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## AN EASY PROOF THAT $\beta N - N - \{p\}$ IS NOT NORMAL

**Abstract.** We give a simple proof that, under CH,  $\beta N - N - \{p\}$  is not normal for any  $p \in \beta N - N$ .

One of the most outstanding open problems in general topology is the question whether  $N^* - \{p\}$  (for any space  $X$  we write  $X^* = \beta X - X$ ) is not normal for any  $p \in N^*$ . Under CH the question has been answered in the affirmative: for non  $P$ -points by Gillman (see [1]) and for  $P$ -points independently by Rajagopalan [8] and Warren [10]. The proof of the non  $P$ -point case uses Parovičenko's [7] characterization of  $N^*$  and the known proofs of the  $P$ -point case do not make use of this characterization. The aim of this note is to give a simple proof that  $N^* - \{p\}$  is not normal under CH. Our proof is different since we use Parovičenko's characterization in the  $P$ -point case and use the  $P$ -point case to solve the non  $P$ -point case.

It will be convenient to call a space  $X$  a *Parovičenko space* if

(a)  $X$  is a zero-dimensional compact space without isolated points of weight  $2^\omega$ ,

(b) every two disjoint open  $F_\sigma$ 's in  $X$  have disjoint closures,

and

(c) every nonempty  $G_\delta$  in  $X$  has nonempty interior.

Notice that (b) implies that every countable subspace is  $C^*$ -embedded.

It is known, [7], [3], that CH is equivalent to the statement that every Parovičenko space is homeomorphic to  $N^*$ .

The following lemma is known. It follows directly from the proof of Gillman's [5] result that, under CH,  $N^* - \{p\}$  is not  $C^*$ -embedded in  $N^*$  for any  $p$ . Since I do not know a reference for it I will give the easy proof.

**LEMMA (CH).** *If  $p \in N^*$ , then there is a Parovičenko space  $X \subset N^*$  containing  $p$  such that  $p$  is a  $P$ -point of  $X$ .*

**Proof.** W.l.o.g.  $p$  is not a  $P$ -point, so take an open  $F_\sigma$   $U \subset N^*$  with  $p \in U^- - U$ . Let, by CH,  $\{C_\alpha : \alpha < \omega_1\}$  enumerate all nonempty clopen subsets of  $N^*$  containing  $p$ . By (b) and (c) we can find for each  $\alpha < \omega_1$  a nonempty clopen  $E_\alpha \subset N^*$  such that

$$E_\alpha \subset \bigcap_{\beta < \alpha} C_\beta - (U \cup \bigcup_{\beta < \alpha} E_\beta)^-.$$

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If  $Y = (\bigcup_{\alpha < \omega_1} E_\alpha)^-$  and if  $y \in Y \cap (U^- - \{p\})$  then for some  $\mu < \omega_1$ ,

$$y \in (\bigcup_{\kappa < \mu} E_\kappa)^-,$$

which contradicts (b). Hence  $Y \cap U^- = \{p\}$ . This implies that  $p$  is a  $P$ -point of  $Y$  for if  $F \subset Y - \{p\}$  is any  $F_\sigma$ , then  $F \cap U^- = \emptyset$  and consequently, by (b),  $F^- \cap U^- = \emptyset$ , i.e.  $p \notin F^-$ .

For each  $\alpha < \omega_1$  take  $p_\alpha \in E_\alpha$  and put  $X = \{p_\alpha : \alpha < \omega_1\}^- - \{p_\alpha : \alpha < \omega_1\}$ . Since  $p$  is a  $P$ -point of  $X$  and since  $\{p_\kappa : \kappa < \mu\}^- - \{p_\kappa : \kappa < \mu\} \approx N^*$  for any  $\omega \leq \mu < \omega_1$ , it easily follows that  $X$  satisfies (c). That  $X$  satisfies (b) is clear since (b) is closed hereditary in normal spaces. This implies that  $X$  is a Parovičenko space, since  $X$  clearly satisfies (a).

By the above Lemma we only need to show that  $N^* - \{p\}$  is not normal for any  $P$ -point  $p \in N^*$ . Since W. Rudin [9] showed that  $N^* - \{p\} \approx N^* - \{q\}$  if  $p, q \in N^*$  are  $P$ -points (under CH), the proof is completed by the following

**EXAMPLE 1.** *There is a Parovičenko space  $X$  having a  $P$ -point  $p$  such that  $X - \{p\}$  is not normal.*

Let  $Z_1 = \{\langle \kappa, \mu \rangle \in (\omega_1 + 1) \times (\omega_1 + 1) : \mu \leq \kappa\}$  and let  $Y = (\omega \times Z)^*$ . We claim that  $Y$  is a Parovičenko space.  $\omega \times Z$  is strongly zero-dimensional, hence so is  $\beta(\omega \times Z)$ , [6, 16.11]. Also,  $\omega \times Z$  is a Lindelöf space with weight  $\omega_1$ , hence  $\omega \times Z$  has  $\omega_1^\omega = 2^\omega$  clopen subsets, hence  $\beta(\omega \times Z)$  has weight  $2^\omega$ . It is clear that  $Y$  has no isolated points.  $Y$  satisfies (b), since  $\omega \times Z$  is  $\sigma$ -compact and locally compact, [6, 14.27]. Finally,  $Y$  satisfies (c), since  $\omega \times Z$  is real compact and locally compact, [4, 3.1].

Let  $\pi : \omega \times Z \rightarrow Z$  be the projection and  $\beta\pi$  its Stone extension.

**CLAIM.**  $\beta\pi^{-1}(\langle \omega_1, \omega_1 \rangle) = (\omega \times \{\langle \omega_1, \omega_1 \rangle\})^-$ .

