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EQUALITY OF THE LEBESGUE AND THE INDUCTIVE DIMENSION FUNCTIONS FOR COMPACT SPACES WITH A UNIFORM CONVEXITY

BY

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If a compact space carries a uniform S_4-convexity with connected convex sets then the dimension functions dim, ind and Ind are all equal to the "convex" dimension of the convex structure. The use of the latter dimension function is essential in the proof that the other ones are equal.

Introduction. It is well-known that for separable metric spaces the dimension functions ind, Ind and dim coincide. Outside of this class these functions diverge. It is the purpose of the present paper to introduce a new class of compact spaces on which the above dimension functions coincide. These spaces have the property of carrying a "*uniform*" *convex structure with connected convex sets and with a natural separation property.*

These convex structures should not be regarded as merely an axiomatic remodelling of linear spaces. For instance, superextensions (see [7]) carry a completely different kind of convex structure which proved useful in obtaining purely topological results (e.g. [8], 6.6 and [13], 3.4). Also, certain constructions like the formation of convex hyperspaces lead to convexities with distinguished properties (see [17] and [18]). Another apparently powerful construction is dealt with in the present paper.

In proving the above quoted result we have heavily borrowed from results concerning "convex dimension theory" as deviced in [15]. This is a dimension theory with a particularly strong geometric flavour, and which seems to be even better behaved than topological dimension theory for separable metric spaces. For instance, convex dimension behaves additively under the formation of products ([15], 3.3), it is characterized by the existence of "convexity preserving" maps onto cubes with "subcube" convexity ([15], 4.4), it is not affected by passing to the closure of a convex set ([15], 3.6), and it satisfies an adapted version of the Sum Theorem ([15], 4.7). Roughly speaking, this dimension function is obtained by a consistent replacement of the word "set" by "convex set" in the definition of "ind". One could of course try to take other (inductive) dimension functions as a model, but within reasonable margins, all these efforts lead to the same "convex dimension" ([15], 2.4). None of the above results requires metrizability; instead connectedness of convex sets is required.

For a compact space X the series of inequalities

$$\dim X \leqslant \operatorname{ind} X \leqslant \operatorname{Ind} X$$

is easily obtained. Our proof of the main theorem consists in showing that the above sequence is bounded above and below by the convex dimension of X (relative to a suitable convexity). The fact that convex dimension does not exceed dim is derived from the corresponding result for separable metric spaces (see [16], 5.2), and involves the formation of metric quotients of X in the sense of convex structures. The inequality with Ind comes from a technical induction procedure.

Formation of quotients of convex structures is studied with some generality in Section 2 below. Section 1 contains the necessary preliminary materials, and the main theorem is derived in Section 3. Some results concerning non-compact spaces are derived as well.

1. Preliminaries.

1.1. Set-theoretic convexity. Let X be a set. A *convexity* on X is a collection $\mathscr{C}$ of subsets of X which is closed under intersection and upward filtered union (in particular, $\emptyset$, $X \in \mathscr{C}$). The members of $\mathscr{C}$ are called *convex sets*, and the pair $(X, \mathscr{C})$ is called a *convex structure*. The (*convex*) *hull* of a set $A \subset X$ is defined in the obvious way:

$$h(A) = \bigcap \{C\colon A \subset C \in \mathscr{C}\}.$$

The hull of a finite set is called a *polytope*, and a set $H \subset X$ is called a *half-space* if both H and $X \setminus H$ are convex.

The following *separation property* will be of use below. A convex structure $(X, \mathscr{C})$ is S_4 if for each pair C, C' of disjoint convex sets there exists a half-space H of X with

$$C \subset H \quad \text{and} \quad C' \subset X \setminus H.$$

It will be assumed throughout that all singletons are convex.

1.2. Uniform convexity. Let (X, μ) be a uniform structure, where μ is defined in terms of diagonal neighborhoods. Let $\mathscr{C}$ be a convexity on X. If U, $V \in \mu$, then V is said to be *associated to U* (*relative to* $\mathscr{C}$) provided that for each $C \in \mathscr{C}$,

$$h(V[C]) \subset U[C].$$

We say that μ is *compatible with* $\mathscr{C}$ if

(1) all polytopes are closed in X (i.e. $(X, \mathscr{C})$ is a *topological convex structure*, cf. [14], 1.2), and

(2) for each $U \in \mu$ there exists an associated $V \in \mu$.

One can easily adapt these definitions to the case where μ is defined in terms of coverings of X (see [16], 2.1). Both approaches are used interchangeably, according to which one is most favourable in given circumstances. Uniformities are assumed to be separating.

A convex structure $(X, \mathscr{C})$ is called *uniformizable* (*metrizable*) if there exists a (metric) uniformity on X compatible with $\mathscr{C}$. It was shown in [16], 2.2, that a uniformizable convex structure is *closure-stable*, that is: the closure of each convex set is again convex.

