Infinite-Dimensional Normed Linear Spaces and Domain Invariance

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The Brouwer invariance of domain property for Euclidean spaces implies that, for open $U \subseteq \mathbb{R}^n$, every injective map $g: U \to \mathbb{R}^n$ is an open imbedding [2]. It is well-known that this property does not hold for infinite-dimensional linear spaces. Indeed, for any infinite-dimensional normed linear space Y we have the following examples:

EXAMPLE 1 ([1, III theorem 6.3]). There exists a homeomorphism $h: Y \to h(Y)$ onto a non-open subset of Y.

PROOF. Let $h_0: S \to H$ be any homeomorphism from the unit sphere S onto a closed hyperplane H. Then for any $y \in Y \setminus H$, h_0 may be extended to a homeomorphism h of Y onto the non-open set $(H + (-\infty, 1)\cdot y) \cup \{y\}$ with $h(0) = y$. $\square$

EXAMPLE 2 (D.W. CURTIS). There exists a bijective map $g: Y \to Y$ such that $g|Y \setminus K$ is not a homeomorphism for any compact K.

PROOF. Since the unit sphere S is non-compact, there exists a map $\lambda: S \to (0,1]$ such that $\inf \lambda(S) = 0$. Define $f: Y \to Y$ by the formulas

$$\begin{cases} f(y) = \lambda\left(\frac{y}{\|y\|}\right)\cdot y & (y \neq 0), \\ f(0) = 0. \end{cases}$$

Clearly, f is a bijective map of Y, but is not a homeomorphism since $0 \notin \operatorname{int} f(B)$ for any bounded set B. Note that $f|Y \setminus \{0\}$ is a homeomorphism.

Using copies of f on a discrete sequence of open balls in Y, we may construct a bijective map $g:Y\to Y$ such that for any compact $K\subseteq Y$, $g|Y\setminus K$ is not a homeomorphism. In fact, there exists an open $U\subseteq Y\setminus K$ such that for any compact $J\subseteq U$, $g(U\setminus J)$ is non-open. However, there is a dense open $V\subseteq Y$ such that $g|V$ is an open imbedding. $\square$

A space X is a *Baire space* if the intersection of any countable family of dense open subsets of X is dense. A function $f:Y\to Y$ on a linear space has *countable type* if there exists a countable set Z in Y such that for each $y\in Y$,

$$f(y)\in \operatorname{span}(\{y\}\cup Z).$$

If each map $f:Y\to Y$ on a topological linear space has countable type, we say that Y has *countable type for maps.*

Clearly $\mathbb{R}^n$ and, more generally, each $\aleph_0$-dimensional topological linear space (i.e. a topological linear space with a countable Hamel basis) has countable type for maps. Consequently, there exist infinite-dimensional topological linear spaces having countable type for maps. However, not all topological linear spaces have this property, see Example 3.

LEMMA 1. *Let Y be a topological linear space and let $A:Y\to Y$ be a linear operator with countable type. Then $A=\lambda I+B$, for some scalar λ and linear operator B with $\aleph_0$-dimensional range.*

PROOF. Let $Z\subseteq Y$ be a countable set such that $A(y)\in\operatorname{span}(\{y\}\cup Z)$ for each y. Let $E\subseteq Y$ be a complementary linear subspace for span Z, and consider any linearly independent set $\{e_1,e_2\}\subseteq E$. There exist scalars λ_1, λ_2 and λ, and elements $s_1,s_2,s\in\operatorname{span} Z$, such that

$$A(e_1)=\lambda_1\cdot e_1+s_1;$$

$$A(e_2)=\lambda_2\cdot e_2+s_2; \text{ and}$$

$$A(e_1+e_2)=\lambda\cdot(e_1+e_2)+s.$$

Using $A(e_1+e_2)=A(e_1)+A(e_2)$, we obtain

$$(\lambda-\lambda_1)\cdot e_1+(\lambda-\lambda_2)\cdot e_2=s_1+s_2-s.$$

Since $E\cap\operatorname{span} Z=\{0\}$, $(\lambda-\lambda_1)\cdot e_1+(\lambda-\lambda_2)\cdot e_2=0$ and since $\{e_1,e_2\}$ is linearly independent, $\lambda_1=\lambda=\lambda_2$. This implies that for any linearly independent set F in E, $(A-\lambda I)(e)\in\operatorname{span} Z$ for each $e\in F$. It follows that $(A-\lambda I)(E)\subseteq$ span Z. Since $Y=E+$ span Z, and $(A-\lambda I)(\text{span } Z)\subseteq$ span Z, we obtain $A-\lambda I:Y\to$ span Z. $\square$

The following result is well-known.

