^X$ is a Baire space **[dG]** and in a Baire space any pair of dense G_δ sets has nonvoid intersection. □

We next examine the situation where $C_\pi(X)$ is a $G_{\delta\sigma}$ subset of $\mathbf{R}^X$. For completely Hausdorff spaces this also occurs only when X is discrete, but the proof is surprisingly complicated compared to the proof of Theorem 1.

2. PROPOSITION. *The following properties of a completely regular space X are equivalent*:

(a) *given any collection $\{(D_\alpha, C_\alpha) | \alpha \in A\}$ where $D_\alpha \subset C_\alpha$, D_α is a zero set, C_α is a cozero set and $C_\alpha \cap C_\beta = \varnothing$ whenever $\alpha \neq \beta$, the set $\bigcup\{D_\alpha | \alpha \in A\}$ is a zero set*;

(b) *the union of any disjoint collection of zero sets is again a zero set*;

(c) *X is discrete.*

PROOF. Obviously (c) $\Rightarrow$ (b) $\Rightarrow$ (a). To prove the converses, we need to know that (a) implies

$$(*) \text{ Each zero set in } X \text{ is a cozero set in } X.$$

Assume (a) holds and let C be a cozero set in X. Then there is a continuous f: $X \to [0,1]$ having $C = f^{-1}[(0,1]]$. Define subsets E_n and E_n' of $(0,1]$ as follows:

$$E_n = [2^{-(2n-1)}, 2^{-(2n-2)}], \qquad E_n' = [2^{-2n}, 2^{-(2n-1)}].$$

Then

$$(0,1] = \bigcup\{E_n \cup E_n' | n \geqslant 1\}.$$

Let $D_n = f^{-1}[E_n]$ and $C_n = f^{-1}[(3/2^{2n+1}, 3/2^{2n-1})]$. Then (a) may be applied to the collection $\{(D_n, C_n) | n \geqslant 1\}$ to show that $S = \bigcup\{f^{-1}[E_n] | n \geqslant 1\}$ is a zero set. Analogously, $T = \bigcup\{f^{-1}[E_n'] | n \geqslant 1\}$ is also a zero set. But then so is $S \cup T = f^{-1}[(0,1]]$, i.e., each cozero subset of X is a zero set. Assertion $(*)$ follows by taking complements.

To prove (b), given any disjoint family $\{D_\alpha | \alpha \in A\}$ of zero sets, we let $C_\alpha = D_\alpha$. By $(*)$, each C_α is a cozero set so that we may apply (a) to the collection $\{(D_\alpha, C_\alpha) | \alpha \in A\}$ to conclude that $\bigcup\{D_\alpha | \alpha \in A\}$ is a zero set, as required.

Finally, assume (b) and let $p \in X$. Suppose $\{p\}$ is not open. Let $\mathscr{D}$ be a maximal disjoint collection of zero sets, each contained in $X - \{p\}$. Let W be any neighborhood of p. Because $\{p\} \neq W$, if $W \cap \bigcup\mathscr{D} = \varnothing$, then we could find a zero set $Z \subset W - \{p\}$ contrary to maximality of $\mathscr{D}$. Hence $\bigcup\mathscr{D}$ is dense in X. But, from (b), $\bigcup\mathscr{D}$ is a zero set so $\bigcup\mathscr{D}$ is both closed and dense in X contrary to $\bigcup\mathscr{D} \subset X - \{p\}$. Hence X must be discrete. □

3. REMARK. In general, assertion $(*)$ in the proof of Lemma 2 is not equivalent to discreteness of X: consider $X = [0, \omega_1]$, in which each countable ordinal is isolated, while ω_1 has its usual neighborhoods. However, if each point of X is a limit point of a countable set, or if each point of X is a G_δ in X, then $(*)$ does imply that X is discrete.