If X is compact Hausdorff, and if μ is the unique uniformity of X, then a topological convexity $\mathscr{C}$ on X is *compatible with* μ iff the following conditions are satisfied:

(1) the collection $\mathscr{C}^*$ of all nonempty convex closed subsets of X is closed in the hyperspace of X;

(2) for each $C \in \mathscr{C}^*$ and for each open set $O \supset C$ there exists a $D \in \mathscr{C}^*$ with $C \subset \operatorname{int} D \subset D \subset O$

(see [16], 2.5). The properties (1) and (2) are also equivalent to the *continuity of the convex closure operator* of $(X, \mathscr{C})$ (see [4], III.1).

1.3. Convex dimension. If $(X, \mathscr{C})$ is a set-theoretic convex structure and if $Y \subset X$, then the family

$$\mathscr{C} \upharpoonright Y = \{C \cap Y\colon\ C \in \mathscr{C}\}$$

is again a convexity which is called the *trace* of $\mathscr{C}$ on Y. This construction is usually applied in case Y is a convex subset. Then, if μ is a uniformity on X compatible with $\mathscr{C}$, $\mu \upharpoonright Y$ is a uniformity on Y compatible with $\mathscr{C} \upharpoonright Y$.

Let $\mathscr{C}$ now be a topological convexity on X. The (convex) *small inductive dimension of* $(X, \mathscr{C})$ is the number $\operatorname{cind}(X, \mathscr{C}) \in \{-1, 0, 1, \ldots, \infty\}$ determined by the following axioms:

(1) $\operatorname{cind}(X, \mathscr{C}) = -1$ iff $X = \emptyset$,

(2) $\operatorname{cind}(X, \mathscr{C}) \leqslant n+1$ (where $n < \infty$) iff for each convex closed $C \subset X$ and for each $x \in X \setminus C$ there exist two convex closed sets $C_1, C_2 \subset X$ with $C \subset C_1 \setminus C_2$, $x \in C_2 \setminus C_1$, $C_1 \cup C_2 = X$, and $\operatorname{cind}(C_1 \cap C_2, \mathscr{C} \upharpoonright C_1 \cap C_2) \leqslant n$.

In [15], 2.3, various other types of convex inductive dimensions have been described. It was shown in [15], 2.4, that all these dimension functions coincide for uniformizable S_4-convexities with connected convex sets. In this way there is no need to consider other inductive dimension functions for (suitable) convexities.

For the reader's convenience, we reformulate some results of [15] in terms of uniformizable convexities. Recall that a *hyperplane* of a topological convexity is the boundary of an open half-space. Note that a hyperplane in a closure-stable convexity is always a convex (closed) subset. If $(X, \mathscr{C})$ and $(X', \mathscr{C}')$ are (set-theoretic) convex structures, then $f\colon X \to X'$ is *convexity preserving* (*CP*) *relative to* $\mathscr{C}$ *and* $\mathscr{C}'$ provided $f^{-1}(C') \in \mathscr{C}$ for each $C' \in \mathscr{C}'$. In this case we write

$$f\colon (X, \mathscr{C}) \to (X', \mathscr{C}').$$

1.4. THEOREM. *Let* $\mathscr{C}$ *be a uniformizable* S_4*-convexity on* X *with connected convex sets. Then* $\operatorname{cind}(X, \mathscr{C}) \leqslant n+1$ *iff for each hyperplane* H *of* X, $\operatorname{cind}(H, \mathscr{C} \restriction H) \leqslant n$.

1.5. THEOREM. *Let* $\mathscr{C}$ *be a uniformizable* S_4*-convexity on* X *with connected convex sets and with compact polytopes. If* $n < \infty$, *then the following assertions are equivalent*:

(1) $\operatorname{cind}(X, \mathscr{C}) \geqslant n$,

(2) *there is a CP map* $f\colon (X, \mathscr{C}) \to [0, 1]^n$ *which is onto* (the n-cube is equipped with the "subcube" convexity, i.e. the subcubes are the convex sets).

By a *map* we mean a continuous function. We finally quote a basic result from [16]:

1.6. THEOREM. *Let* X *be a separable metric space, and let* $\mathscr{C}$ *be a metrizable* S_4*-convexity on* X *with connected convex sets and with compact polytopes. Then*

$$\operatorname{cind}(X, \mathscr{C}) = \dim X.$$

For details concerning the Lebesgue covering dimension dim, or concerning other topological dimension functions, see [3], Chapter VII.

2. Uniform quotients of convexities. We now describe a simple though very effective procedure to obtain quotients of convex structures the basic properties of which can easily be controlled from the original convex structure.