LEMMA 2. *Let $B:Y\to Y$ be a bounded linear operator on a Baire topological linear space. If B has $\aleph_0$-dimensional range, it has finite-dimensional range.*

PROOF. Write $B(Y)$ as $\bigcup_1^\infty F_n$, where each F_n is a finite-dimensional linear subspace of $B(Y)$. Observing that each F_n is closed in $B(Y)$ and that B is continuous, it follows that each $B^{-1}(F_n)$ is closed in Y. Since clearly $\bigcup_1^\infty B^{-1}(F_n)=Y$ and since Y is a Baire space, one of the $B^{-1}(F_n)$'s has non-empty interior, say $B^{-1}(F_{n_0})$. Every proper closed linear subspace of any topological linear space has empty interior. We conclude that $B^{-1}(F_{n_0})=Y$. □

A topological linear space Y is said to have *few operators* if every bounded linear operator $A:Y\to Y$ has the form $A=\lambda I+B$, for some scalar λ and some operator B with finite-dimensional range.

In [4] the author constructed an infinite-dimensional pre-Hilbert space with few operators (a Banach space B of uncountable weight such that each bounded linear operator $A:B\to B$ has the form $A=\lambda I+E$, for some scalar λ and some operator E with separable range, was earlier constructed under $V=L$ by SHELAH [7]). Lemmas 1 and 2 yield

THEOREM 1. *Let Y be a Baire topological linear space with countable type for maps. Then Y has few operators;*

and

EXAMPLE 3. *Hilbert space ℓ^2 does not have countable type for maps.*

PROOF. Let $E:\ell^2\to\ell^2$ be a bounded linear operator which is of the form $\lambda I+B$, for some scalar λ and some operator B with finite-dimensional range. If $\lambda=0$ then $E=B$ which implies that E has finite-dimensional range. Suppose that $\lambda\neq 0$. If $E(x)=0$ then $x=\frac{1}{\lambda}\cdot B(x)$, which belongs to the range of B. Consequently, in this case the kernel of E is finite-dimensional.

By Theorem 1 and the above remarks it suffices to construct a bounded linear operator $A:\ell^2\to\ell^2$ such that neither the range nor the kernel of A is finite-dimensional. This is a triviality of course. Indeed, define $A:\ell^2\to\ell^2$ by

$$A(x_1,x_2,x_3,\ldots)=(x_1,0,x_3,0,\ldots).$$

Then A is clearly as required. □

Since each $\aleph_0$-dimensional topological linear space has countable type for maps and Hilbert space ℓ^2 has not, the question naturally arises whether there are topological linear spaces having countable type for maps but which are not $\aleph_0$-dimensional. In [5] the author proved

THEOREM 2 ([5]). *Each separable Banach space B contains a linear subspace Y such that*

(a) *Y is dense in X;*
(b) *Y is a Baire space; and*
(c) *Y has countable type for maps.*

PROOF (sketch). If B is finite-dimensional then $Y=B$ is as required. Therefore assume that B is infinite-dimensional.

Let $g:A\to B$ be a function defined on a subset of B. A subset P of A is said to be *g-independent* if the following conditions are satisfied:

(1) $g|P$ is injective;

(2) $P\cap g(P)=\emptyset$; and

(3) $P\cup g(P)$ is linearly independent.

Via a standard procedure it is possible to prove that if A is a G_δ-subset of B which contains an uncountable g-independent subset then A contains a g-independent Cantor set.

Now let $\mathcal{H}$ denote the collection of all homeomorphisms $h:K_1\to K_2$ between Cantor sets in B such that K_1 is h-independent. It is possible to construct a linear subspace Y of B with the following property

(*) for each $h\in\mathcal{H}$, there exists $x\in\operatorname{dom} h$ such that $x\in Y$ but $h(x)\notin Y$.

Then Y is as required.

By (*) it easily follows that Y intersects every linearly independent Cantor set in B. Since B is infinite-dimensional, every dense G_δ-subset of B contains a linearly independent Cantor set and consequently intersects Y. This implies that Y is a Baire space.

If Y were not dense then the closure of Y would be a proper closed linear subspace of B which therefore would have to be nowhere dense which is impossible since Y intersects every dense G_δ-subset of B. This proves (a) and (b).

For (c), let $f:Y\to Y$ be a map. Since B is complete, f extends to a map $g:A\to B$, for some G_δ-subset A of B. Suppose that A contains an uncountable g-independent set. Then A contains a g-independent Cantor set K. Then $g|K$ is a member of the collection $\mathcal{H}$, and by (*) there exists $x\in K\cap Y$ such that $g(x)\notin Y$. But this contradicts the fact that $g(x)=f(x)\in Y$. Thus every g-independent subset of A is countable, and in particular, every f-independent set is countable.