In order to apply Proposition 2 to the case where $C_\pi(X)$ is a $G_{\delta\sigma}$ subset of $\mathbf{R}^X$, we need two rather technical lemmas. In what follows, given $f \in \mathbf{R}^X$ and $B \subset X$,

$$E(f, B) = \{ g \in \mathbf{R}^X | g(x) = f(x) \text{ for each } x \in B \}.$$

4. LEMMA. *Let A be any set and let H be a G_δ-subset of $\mathbf{R}^A$. Suppose B_0 is a fixed finite subset of A. Let e: $B_0 \to \mathbf{R}$ be a fixed function and suppose δ: $A \to [0, \infty)$ is given. Let*

$$P = \{ g \in \mathbf{R}^A | 0 \leqslant g(x) \leqslant \delta(x) \text{ for } x \in A - B_0 \} \cap E(e, B_0).$$

Then either

(a) *there is a finite set $B \subset A - B_0$ and a function $h \in P$ having $P \cap E(h, B) \cap H = \varnothing$; or*

(b) *there are functions $g_0, g_1 \in H$ having $g_0(x) + g_1(x) = \delta(x)$ for each $x \in A - B_0$.*

PROOF. Write $H = \bigcap\{U_n | n \geqslant 1\}$ where $U_1 \supset U_2 \supset \cdots$ are open in $\mathbf{R}^A$. Suppose (a) fails. Then for each $h \in P$ and each finite $B \subset A - B_0$, $P \cap E(h, B) \cap H \neq \varnothing$.

Fix $f_1 \in P$ and let $B = \varnothing$. There is a function $\hat{f}_1 \in P \cap H \cap E(f_1, \varnothing)$. Because $\hat{f}_1 \in H \subset U_1$, there is a finite set $B_1 \supset B_0$ such that $E(\hat{f}_1, B_1) \subset U_1$.

Suppose $n \geqslant 1$ and we have constructed finite sets $B_1, \ldots, B_n$ and functions f_k, $\hat{f}_k$ for $1 \leqslant k \leqslant n$ satisfying:

(a) $B_0 \subset B_1 \subset \cdots \subset B_n$,

(b) if $k + 1 \leqslant n$ then

$$f_{k+1} = \begin{cases} e(x) & \text{if } x \in B_0, \\ \delta(x) - \hat{f}_k(x) & \text{if } x \in B_k - B_0, \\ 0 & \text{otherwise,} \end{cases}$$

(c) $f_k, \hat{f}_k \in P$ for $1 \leqslant k \leqslant n$,

(d) $\hat{f}_{k+1} \in P \cap H \cap E(f_{k+1}, B_k)$ if $k + 1 \leqslant n$,

(e) $E(\hat{f}_k, B_k) \subset U_k$ if $k \leqslant n$.

We obtain $f_{n+1}, \hat{f}_{n+1}$ and B_{n+1} as follows. Define

$$f_{n+1}(x) = \begin{cases} e(x) & \text{if } x \in B_0, \\ \delta(x) - \hat{f}_n(x) & \text{if } x \in B_n - B_0, \\ 0 & \text{otherwise.} \end{cases}$$

Because $\hat{f}_n \in P$, we know that $0 \leqslant \hat{f}_n(x) \leqslant \delta(x)$ for each $x \in A - B_0$. Hence $0 \leqslant \delta(x) - \hat{f}_n(x) \leqslant \delta(x)$ for each $x \in B_n - B_0$, so that $f_{n+1} \in P$. Since (a) fails, $P \cap H \cap E(f_{n+1}, B_n) \neq \varnothing$. Choose $\hat{f}_{n+1} \in P \cap H \cap E(f_{n+1}, B_n)$. Since $\hat{f}_{n+1} \in H \subset U_{n+1}$, there is a finite set $B_{n+1} \supset B_n$ such that $E(\hat{f}_{n+1}, B_{n+1}) \subset U_{n+1}$. Therefore $f_{n+1}, \hat{f}_{n+1}$ and B_{n+1} are defined, as required.