2.1. Compatible subuniformity. Let $(X, \mu, \mathscr{C})$ be a uniform convex structure, where μ is defined in terms of diagonal neighborhoods. By a *compatible subuniformity of* μ we mean a subuniformity μ' of μ (which need not generate the μ-topology, and which need not even be separating) such that for each $U \in \mu'$ there is an associated $V \in \mu$ which is in μ' (terminology of 1.2).

For any subuniformity μ' of μ there is a corresponding equivalence relation $\sim_{\mu'}$ on X the graph of which is simply the set $\bigcap \mu'$. Put $\tilde{X} = X/\sim_{\mu'}$ and let $q\colon X \to \tilde{X}$ denote the resulting quotient function. We also put

$$\tilde{\mu} = \{U\colon (q \times q)^{-1}(U) \in \mu'\} \quad \text{and} \quad \tilde{\mathscr{C}} = \{D\colon q^{-1}(D) \in \mathscr{C}\}.$$

Note that q is CP relative to $\mathscr{C}$ and $\tilde{\mathscr{C}}$.

2.2. THEOREM. *Let* $(X, \mu, \mathscr{C})$ *be a uniform convex structure and let* $\mu' \subset \mu$ *be a compatible subuniformity. Then* (with the above notation) *the following is true*:

(1) $(\tilde{X}, \tilde{\mu}, \tilde{\mathscr{C}})$ *is a uniform convex structure,*
(2) *for each* $C \in \mathscr{C}$, $q(C) \in \tilde{\mathscr{C}}$,
(3) $(\tilde{X}, \tilde{\mathscr{C}})$ *has compact polytopes if* $(X, \mathscr{C})$ *has,*
(4) *if* $(X, \mathscr{C})$ *is* S_4, *then so is* $(\tilde{X}, \tilde{\mathscr{C}})$.

Proof. For $V \in \mu'$ we put

$$\tilde{V} = (q \times q)(V).$$

Note that $\tilde{V} \in \tilde{\mu}$. Let $U \in \tilde{\mu}$, and choose $V \in \mu'$ such that

$$V \circ V \circ V \subset (q \times q)^{-1}(U).$$

Let $(v_1, v_2) \in \tilde{V} \circ \tilde{V}$. Fix $v \in \tilde{X}$ with $(v_1, v) \in \tilde{V}$ and $(v, v_2) \in \tilde{V}$. Then there exist x_1, y, y', x_2 in X with

$$q(x_1) = v_1, \quad q(y) = v = q(y'), \quad q(x_2) = v_2,$$

and such that $(x_1, y) \in V$ and $(y', x_2) \in V$. Then also $(y, y') \in V$ since $y \sim_{\mu'} y'$, and it follows that

$$(x_1, x_2) \in V \circ V \circ V \subset U.$$

This shows that $\tilde{V} \circ \tilde{V} \subset U$, and it easily follows that $\tilde{\mu}$ is a uniformity on $\tilde{X}$.

It is clear that $\tilde{\mathscr{C}}$ is a convexity on $\tilde{X}$. Singletons in $\tilde{X}$ are convex by a more general argument below. Let $C \subset X$ be convex. Its saturation relative to $\sim_{\mu'}$ is denoted by sat(C). One easily sees that

$$q^{-1}(q(C)) = \operatorname{sat}(C) = \bigcap \{U[C]: U \in \mu'\}.$$

By the compatibility requirement on μ' we can fix an associated $U' \in \mu'$ for each $U \in \mu'$. Then $\operatorname{sat}(C) = \bigcap \{h(U'[C]): U \in \mu'\}$, which is a convex set. Therefore, by construction, $q(C)$ is convex in $\tilde{X}$. This establishes (2).

Note that for each subset $A \subset X$,

$$(*) \qquad q(h(A)) \subset \tilde{h}(q(A)) \quad (\tilde{h} \text{ is the hull operator of } (\tilde{X}, \tilde{\mathscr{C}}))$$

since q is CP. By the above argument, $q(h(A))$ is a convex set including $q(A)$, whence the inclusion $(*)$ becomes equality. Then (3) is obtained by taking A to be any finite set.

We now show that $\tilde{\mu}$ is compatible with $\tilde{\mathscr{C}}$. To this end, let $U \in \tilde{\mu}$ and let $V \in \mu'$ be associated to $(q \times q)^{-1}(U)$. Then choose $W \in \mu'$ such that

$$W \circ W \circ W \subset V.$$

We first show that for each $D \in \tilde{\mathscr{C}}$

$$(**) \qquad q^{-1}(\tilde{W}[D]) \subset V[q^{-1}(D)].$$

Indeed, if $q(x)\in \tilde{W}[D]$, then there is a point $y\in q^{-1}(D)$ with $(q(x), q(y))\in \tilde{W}$. Hence there exist $x', y'\in X$ with

$$x' \sim_{\mu'} x, \quad y' \sim_{\mu'} y, \quad (x', y')\in W.$$

Then $(x, y)\in W\circ W\circ W\subset V$, whence $x\in V[q^{-1}(D)]$. It follows from (**) that $h(q^{-1}(\tilde{W}[D]))\subset h(V[q^{-1}(D)])\subset (q\times q)^{-1}(U)[q^{-1}(D)] = q^{-1}(U[D])$, and hence that

$$q\big(h(q^{-1}(\tilde{W}[D]))\big)\subset U[D].$$

By (2) we find that

$$q\big(h(q^{-1}(\tilde{W}[D]))\big) = \tilde{h}\big(q(q^{-1}(\tilde{W}[D]))\big) = \tilde{h}(\tilde{W}[D]),$$

showing that $\tilde{W}$ is an associated refinement of U.