By (*) Y cannot contain any linearly independent Cantor set. This easily implies that for every countable set P there exists a countable set $\hat{P}$ such that $f(\operatorname{span} P)\subseteq\operatorname{span}\hat{P}$. Now let $Q\subseteq Y$ be a maximal f-independent set. If $Q=\emptyset$ then $f(y)\in\operatorname{span}\{y\}$ for each $y\neq 0$, and f obviously has countable type. Otherwise, construct a tower (P_i) of countable sets by taking $P_1=Q$ and $P_{n+1}=P_n\cup\hat{P}_n$ for each $n\geqslant 1$. Take $Z=\bigcup_1^\infty P_n$. It can be shown that $f(y)\in\operatorname{span}(\{y\}\cup Z)$ for every $y\in Y$. □

COROLLARY 1 ([4]). *There exists an infinite-dimensional pre-Hilbert space with few operators.*

PROOF. By Theorem 2 there is a dense linear subspace $Y\subseteq\ell^2$ such that Y is a

Baire space and has countable type for maps. Then Y, being dense in ℓ^2, is clearly infinite-dimensional. Also, Y has few operators by Theorem 1. $\square$

Let Y be a topological linear space. Suppose that, for every injective map $g:U\to Y$ with domain an open subset of Y, there exists a nonempty open $V\subseteq U$ such that $g|V$ is an open imbedding. Then we say that Y has *restricted domain invariance.*

We may suppose that V is dense in U, since the condition can be applied to every restriction $g|W$ to an open nonempty $W\subseteq U$. In addition, for Y a normed linear space, it suffices to verify the condition for every injective map $f:Y\to Y$, since there is an open imbedding of Y into every nonempty open subset.

The reader naturally wonders what the relation is between the title of our paper and the results derived or mentioned so far. This is cleared by the following

THEOREM 3 ([5]). *Let Y be a normed linear space with the Baire property and with countable type for maps. Then Y has restricted domain invariance.*

PROOF (sketch). We may assume Y is infinite-dimensional. By the above remark, we need only to consider an injective map $f:Y\to Y$. Let Z be a countable subset of Y such that $f(y)\in\operatorname{span}(\{y\})\cup Z)$ for each y. There exists a tower (A_n) of compacta such that span A_n is finite-dimensional for each n, and $\bigcup_1^\infty A_n=$ span Z. For each n, set

$$Y_n=\{y\in Y|\text{for some }\lambda\in[-n,n],f(y)-\lambda\cdot y\in A_n\}.$$

It is easily seen that each Y_n is closed and that $\bigcup_1^\infty Y_n=Y$. Since Y is a Baire space, some Y_n has nonempty interior and since Y is infinite-dimensional, there exists a nonempty open set $W\subseteq Y_n\setminus\operatorname{span}A_n$. For each $w\in W$ there is a unique $\lambda(w)\in[-n,n]$ such that $f(w)-\lambda(w)\cdot w\in A_n$; furthermore, the assignment $w\to\lambda(w)$ is continuous. It is possible to show that there exists a nonempty open $V\subseteq W$ with either $\lambda(V)\subseteq[-n,0)$ or $\lambda(V)\subseteq(0,n]$.

For convenience, assume that $\lambda(V)\subseteq(0,n]$ and take $p\in V$ arbitrarily. We may assume that $f(p)=p$. Let $E=\operatorname{span}(\{p\}\cup A_n)$. By using, among other things, the Brouwer invariance of domain property for E, it can be shown that $f(V)$ is a neighborhood of $f(p)=p$. $\square$

Observe that by Examples 1 and 2, Theorem 3 is 'best possible'.

COROLLARY 2 ([5]). *There exists an infinite-dimensional pre-Hilbert space X such that X is not homeomorphic to $X\times\mathbb{R}$.*

PROOF. By Theorems 2 and 3 there is a dense linear subspace $X\subseteq\ell^2$ having restricted domain invariance. X is clearly infinite-dimensional. We claim that X is as required. To the contrary, assume that $\phi:X\to X\times\mathbb{R}$ is a homeomorphism.

Define $\psi : X \to X$ by

$$\psi(x) = \phi^{-1}(x, 0).$$

Then ψ is clearly an imbedding of X onto a subset of X with empty interior. But this contradicts restricted domain invariance. □

The above result generalizes POL [6] and answers Question LS12 in GEOGHEGAN [3].

COROLLARY 3. *Every separable Banach space contains a dense linear subspace Y such that:*
(a) *Y is a Baire space;*
(b) *Y has restricted domain invariance; and*
(c) *Y has few operators.* □

I am indebted to D.W. CURTIS for many helpful comments.

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