Fix n and $x \in B_n - B_0$. Then $\hat{f}_{n+1}(x) = f_{n+1}(x) = \delta(x) - \hat{f}_n(x)$, and $\hat{f}_{n+2}(x) = f_{n+2}(x) = \delta(x) - \hat{f}_{n+1}(x)$. Therefore, $\hat{f}_{n+2}(x) = \delta(x) - (\delta(x) - \hat{f}_n(x)) = \hat{f}_n(x)$. It follows that for $x \in B_{2n}$, $\hat{f}_{2k}(x) = \hat{f}_{2n}(x)$ for every $k \geqslant n$. Therefore we may define a function

$$g_0(x) = \begin{cases} \hat{f}_{2n}(x) & \text{if } x \in B_{2n} \text{ for some } n, \\ 0 & \text{if } x \in A - \bigcup_{n=1}^{\infty} B_{2n} = A - \bigcup_{n=0}^{\infty} B_n. \end{cases}$$

Fix n and $x \in B_{2n}$. Since $g_0(x) = \hat{f}_{2n}(x)$ we see that $g_0 \in E(\hat{f}_{2n}, B_{2n}) \subset U_{2n}$. Since $U_1 \supset U_2 \supset \cdots$ we have $g_0 \in \bigcap_{n=1}^{\infty} U_n = H$. Now define

$$g_1(x) = \begin{cases} \delta(x) - g_0(x) & \text{if } x \in A - B_0, \\ e(x) & \text{if } x \in B_0. \end{cases}$$

Fix n and $x \in B_{2n+1}$. If $x \in B_0$ then $g_1(x) = e(x) = \hat{f}_{2n+1}(x)$, so suppose $x \in B_{2n+1} - B_0$. Then $x \in B_{2n+2}$, so that $g_0(x) = \hat{f}_{2n+2}(x)$ and $\hat{f}_{2n+2}(x) = \delta(x) - \hat{f}_{2n+1}(x)$, because $x \in B_{2n+1} - B_0$. Thus $g_0(x) = \delta(x) - \hat{f}_{2n+1}(x)$ for each $x \in B_{2n+1} - B_0$, and so $\hat{f}_{2n+1}(x) = \delta(x) - g_0(x) = g_1(x)$. Therefore $g_1 \in E(\hat{f}_{2n+1}, B_{2n+1}) \subset U_{2n+1}$. Hence $g_1 \in \bigcap_1^{\infty} U_n = H$. By definition of g_1, if $x \in A - B_0$, $g_0(x) + g_1(x) = \delta(x)$, as required. □

5. LEMMA. *Let A be any set, and let G be a $G_{\delta\sigma}$ subset of $\mathbf{R}^A$ which is an additive subgroup of $\mathbf{R}^A$. Suppose G contains every function $w \in \mathbf{R}^A$ such that $\{x \in A \mid |w(x)| \geqslant \varepsilon\}$ is finite for each $\varepsilon > 0$. Then each bounded function from A to $\mathbf{R}$ belongs to G.*

PROOF. Since each bounded function from A to $\mathbf{R}$ is the difference of two nonnegative bounded functions, it will be enough to show that each bounded f: $A \to [0, \infty)$ belongs to G. Suppose f: $A \to [0, \infty)$ is bounded and does not belong to G. Write $G = \bigcup_1^{\infty} H_n$ where $H_1 \subset H_2 \subset \cdots$ and each H_n is a G_δ subset of $\mathbf{R}^A$.

Let $P_1 = \{g \in \mathbf{R}^A \mid 0 \leqslant g(x) \leqslant f(x) \text{ for all } x \in A\}$. We apply Lemma 4 to P_1 and H_1. If there are functions $g_0, g_1 \in H_1$ with $g_0 + g_1 = f$, it would follow that $g_0 + g_1 = f$ is also in G. Hence there is a finite set $A_1 \subset A$ and a function $h_1 \in P_1$ such that $P_1 \cap H_1 \cap E(h_1, A_1) = \varnothing$.