We finally derive statement (4). Let $D_1, D_2\in \tilde{\mathscr{C}}$ be disjoint convex sets. As $(X, \mathscr{C})$ is S_4, we find a half-space H with

$$q^{-1}(D_1)\subset H \quad \text{and} \quad q^{-1}(D_2)\subset X\setminus H.$$

Then

$$D_1\subset q(H)\setminus q(X\setminus H), \quad D_2\subset q(X\setminus H)\setminus q(H) \quad \text{and} \quad q(H)\cup q(X\setminus H) = \tilde{X}$$

(i.e. the pair $q(H), q(X\setminus H)$ *screens* D_1 *from* D_2 *in* $\tilde{X}$), and $q(H)$ and $q(X\setminus H)$ are convex by (2). It then follows from [14], 2.2, that $(\tilde{X}, \tilde{\mathscr{C}})$ is S_4. ■

The resulting map

$$q\colon (X, \mu, \mathscr{C})\to (\tilde{X}, \tilde{\mu}, \tilde{\mathscr{C}})$$

will henceforth be called the *uniform quotient map* (obtained from μ').

3. Main theorem. We begin with a modification of a factorization theorem of Mardesić [6]. Our proof is largely based on an argument of Arhangel'skiĭ in [1].

3.1. THEOREM. *Let X be compact, and let $\mathscr{C}$ be a uniformizable convexity on X. If $g\colon X\to Y$ is a mapping, then there exist a uniform quotient map*

$$q\colon (X, \mathscr{C})\to (\tilde{X}, \tilde{\mathscr{C}})$$

and a map $\tilde{g}\colon \tilde{X}\to Y$ such that

(1) $\tilde{g}\circ q = g$,

(2) $\dim \tilde{X}\leqslant \dim X$,

(3) *weight of $\tilde{X}\leqslant$ weight of Y.*

Proof. Let w denote the weight of Y. Then there is a base v for the canonical covering uniformity of Y such that v has cardinality w. Put

$$\mu_0 = \{g^{-1}(\mathscr{V})\colon \mathscr{V}\in v\}.$$

We construct μ_{p+1} from μ_p as follows. Let $\mathscr{V}_1, \mathscr{V}_2 \in \mu_p$. We first fix a finite open covering $\mathscr{V}$ of X with the following properties:

(4) $\mathscr{V}$ is a common star refinement of $\mathscr{V}_1$ and $\mathscr{V}_2$;

(5) $\mathscr{V}$ is a common associated refinement of $\mathscr{V}_1$ and $\mathscr{V}_2$;

(6) $\mathscr{V}$ has degree $\leqslant n+1$, where $n \leqslant \infty$ equals $\dim X$.

Let $\mathscr{V} = \{V_1, \ldots, V_k\}$. By [3], Theorem 7.1.7, there exists an open cover $\mathscr{W} = \{W_1, \ldots, W_k\}$ of X such that

(7) $W_i \subset \overline{W}_i \subset V_i$ for each $i = 1, \ldots, k$.

Note that $\mathscr{W}$ also has degree $\leqslant n+1$. Then put

$$W'(x) = \bigcap \{V_i\colon x \in V_i\} \setminus \bigcup \{\overline{W}_j\colon x \notin \overline{W}_j\},$$

for each $x \in X$. We find that

$$\mathscr{W}' = \{W'(x)\colon x \in X\}$$

is a (finite) open cover of X. If $W'(x)$ meets $\overline{W}_i$, then $x \in \overline{W}_i \subset V_i$, whence $W'(x) \subset V_i$ and

(8) $\mathrm{st}(\overline{W}_i, \mathscr{W}') \subset V_i$.

Having fixed two covers $\mathscr{W}$ and $\mathscr{W}'$ as in (7) and (8), we let μ_{p+1} consist of all covering $\mathscr{W}$, $\mathscr{W}'$ obtained from the various pairs $\mathscr{V}_1$, $\mathscr{V}_2$ in μ_p.