Clearly  $(\omega \times \{\langle \omega_1, \omega_1 \rangle\})^- \subset \beta\pi^{-1}(\langle \omega_1, \omega_1 \rangle)$ . Take

$$x \in \beta\pi^{-1}(\langle \omega_1, \omega_1 \rangle) - (\omega \times \{\langle \omega_1, \omega_1 \rangle\})^-.$$

Let  $C$  be a clopen neighborhood of  $x$  in  $\beta(\omega \times Z)$  which misses  $\omega \times \{\langle \omega_1, \omega_1 \rangle\}$ . Then  $x \in (C \cap (\omega \times Z))^-$  and consequently

$$\beta\pi(x) \in \beta\pi((C \cap (\omega \times Z))^-) \subset (\pi(C \cap (\omega \times Z)))^-.$$

$\pi(C \cap (\omega \times Z))$  is  $\sigma$ -compact because  $C \cap (\omega \times Z)$  is  $\sigma$ -compact, and  $\langle \omega_1, \omega_1 \rangle \notin \pi(C \cap (\omega \times Z))$ , and  $\langle \omega_1, \omega_1 \rangle \notin (\pi(C \cap (\omega \times Z)))^-$ , which is a contradiction since  $\beta\pi(x) = \langle \omega_1, \omega_1 \rangle$ .

We conclude that  $Y - (\omega \times \{\langle \omega_1, \omega_1 \rangle\})^*$  admits a perfect map onto  $Z - \{\langle \omega_1, \omega_1 \rangle\}$ , and hence is not normal.

Let  $X = Y / (\omega \times \{\langle \omega_1, \omega_1 \rangle\})^*$  be the quotient space obtained from  $Y$  by collapsing  $(\omega \times \{\langle \omega_1, \omega_1 \rangle\})^*$  to a point. We claim that  $X$  and  $p = \{(\omega \times \{\langle \omega_1, \omega_1 \rangle\})^*\}$  are as required. First, since  $\langle \omega_1, \omega_1 \rangle$  is a  $P$ -point of  $Z$ , it easily follows that  $(\omega \times \{\langle \omega_1, \omega_1 \rangle\})^-$  is a  $P$ -set of  $\beta(\omega \times Z)$  (a subset  $A$  of a space  $S$  is called a  $P$ -set whenever the intersection of countably many neighborhoods of  $A$  is again a neigh-

borhood of  $A$ ), hence  $(\omega \times \{\langle \omega_1, \omega_1 \rangle\})^*$  is a  $P$ -set of  $(\omega \times Z)^*$  and consequently,  $p$  is a  $P$ -point of  $X$ . Second,  $X$  is a Parovičenko space. This follows from the fact that  $Y$  is a Parovičenko space and that  $(\omega \times \{\langle \omega_1, \omega_1 \rangle\})^*$  is a nowhere dense closed  $P$ -set of  $Y$ .

The above example suggests the question whether it can be proved in ZFC that for any Parovičenko space  $X$  and for any  $p \in X$  it is true that  $X - \{p\}$  is not normal. Unfortunately, this is not possible, as the following example shows.

EXAMPLE 2. *There is a compact zero-dimensional space  $X$  without isolated points of weight  $\omega_2 \cdot 2^\omega$  which satisfies (b) and (c), having a  $P$ -point  $p$  such that  $X - \{p\}$  is both normal and  $C^*$ -embedded in  $X$ .*

Let  $P = \{\alpha \leq \omega_2 : \text{cf}(\alpha) \geq \omega_1\}$ . Put  $Y = \beta P$  and  $X = Y - \{y \in Y : y \text{ is isolated}\}$ . Van Douwen [2] showed that  $Y - \{\omega_2\}$  is almost compact, i.e. if  $A$  and  $B$  are disjoint closed subsets of  $Y - \{\omega_2\}$  then one of them is compact, and that  $Y$  has weight  $\omega_2 \cdot 2^\omega$ . That implies that  $X - \{\omega_2\}$  is almost compact, and hence is normal and  $C^*$ -embedded in  $X$ . Since  $X$  has clearly weight  $\omega_2 \cdot 2^\omega$  it remains to be shown that  $X$  satisfies (b) and (c). That  $X$  satisfies (b) is trivial since  $Y$  satisfies (b) ([2]). Let  $G$  be any nonempty closed  $G_\delta$  of  $X$ . If  $G \cap P \neq \emptyset$ , then  $\text{int } G \neq \emptyset$ , since  $P \cap X$  consists of  $P$ -points of  $X$ . Hence  $G \subset \{\xi \in P : \xi \leq \alpha\}^-$  for certain  $\alpha < \omega_2$ . By transfinite induction it is easy to show that  $X \cap \{\xi \in P : \xi \leq \alpha\}^-$  has the property that each nonempty  $G_\delta$  has nonempty interior. Since all these sets are clopen in  $X$  it follows that  $G$  has nonempty interior.

Since the space of the above example is a Parovičenko space if CH fails our claim follows. It is interesting that such a space exists since it shows that the properties (a), (b) and (c) of  $N^*$  are not enough to prove that  $N^* - \{p\}$  is not normal for any  $p \in N^*$  in ZFC alone.

Since, as remarked earlier, under CH,  $N^* - \{p\}$  is neither normal nor  $C^*$ -embedded in  $N^*$ , we have also obtained the following result.

THEOREM. *Each of the following statements is equivalent to CH:*

- (a) *if  $X$  is any Parovičenko space and  $p \in X$  then  $X - \{p\}$  is not normal;*
- (b) *if  $X$  is any Parovičenko space and  $p \in X$  then  $X - \{p\}$  is not  $C^*$ -embedded in  $X$ .*

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