For induction hypothesis suppose $n \geqslant 1$ and that we have pairwise disjoint finite sets $A_1, \ldots, A_n$ and functions $h_1, \ldots, h_n$ satisfying:

(a) $h_{k+1}|_B = h_k|_B$ provided $k + 1 \leqslant n$ where $B = A_1 \cup \cdots \cup A_k$;

(b) $h_k \in P_k = \{g \in \mathbf{R}^A \mid 0 \leqslant g(x) \leqslant (1/k)f(x) \text{ for } x \in A - (A_1 \cup \cdots \cup A_{k-1})\} \cap (\bigcap_{j=1}^{k-1} E(h_j, A_j))$ provided $k \leqslant n$;

(c) $H_k \cap P_k \cap E(h_k, A_k) = \varnothing$ provided $k \leqslant n$.

To obtain h_{n+1} and A_{n+1} we proceed as follows. Let $B_0 = A_1 \cup \cdots \cup A_n$ and define e: $B_0 \to [0, \infty)$ by $e(x) = h_k(x)$ when $x \in A_k$. Let

$$P_{n+1} = \left\{g \in \mathbf{R}^A \mid 0 \leqslant g(x) \leqslant (n+1)^{-1} f(x) \text{ for } x \in A - B_0\right\} \cap E(e, B_0).$$

Apply Lemma 4 to P_{n+1} and H_{n+1}. If there are functions $g_0, g_1 \in H_{n+1} \subset G$ having $g_0(x) + g_1(x) = (n+1)^{-1} f(x)$ for each $x \in A - B_0$, define

$$d(x) = \begin{cases} f(x) - (n+1)(g_0(x) + g_1(x)) & \text{for } x \in B_0, \\ 0 \quad \text{if } x \in A - B_0. \end{cases}$$

Since $\{x \in A \mid d(x) \neq 0\}$ is finite, $d \in G$. Therefore $d + (n+1)(g_0 + g_1) \in G$. But for each $x \in A$, $d(x) + (n+1)(g_0(x) + g_1(x)) = f(x)$, contradicting $f \notin G$. Hence there is a function $h_{n+1} \in P_{n+1}$ and a finite set $A_{n+1} \subset A - B_0$ such that $\varnothing = H_{n+1} \cap P_{n+1} \cap E(h_{n+1}, A_{n+1})$. Since $h_{n+1} \in P_{n+1} \subset E(e, B_0)$, we have $h_{n+1}|_{A_k} = h_k|_{A_k}$ whenever $k \leqslant n$. Thus the induction continues.

The induction generates pairwise disjoint finite sets $A_1, A_2, \ldots$ and functions h_k such that if $m < n$ then $h_n|_{A_m} = h_m|_{A_m}$. Therefore the function

$$h(x) = \begin{cases} h_k(x) & \text{if } x \in A_k \text{ for some } k, \\ 0 & \text{if } x \in A - \bigcup\{A_k | k \geq 1\} \end{cases}$$

is well defined. We show that $h \in G$. Fix a number $M > 0$ with $|f(x)| \leq M$ for each x. Fix $\varepsilon > 0$ and n so large that $M/n < \varepsilon$. Suppose $x \in A_k$ for some $k > n$. Then $h(x) = h_k(x)$. Since h_k belongs to $\{g \in \mathbf{R}^A | 0 \leq g(y) \leq (1/k)f(y) \text{ for } y \in A - (A_1 \cup \cdots \cup A_{k-1})\}$, and since $A_k \cap (A_1 \cup \cdots \cup A_{k-1}) = \varnothing$, we have $0 \leq h_k(x) \leq (1/k)f(x) \leq (1/k)M < \varepsilon$. Therefore $\{x \in A | |h(x)| \geq \varepsilon\} \subset A_1 \cup \cdots \cup A_n$, which is finite. It follows that $h \in G$. But then $h \in H_n$ for some n, so that $h \in H_n \cap P_n \cap E(h_n, A_n)$, which is impossible by induction hypothesis. That contradiction completes the proof. □

We are now prepared to prove our main theorem.