By (5), the collection $\bigcup_{p=0}^{\infty} \mu_p$ is a base for a uniformity $\mu' \subset \mu$, which by (6) is compatible with $\mathscr{C}$. In each step of the above inductive construction, we can take care that the cardinality of μ_p remains equal to w, whence μ' has a base of cardinality w. Let $q\colon (X, \mathscr{C}) \to (\tilde{X}, \tilde{\mathscr{C}})$ be the uniform quotient map obtained from μ'. Then the weight of $\tilde{X}$ is at most w (establishing (3)), and as $\mu_0 \subset \mu'$ we find that g factors through q as was required in (1).

We now take a look at the dimension of $\tilde{X}$. Let $\mathscr{U}$ be a finite open cover of $\tilde{X}$. By compactness of X there is a finite open cover $\mathscr{V}' \in \mu'$ with $q(\mathscr{V}')$ refining $\mathscr{U}$. By construction of the collections μ_p, there exist $\mathscr{W}$, $\mathscr{W}'$, $\mathscr{V}$ in μ' with $\mathscr{V} \leqslant \mathscr{V}'$ and

$$\mathscr{V} = \{V_1, \ldots, V_k\}, \quad \mathscr{W} = \{W_1, \ldots, W_k\} \text{ (injective indexation)},$$

$$W_i \subset \overline{W}_i \subset V_i \quad (i = 1, \ldots, k),$$

$$\mathrm{st}(W_i, \mathscr{W}') \subset V_i \quad (i = 1, \ldots, k),$$

and such that the degree of $\mathscr{V}$ is at most $n+1$.

Then $q(\mathscr{W})$ is a refinement of $\mathscr{U}$, and we show that the degree of $q(\mathscr{W})$ is at most $n+1$. Indeed, suppose that $\mathscr{W}_0 \subset \mathscr{W}$ is a subcollection with more that $n+1$ members such that $\bigcap q(\mathscr{W}_0) \neq \emptyset$. Then there exists a point $x(W) \in W$ for each $W \in \mathscr{W}_0$, such that $x(W_{i_1})$ and $x(W_{i_2})$ are equivalent (under $\sim_{\mu'}$) for each $W_{i_1}, W_{i_2} \in \mathscr{W}_0$. For any $W_{i_0} \in \mathscr{W}_0$ we then find that (since $\mathscr{W}' \in \mu'$)

$$\{x(W)\colon W \in \mathscr{W}_0\} \subset \mathrm{st}(W_{i_0}, \mathscr{W}') \subset V_{i_0}$$

by (8). Hence the collection

$$\mathscr{V}_0 = \{V_i\colon\ W_i \in \mathscr{W}_0\}$$

has more than $n+1$ members (injective indexation), and

$$\{x(W)\colon\ W \in \mathscr{W}_0\} \subset \bigcap \mathscr{V}_0,$$

a contradiction. This establishes (2). ■

3.2. MAIN THEOREM. *Let X be compact and let $\mathscr{C}$ be a uniformizable S_4-convexity on X with connected convex sets. Then*

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X = \operatorname{cind}(X, \mathscr{C}).$$

Proof. We will derive the following sequence of inequalities:

$$\operatorname{cind}(X, \mathscr{C}) \overset{(1)}{\leqslant} \dim X \overset{(2)}{\leqslant} \operatorname{ind} X \overset{(3)}{\leqslant} \operatorname{Ind} X \overset{(4)}{\leqslant} \operatorname{cind}(X, \mathscr{C}).$$

Note that (3) is a triviality, and that (2) follows from the fact that X is Lindelöf, [3], 7.2.7.

Proof of (1). We will show that for each n with $0 \leqslant n < \infty$,

$$n \leqslant \operatorname{cind}(X, \mathscr{C}) \Rightarrow n \leqslant \dim X.$$

Assume that $\operatorname{cind}(X, \mathscr{C}) \geqslant n$. Then by 1.5 there exists an onto CP map

$$g\colon\ X \to [0, 1]^n.$$

By Theorem 3.1 we obtain a uniform quotient map

$$q\colon\ (X, \mathscr{C}) \to (\tilde{X}, \tilde{\mathscr{C}})$$

together with a factorization

$$\tilde{g}\colon\ \tilde{X} \to [0, 1]^n$$

of g, such that $\tilde{X}$ is metrizable (countable weight), and $\dim \tilde{X} \leqslant \dim X$. For each convex set $C \subset [0, 1]^n$ we find that $\tilde{g}^{-1}(C) \subset \tilde{X}$ is convex since

$$q^{-1}(\tilde{g}^{-1}(C)) = g^{-1}(C)$$

is convex in X. Hence $\tilde{g}$ is a CP map, which is obviously onto. By 1.5 again,

$$\operatorname{cind}(\tilde{X}, \tilde{\mathscr{C}}) \geqslant n,$$

whereas by 1.6,

$$\operatorname{cind}(\tilde{X}, \tilde{\mathscr{C}}) = \dim \tilde{X} \leqslant \dim X.$$

It follows that $n \leqslant \dim X$, establishing the above implication.

Proof of (4). The following statements $P(n)$, $Q(n)$ will be obtained simultaneously by induction on $n = 0, 1, 2, \ldots$:

$P(n)$: *if* $\mathscr{C}$ *is a uniformizable* S_4*-convexity with connected convex sets on the compact space* X, *then* $\operatorname{cind}(X, \mathscr{C}) \leqslant n \Rightarrow \operatorname{Ind} X \leqslant n$.

$Q(n)$: *if* $\mathscr{C}$ *is a uniformizable* S_4*-convexity with connected convex sets on the compact space* X, *and if* $C_1, \ldots, C_p$ *are convex closed sets of* X *with* $\operatorname{Ind} C_k \leqslant n$ *for* $k = 1, \ldots, p$, *then* $\operatorname{Ind} \bigcup_{k=1}^{p} C_k \leqslant n$.

In the sequel the boundary $\bar{A} \setminus \operatorname{int} A$ of a set A in a space X will be denoted by $\partial_X(A)$ or $\partial(A)$. Reference to the convexity $\mathscr{C}$ will often be supressed from our notation.

We first note that a connected (convex) set A with $\operatorname{Ind} A \leqslant 0$ (resp. $\operatorname{cind} A \leqslant 0$) can have at most one point. Hence $P(0)$ and $Q(0)$ are trivialities. We proceed by induction, assuming the above statements to hold for all $m \leqslant n$. Suppose that $\operatorname{cind}(X) \leqslant n+1$, let $A \subset X$ be closed, and let $P \supset A$ be an open set. It follows from the compactness of X and from the "Hahn–Banach Theorem" ([16], 2.6) that the open half-spaces of X form a subbase for the topology of X. Hence we can find a number of open half-spaces O_{ij}, $i = 1, \ldots, q$, $j = 1, \ldots, r$, such that

$$A \subset O \subset \bar{O} \subset P,$$

where

$$O = \bigcup_{i=1}^{q} \bigcap_{j=1}^{r} O_{ij}.$$

Then

$$\partial(O) \subset \bigcup_{i=1}^{q} \partial\Big(\bigcap_{j=1}^{r} O_{ij}\Big) \subset \bigcup_{i=1}^{q} \bigcup_{j=1}^{r} \partial(O_{ij}).$$

By Theorem 1.4, $\operatorname{cind}(\partial O_{ij}) \leqslant n$, and hence by $P(n)$,

$$\operatorname{Ind}(\partial O_{ij}) \leqslant n.$$

Using $Q(n)$, we find that

$$\operatorname{Ind}(\partial O) \leqslant \operatorname{Ind}\Big(\bigcup_{i=1}^{q} \bigcup_{j=1}^{r} \partial O_{ij}\Big) \leqslant n$$

showing that $\operatorname{Ind} X \leqslant n+1$.

This establishes $P(n+1)$. Suppose next that $C_1, \ldots, C_p$ are convex closed sets in X with $\operatorname{Ind} C_k \leqslant n+1$ for $k = 1, \ldots, p$. Let

$$A \subset \bigcup_{k=1}^{p} C_k$$

be a closed set, and let $P \supset A$ be a relatively open subset of $C = \bigcup_{k=1}^{p} C_k$. As above we obtain open half-spaces O_{ij}, $i = 1, \ldots, q$, $j = 1, \ldots, r$, of X such that

$$A \subset O \subset \bar{O} \subset P,$$

where

$$O = \bigcup_{i=1}^{q} \bigcap_{j=1}^{r} O_{ij} \cap C.$$

Then

$$\partial_C(O) \subset \bigcup_{i=1}^{q} \bigcup_{j=1}^{r} \partial_C(O_{ij} \cap C),$$

where

$$\partial_C(O_{ij} \cap C) \subset \bigcup_{k=1}^{p} \partial_{C_k}(O_{ij} \cap C_k).$$

As $\operatorname{Ind} C_k \leqslant n+1$, we obtain from the inequalities (1), (2), and (3) that $\operatorname{cind} C_k \leqslant n+1$. Also, $\partial_{C_k}(O_{ij} \cap C_k)$ is a (probably empty) relative hyperplane of C_k, whence by Theorem 1.4

$$\operatorname{cind} \partial_{C_k}(O_{ij} \cap C_k) \leqslant n.$$

Hence by $P(n)$,

$$\operatorname{Ind} \partial_{C_k}(O_{ij} \cap C_k) \leqslant n,$$

and by $Q(n)$,

$$\operatorname{Ind} \partial_C(O) \leqslant \operatorname{Ind} \bigcup_{i=1}^{q} \bigcup_{j=1}^{r} \bigcup_{k=1}^{p} \partial_{C_k}(O_{ij} \cap C_k) \leqslant n.$$

This shows that

$$\operatorname{Ind} \bigcup_{k=1}^{p} C_k \leqslant n+1,$$

completing the inductive proof.