6. THEOREM. *For any space X, if $C_\pi(X)$ is a dense $G_{\delta\sigma}$ subset of $\mathbf{R}^X$, then X is discrete.*

PROOF. Suppose Theorem 6 has been proved for all completely regular spaces, and suppose that $\mathscr{T}$ is a topology on X such that $C_\pi(X, \mathscr{T})$ is a dense $G_{\delta\sigma}$ subset of $\mathbf{R}^X$. Then $(X, \mathscr{T})$ is completely Hausdorff (cf. §1). Let $\mathscr{S}$ be the topology on X having the family $\{f^{-1}[U] | f \in C_\pi(X, \mathscr{T}) \text{ and } U \text{ is open in } \mathbf{R}\}$ as a subbase. Then $\mathscr{S} \subset \mathscr{T}$ and $(X, \mathscr{S})$ is completely regular and Hausdorff. Furthermore $C_\pi(X, \mathscr{T})$ and $C_\pi(X, \mathscr{S})$ are identical subsets of $\mathbf{R}^X$ so that $C_\pi(X, \mathscr{S})$ is a dense $G_{\delta\sigma}$ subset of $\mathbf{R}^X$. But then $(X, \mathscr{S})$ is a discrete space. Because $\mathscr{S} \subset \mathscr{T}$, so is $(X, \mathscr{T})$. Therefore it will be enough to prove Theorem 6 for an arbitrary completely regular space X.

According to Proposition 2, it will be enough to show that $\bigcup\{D_\alpha : \alpha \in A\}$ is a zero set provided D_α is a zero set contained in a cozero set C_α, where $\{C_\alpha | \alpha \in A\}$ is a disjoint family. Choose a continuous function $f_\alpha: X \to [0,1]$ having $f_\alpha^{-1}[\{1\}] = D_\alpha$ and $\mathrm{cl}_X(f_\alpha^{-1}[(0,1]]) \subset C_\alpha$. Define $h(x) = \Sigma\{f_\alpha(x) | \alpha \in A\}$. Since the sets C_α are pairwise disjoint, h is well defined and $h[X] \subset [0,1]$. Since $h^{-1}[\{1\}] = \bigcup\{D_\alpha | \alpha \in A\}$, it will be enough to prove that h is continuous.

Define a function $\Phi: \mathbf{R}^A \to \mathbf{R}^X$ by the rule that if $u \in \mathbf{R}^A$ then $\Phi(u) = \Sigma\{u(\alpha)f_\alpha : \alpha \in A\}$. (Because $\{C_\alpha | \alpha \in A\}$ is a disjoint collection, $\Phi(u) \in \mathbf{R}^X$.) Clearly Φ is one-to-one. In order to show that Φ is continuous, fix a finite set $F \subset X$ and a basic neighborhood $N(\Phi(u), F, \varepsilon) = \{g \in \mathbf{R}^X | |\Phi(u)(x) - g(x)| < \varepsilon \text{ for each } x \text{ in } F\}$ of $\Phi(u)$ in $\mathbf{R}^X$. Let $A_0 = \{\alpha \in A | C_\alpha \cap F \neq \varnothing\}$. Then A_0 is finite and the set

$$M(u, A_0, \varepsilon) = \left\{v \in \mathbf{R}^A \middle| |u(\alpha) - v(\alpha)| < \varepsilon \text{ for each } \alpha \in A_0\right\}$$

is a basic neighborhood of u in $\mathbf{R}^A$. Fix $v \in M(u, A_0, \varepsilon)$ and consider $|\Phi(u)(x) - \Phi(v)(x)|$, where $x \in F$. If $x \notin \bigcup\{C_\alpha | \alpha \in A\}$, then $f_\alpha(x) = 0$ for each $\alpha \in A$, so that $\Phi(u)(x) = 0 = \Phi(v)(x)$. If $x \in C_\beta$ for some $\beta \in A$, then $\beta \in A_0$ so that

$$|\Phi(u)(x) - \Phi(v)(x)| = |u(\beta)f_\beta(x) - v(\beta)f_\beta(x)| \leq |u(\beta) - v(\beta)| < \varepsilon$$

because $f_\beta[X] \subset [0,1]$. Hence, $\Phi[M(u, A_0, \varepsilon)] \subset N(\phi(u), F, \varepsilon)$, as required.