The inequality (4) now follows from the statements $P(n)$, $n = 0, 1, \ldots$ ■

The compactness condition on the space X is rather essential both in the proof of Theorem 3.1 as in the proof of the inequalities (1) and (4). Nevertheless, it is possible to derive some results concerning arbitrary convex subsets of a compact space:

3.3. COROLLARY. *Let X be a compact space equipped with a uniformizable S_4-convexity with connected convex sets. If $C \subset X$ is convex, then* $\operatorname{ind} C = \operatorname{cind} C$. *If C is Lindelöf moreover, then also* $\dim C = \operatorname{cind} C$.

Proof. If A is a subspace of the regular space B, then $\operatorname{ind} A \leqslant \operatorname{ind} B$ ([3], 7.1.1) and if A is a closed subspace of the normal space B, then $\dim A \leqslant \dim B$ ([3], 7.1.8). Let $C \subset X$ be convex, and let $P \subset C$ be a polytope. Then

(1) $$\operatorname{ind} P \leqslant \operatorname{ind} C \leqslant \operatorname{ind} \bar{C}.$$

If C is Lindelöf then C is also normal ([3], 3.8.2), whence by the above quoted result and by [3], 7.2.7,

(2) $$\dim P \leqslant \dim C \leqslant \operatorname{ind} C.$$

We distinguish between the following cases:

Case I: $\operatorname{cind} C = n < \infty$. Then by [15], 4.5, there is a polytope $P \subset C$ with $\operatorname{cind} P = \operatorname{cind} C$, whereas by [15], 3.6, $\operatorname{cind} C = \operatorname{cind} \bar{C}$. As P and $\bar{C}$ are compact, we can apply Theorem 3.2 to obtain

(3) $$\operatorname{ind} P = \operatorname{cind} P = \operatorname{cind} \bar{C} = \operatorname{ind} \bar{C},$$

and hence $\operatorname{ind} C = \operatorname{ind} \bar{C}$ by (1). If C is Lindelöf we then find from 3.2 that

$$\dim P = \operatorname{cind} P = \operatorname{cind} C = \operatorname{ind} C,$$

whence $\dim C = \operatorname{ind} C$ by (2).

Case II: $\operatorname{cind} C = \infty$. By [15], 4.5, again, there is a sequence of polytopes $P_n \subset C$ with $\operatorname{cind} P_n \geqslant n$. Then for each $n \in N$, 3.2 implies that

$$n \leqslant \operatorname{cind} P_n = \operatorname{ind} P_n \leqslant \operatorname{ind} C,$$

$$n \leqslant \operatorname{cind} P_n = \dim P_n \leqslant \dim C \text{ (} C \text{ normal, e.g. Lindelöf)},$$

whence $\operatorname{ind} C = \infty = \dim C$. ■

3.4. Problem. In order to obtain better results for "nice" convexities on noncompact spaces, one is naturally lead to consider the following problem. Let $(X, \mathscr{C})$ be an S_4 uniform convex structure. *Do there exist a compact space $\tilde{X}$ and a uniform S_4-convexity $\tilde{\mathscr{C}}$ on $\tilde{X}$ such that $(X, \mathscr{C})$ embeds as a convex set (with trace convexity) of $(\tilde{X}, \tilde{\mathscr{C}})$?* **(P 1287)**

Note that $\tilde{\mathscr{C}}$ is closure stable, and hence that $\mathrm{cl}_{\tilde{X}}(X)$ is convex in $\tilde{X}$. Hence it may be assumed that X is dense in $\tilde{X}$. It then follows from the continuity of the convex closure operator of $\tilde{\mathscr{C}}$ that all convex sets in $\tilde{X}$ are connected if the same is true for X. Also note that convexity of X in $\tilde{X}$ implies that $\mathscr{C}$-polytopes should be compact, and density of X in $\tilde{X}$ implies that $\operatorname{cind} X = \operatorname{cind} \tilde{X}$. Finally, it follows from the continuity of the convex closure operator on $\tilde{X}$ that the convex closed sets of $\tilde{X}$ form a closed subbase for the topology. In the terminology of [14], 1.5, $\tilde{X}$ has the weak topology relative to $\tilde{\mathscr{C}}$. This property is inherited by subspaces, and it follows that X must have the weak topology relative to $\mathscr{C}$. Even in case $(X, \mathscr{C})$ is $\boldsymbol{R}^n$ with linear convexity ($n > 1$), no solution is known (and, as a matter of fact, we have some doubts on an affirmative solution)(1).

(1) For updated information, see [19].

A different notion of "compactification" has already been considered in [9], 2.6. It appears that the "compact" objects in the category of topological convexities are the binary ones (*binary* means that every finite collection of pairwise intersecting convex sets has a common point). The following results may motivate this point of view.

(1) *If $(X, \mathscr{C})$ is a binary topological convexity with compact polytopes such that every pair of points in X can be "screened" with convex closed sets, then the same is true for every pair of disjoint convex closed sets* ([14], 2.9; compare with the implication "$T_2 \Rightarrow T_4$" for compact spaces);

(2) *A CP image of a binary convexity is again binary* (this can be seen with the notion of "triple-convexity", see [10], p. 992);

(3) *Binary convexities with enough separation properties are "maximal" binary* ([9], 1.6; compare with the fact that compact T_2-spaces are maximal compact);

(4) *Every "sufficiently separated" $(X, \mathscr{C})$ extends to a compact binary and S_4-convexity space $\lambda(X, \mathscr{C})$* (its "superextension") *which is a compactification of $(X, \mathscr{C})$ in the following (categorical) sense: every CP map*

$$(X, \mathscr{C}) \to (X', \mathscr{C}')$$

extends uniquely to a CP map

$$\lambda(X, \mathscr{C}) \to \lambda(X', \mathscr{C}')$$

([9], 2.6). We note that X need not be dense in $\lambda(X, \mathscr{C})$ (X may be compact for instance), nor need it be convex in $\lambda(X, \mathscr{C})$. Also, *$\lambda(X, \mathscr{C})$ is isomorphic to $(X, \mathscr{C})$ if X is compact and $\mathscr{C}$ is binary.*

If $(X, \mathscr{C})$ is a binary convexity with X not necessarily compact, then X is a dense convex subset of $\lambda(X, \mathscr{C})$ if its polytopes are compact: this follows from the fact that $\lambda(X, \mathscr{C})$ is a Wallman-type extension of X (ultrafilters are replaced by maximal pairwise intersecting families of convex closed sets): no new points will be added to compact polytopes. Then X is convex in $\lambda(X, \mathscr{C})$ as an upward filtered union of $\lambda(X, \mathscr{C})$-polytopes. Density of X follows from the fact that a maximal pairwise intersecting family of convex sets is also finitely intersecting by binarity.

In this case the above suggested compactification problem theorefore has an affirmative solution, and thus we have the following consequence of 3.3:

3.5. Corollary. *Let X be a connected space with the weak topology relative to a uniform binary S_4-convexity with compact polytopes. Then* $\operatorname{ind} X = \operatorname{cind} X$. *If X is Lindelöf moreover, then also* $\dim X = \operatorname{cind} X$.

Proof. All convex sets in X are connected by [14], 2.9. ■

By way of example, let us mention the following result. If X is a locally connected tree-like space, then the collection of all connected subsets of X yields a binary S_4-convexity with compact polytopes, [14], 2.10, which is uniformizable if X is, [14], 2.2. Relative to the weak topology, X is still a tree-like space, which is completely regular and hence uniformizable by [14], 1.6.

Each of the above quoted properties can be passed on to convex products with finitely many factors (a product of convex structures is the "coarsest" convexity on the product set making all projections CP). Hence if X is a product of n locally connected, nontrivial tree-like spaces with the weak topology, then

$$\operatorname{ind} X = \operatorname{cind} X = n$$

(convex dimension is additive under formation of products, [15], 3.3), and if X is Lindelöf moreover, then also $\dim X = n$.

For the case where X is a product of totally ordered connected spaces (which have the weak topology by definition), this result was obtained in [2], 12.1. See also [5], Theorem 1, for a comparable result concerning products of one-dimensional compacta.

One other area to apply our results is the theory of linear spaces. If X is a locally convex linear space then the natural (translation invariant) uniformity on X is compatible with the linear convexity, [16], 2.2. Also, this linear convexity is S_4, see [11], 2.3. These properties are inherited by convex subsets. Hence if C is a compact linearly convex subset of X, then by 3.2,

$$\operatorname{ind} C = \operatorname{Ind} C = \dim C.$$

However, this result is not very deep: it is fairly easy to see that a finite-dimensional (in any sense) linearly convex set is separable and metrizable.

As is well known, there exist compact metric spaces X and Y with

$$(*) \qquad \dim(X \times Y) < \dim X + \dim Y.$$

The first example of this kind was discovered by Pontrjagin, [11]. Since then there has been a strong interest in obtaining conditions on X and Y in order that (*) does not occur. One procedure is to try to obtain equality of dim with an additively behaved dimension function like, for instance, cohomological dimension over the integers modulo a prime. We are now able to present a quite different candidate: convex dimension. It is true, of course, that convex dimension theory attains its full strength for sufficiently nice convexities only, and consequently that the class of involved spaces is relatively small. However, these spaces are behaving in much the same way as absolute retracts (a metric space as in theorem 3.2 is actually an AR by [16], 5.1), and even for such spaces the question whether or not (*) can occur, seems to be relevant.

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