Since $C_\pi(X)$ is a $G_{\delta\sigma}$ subset of $\mathbf{R}^X$, it follows that the set $H = \Phi^{-1}[C_\pi(X)]$ is a $G_{\delta\sigma}$ subset of $\mathbf{R}^A$. Obviously, H is an additive subgroup of $\mathbf{R}^A$. Suppose $u \in \mathbf{R}^A$ has the property that for each $\varepsilon > 0$ the set $\{\alpha \in A \mid |u(\alpha)| \geqslant \varepsilon\}$ is finite. Consider the function $g = \Phi(u) \in \mathbf{R}^X$. To show that $g \in C_\pi(X)$, fix $x \in X$. If $x \in \bigcup\{C_\alpha \mid \alpha \in A\}$, there is a unique α with $x \in C_\alpha$. Since $g|_{C_\alpha} = u(\alpha)f_\alpha|_{C_\alpha}$ and C_α is a neighborhood of x, g is continuous at x. Hence, suppose $x \notin \bigcup\{C_\alpha \mid \alpha \in A\}$. Then $g(x) = 0$. Fix $\varepsilon > 0$ and consider the set $S = g^{-1}[(-\varepsilon, \varepsilon)]$. By assumption, $B = \{\alpha \in A \mid |u_\alpha| \geqslant \varepsilon\}$ is finite, say $B = \{\alpha_1, \dots, \alpha_n\}$. Let

$$W = X - \bigcup\left\{\operatorname{cl} f_{\alpha_i}^{-1}[(0,1]] \mid 1 \leqslant i \leqslant n\right\}.$$

Then W is open and, since $\operatorname{cl} f_{\alpha_i}^{-1}[(0,1]] \subset C_{\alpha_i}$, $x \in W$. Let $z \in W$. If $z \notin \bigcup\{C_\alpha \mid \alpha \in A\}$, then $g(z) = 0$, so $z \in S$. If $z \in \bigcup\{C_\alpha \mid \alpha \in A\}$, there is a unique $\beta \in A$ having $z \in C_\beta$ and then $g(z) = u(\beta)f_\beta(z)$. There are two cases. If $\beta \in B$, then $z \in C_\beta - \operatorname{cl}[f_\beta^{-1}[(0,1]]]$ so that $f_\beta(z) = 0$. Then $g(z) = 0$ and so $z \in S$. Finally, if $\beta \notin B$ then $|u(\beta)| < \varepsilon$ so that $|u(\beta)f_\beta(z)| \leqslant |u(\beta)| < \varepsilon$, because $f_\beta[X] \subset [0,1]$, so that $z \in S$. Therefore W is a neighborhood of x which is contained in $S = g^{-1}[(-\varepsilon, \varepsilon)]$. Hence g is continuous. Because g is continuous, $u \in \Phi^{-1}[C_\pi(X)] = H$. Now apply Lemma 5 to H, concluding that the constant function 1 belongs to H. But then $\Phi(1) \in C_\pi(X)$, i.e., the function $\Sigma\{f_\alpha \mid \alpha \in A\}$ is continuous, as required to prove Theorem 6. □

If X happens to be countable, then $\mathbf{R}^X$ is metrizable, so that each F_σ set is a $G_{\delta\sigma}$ set, showing that $C_\pi(X)$ is not an F_σ unless X is discrete. Different techniques show that the same conclusion holds even if X is not countable.

7. THEOREM. *For any space X, if $C_\pi(X)$ is a dense F_σ subspace of $\mathbf{R}^X$ then X is discrete.*

PROOF. As in the proof of Theorem 6, it will be enough to prove this result for an arbitrary completely regular space X.

Assume that X has a nonisolated point p and that $C_\pi(X) = \bigcup_{i=0}^\infty F_i$, where $F_0 = \varnothing$ and every F_i is closed in $\mathbf{R}^X$. We shall construct inductively a monotone sequence $f_0 \leqslant f_1 \leqslant f_2 \leqslant \cdots$ in $[0,1]^X$ and subsets $U_0 \supset \overline{U}_1 \supset U_1 \supset \overline{U}_2 \supset U_2 \supset \cdots$ of X with the properties:

(a) U_i is an open neighborhood of p in X,

(b) $f_i(p) = 1$,

(c) $f_i(U_i - \{p\}) = \{1 - 2^{-i}\}$,

(d) $f_i|_{X-\{p\}}$ is continuous,

(e) $f_{i+1}|_{X-U_i} = f_i|_{X-U_i}$, and

(f) every $f \in \mathbf{R}^X$ that coincides with f_i on $(X - U_i) \cup \{p\}$ is not an element of F_i.

Assume now that we have proved the existence of these sequences. Obviously, the pointwise limit of $(f_i)_{i=1}^\infty$ exists. Call this limit f and note that $f|_{X-U_i} = f_i|_{X-U_i}$ for every i and that $f(\bigcap_{i=0}^\infty U_i) = \{1\}$. This means, according to the induction hypothesis, that $f \notin \bigcup_{i=0}^\infty F_i$. We shall see, however, that f is continuous. If $x \notin \bigcap_{i=0}^\infty U_i$, then for some i, $x \notin \overline{U}_i$ and $f|_{X-\overline{U}_i} = f_i|_{X-\overline{U}_i}$. Since $f_i|_{X-\{p\}}$ is continuous, f is continuous at x. If $x \in \bigcap_{i=0}^\infty U_i$, then U_i is a neighborhood of x such that $f_i(U_i) \subset \{1 - 2^{-i}, 1\}$.

Consequently, $f_j(U_i) \subset [1 - 2^{-i}, 1]$ for every $j \geqslant i$ and hence $f(U_i) \subset [1 - 2^{-i}, 1]$. This yields the continuity of f at x. Hence we have $f \in C_\pi(X)$, which contradicts $f \notin \bigcup_{i=0}^{\infty} F_i$.

It remains to perform the induction. As basis step we put $f_0(p) = 1$, $f_0(x) = 0$ if $x \neq p$, and $U_0 = X$. Assume now that f_i and U_i have been determined. Let ϕ be a continuous function from X into $[0, 2^{-i}]$ such that $\phi(X - U_i) \subset \{0\}$ and $\phi(p) = 2^{-i}$. Define $f_{i+1}: X \to [0,1]$ by $f_{i+1}(p) = 1$ and

$$f_{i+1}(x) = f_i(x) + \min\{2^{-i-1}, \phi(x)\} \quad \text{if } x \neq p.$$

Note that $V = \phi^{-1}((2^{-i-1}, 2^{-i}])$ is a neighborhood of p with $f_{i+1}(V - \{p\}) = \{1 - 2^{-i-1}\}$. Since p is not isolated, this means that f_{i+1} is not continuous and hence not an element of F_{i+1}. Because $\mathbf{R}^X$ carries the product topology, there exists a finite set $A \subset X$ such that every $f \in \mathbf{R}^X$ that coincides with f_{i+1} on A is not in F_{i+1}. Put $U_{i+1} = (V - A) \cup \{p\}$. One easily verifies that the pair (f_{i+1}, U_{i+1}) satisfies the induction hypothesis. □

III. Examples. We begin by showing that the lowest Borel class to which $C_\pi(X)$ can belong for a nondiscrete completely regular space X is $F_{\sigma\delta} \cap G_{\delta\sigma\delta}$.

8. PROPOSITION. *If X is any countable metric space, then $C_\pi(X)$ is an $F_{\sigma\delta}$ subset of $\mathbf{R}^X$ (and hence a $G_{\delta\sigma\delta}$ subset, too).*

PROOF. Let ρ be a metric on X, and for $x \in X$ and $\varepsilon > 0$ let $B(x, \varepsilon) = \{y \in X | \rho(x, y) < \varepsilon\}$. Then the ε-δ definition of continuity shows that $C_\pi(X)$ is the set

$$\bigcap_{x \in X} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{g \in \mathbf{R}^X | g[B(x, 1/m)] \subset [g(x) - 1/n, g(x) + 1/n]\}.$$

Since each set $\{g \in \mathbf{R}^X | g[B(x,1/m)] \subset [g(x) - 1/n, g(x) + 1/n]\}$ is a closed subset of $\mathbf{R}^X$, $C_\pi(X)$ is an $F_{\sigma\delta}$ subset in $\mathbf{R}^X$, as claimed. □

Our next example shows that Proposition 8 does not characterize metrizability in countable spaces:

9. EXAMPLE. There is a countable, regular, nonmetrizable space X for which $C_\pi(X)$ is an $F_{\sigma\delta}$ subset of $\mathbf{R}^X$.

PROOF. Let $X_k = \{(1/n, 1/nk) | 1 \leqslant n < \infty\} \cup \{(0,0)\}$ for $k \geqslant 1$, and let $X = \bigcup\{X_k | 1 \leqslant k < \infty\}$. Each X_k is topologized in such a way that X_k becomes a convergent sequence with limit $(0,0)$. A subset $S \subset X$ is closed in X if and only if $S \cap X_k$ is closed in X_k for each k. Then X is nonmetrizable (it is not first-countable at $(0,0)$) and a function $f: X \to \mathbf{R}$ is continuous if and only if $f|_{X_k}: X_k \to \mathbf{R}$ is continuous for each $k \geqslant 1$. For $k \geqslant 1$ let $C_k = \{g \in \mathbf{R}^X | g|_{X_k}$ is continuous on $X_k\}$. The proof of Proposition 8 can be modified to show that each C_k is an $F_{\sigma\delta}$ subset of $\mathbf{R}^X$. Hence so is $C_\pi(X) = \bigcap_{k=1}^{\infty} C_k$. □

Our final examples show the importance of the complete Hausdorff property in Theorems 1, 6 and 7.

10. EXAMPLE. There is a nondiscrete regular T_1-space X such that $C_\pi(X)$ is a closed (and hence an F_σ) subset of $\mathbf{R}^X$.

PROOF. Let X be any regular T_1-space such that every continuous, real-valued function on X is constant [E, p. 161]. Then $C_\pi(X)$ is a closed subset of $\mathbf{R}^X$. □

11. EXAMPLE. There is a countable nondiscrete Hausdorff space such that $C_\pi(X)$ is a G_δ subset (and hence a $G_{\delta\sigma}$ subset) of $\mathbf{R}^X$.

PROOF. Let X be a countable, connected, Hausdorff space [W, p. 196]. Then every continuous, real-valued function on X is constant, so that $C_\pi(X)$ is a closed subset of the metrizable space $\mathbf{R}^X$. Hence $C_\pi(X)$ is a G_δ set in $\mathbf{R}^X$. □

BIBLIOGRAPHY

[E] R. Engelking, *General topology*, PWN, Warsaw, 1977.

[dG] J. de Groot, *Subcompactness and the Baire category theory*, Indag. Math. **25** (1963), 761–767.

[LMP] D. Lutzer, J. van Mill and R. Pol, *Descriptive complexity of function spaces*, Trans. Amer. Math. Soc. (to appear).

[LM] D. Lutzer and R. McCoy, *Category in function spaces*. I, Pacific J. Math. **90** (1980), 145–168.

[W] S. Willard, *General topology*, Addison-Wesley, Reading, Mass., 1970.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MIAMI UNIVERSITY, OXFORD, OHIO 45056 (Current address of T. Grilliot and D. Lutzer)

DEPARTMENT OF MATHEMATICS, VRIJE UNIVERSITEIT, AMSTERDAM, THE NETHERLANDS (Current address of Jan van Mill)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AMSTERDAM, AMSTERDAM, THE NETHERLANDS (Current address of J. Dijkstra)