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# LOCAL CONTRACTIBILITY, CELL-LIKE MAPS, AND DIMENSION

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ABSTRACT. We consider the existence of cell-like maps $f\colon I^n \to X$ such that no nonempty open subset of $X$ is contractible in $X$. From the Taylor Example, it is easy to construct such a map for $n = \infty$. We show that there exists such a map for some finite $n$ if (and only if) there exists a dimension raising cell-like map of a compactum.

**1. Introduction.** *All spaces are separable metric.* A compactum $X$ has *trivial shape* if every continuous function from $X$ to a polyhedron is null homotopic. In addition, a continuous surjection $f\colon X \to Y$ between compact spaces is called *cell-like* provided that $f^{-1}(y)$ has trivial shape for every $y \in Y$. One of the most outstanding open problems in geometric topology is whether a cell-like map of a compactum can raise dimension. This problem is known as the "*cell-like dimension raising mapping problem*" and has many equivalent formulations. For example, the cell-like dimension raising mapping problem is equivalent to the problem whether every compactum with finite cohomological dimension is finite dimensional (Edwards; for details, see Walsh [**8**]) and also to the problem whether the cell-like image of a (finite-dimensional) compact manifold is an ANR (Kozlowski [**6**]; see also Ancel [**1**]). Since every finite-dimensional space imbeds in $I^n$ for some $n < \infty$ (here $I$ denotes the interval $[0,1]$), [**4**, 1.11.4], several problems related to cell-like maps are in fact problems concerning cell-like maps defined on the $n$-cubes $I^n$, $n < \infty$.

In [**1**], Ancel asked whether the cell-like image of a compact ANR is locally contractible. It is known that such an image is $\mathrm{LC}^\infty$, [**2**, 12.1], but need not be an ANR, [**7**]. Relevant to Ancel's question is also that Daverman and Walsh [**3**] used the Taylor Example [**7**] to produce a cell-like map $f\colon I^\infty \to X$ such that $X$ is not an ANR but is locally contractible.

We shall establish the following results.

THEOREM. *If for every cell-like map $f\colon I^n \to X$, $n < \infty$, there exists a nonempty open subset of $X$ which is contractible in $X$, then no cell-like map of a compactum raises dimension* (*and conversely*).

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EXAMPLE. *There exists a cell-like image of the Hilbert cube such that no nonempty open subset is contractible in the space.*

I am indebted to Doug Curtis for some helpful comments.

**2. Proofs.** The proof of our theorem is in two steps.

2.1. LEMMA. *Let $n \in \mathbf{N} \cup \{\infty\}$. If for every cell-like map $f\colon I^{n+1} \to X$, there exists a nonempty open subset of $X$ which is contractible in $X$, then every cell-like image of $I^n$ is contractible.*

PROOF. Let $f\colon I^n \to Y$ be cell-like. In addition, let $\{X_i\colon i \in \mathbf{N}\}$ be a null sequence of disjoint compacta in $I^{n+1}$ such that

(1) $X_i$ is homeomorphic to $I^n$ for every $i$;

(2) if $U$ in $I^{n+1}$ is open and nonempty then there is an $i \in \mathbf{N}$ with $X_i \subseteq U$.

(It is of course a triviality to construct such a sequence.) Let $\{Y_i\colon i \in \mathbf{N}\}$ be a sequence of disjoint copies of $Y$ and for each $i \in \mathbf{N}$ let $f_i\colon X_i \to Y_i$ be a copy of the map $f$, i.e. $f_i$ has the form $\eta \circ f \circ \xi^{-1}$ for certain homeomorphisms $\xi\colon I^n \to X_i$ and $\eta\colon Y \to Y_i$. It is easy to prove (and well known) that the collection

$$\mathcal{G} = \{f_i^{-1}(y)\colon y \in Y_i, i \in \mathbf{N}\} \cup \left\{\{x\}\colon x \in I^{n+1} \setminus \bigcup_{i=1}^{\infty} X_i\right\}$$

is an upper-semicontinuous decomposition of $I^{n+1}$. Put $Z = I^{n+1}/\mathcal{G}$ and let $\pi\colon I^{n+1} \to Z$ be the corresponding quotient map. For convenience we shall identify $\pi(X_i)$ and $Y_i$ for every $i$. Clearly, $\pi$ is cell-like. By assumption, there exists an open set $U$ in $Z$ which is contractible in $Z$ and nonempty. Since $\pi^{-1}(U)$ is open in $I^{n+1}$ and nonempty, by (2) there exists an $i \in \mathbf{N}$ with $X_i \subseteq \pi^{-1}(U)$. Consequently, $Y_i = \pi(X_i) \subseteq U$ from which follows that $Y_i$ is contractible in $Z$. We shall prove that $Y_i$ is also a retract of $Z$. It then follows that $Y_i$ is contractible and since $Y_i$ is homeomorphic to $Y$ we conclude that $Y$ is contractible, as required. Let $\mathcal{H}$ be the upper-semicontinuous decomposition of $I^{n+1}$ with nondegenerate elements $\{f_j^{-1}(y)\colon y \in Y_j, j \in \mathbf{N} \setminus \{i\}\}$ and let $T = I^{n+1}/\mathcal{H}$. The corresponding quotient map from $I^{n+1}$ shall be denoted by $\theta$. Clearly, $X_i$ and $\theta(X_i)$ are homeomorphic from which follows, by (2), that there is a retraction $r\colon T \to \theta(X_i)$. There clearly exists a continuous function $\xi\colon T \to Z$ such that

(3) $\xi(\theta(X_i)) = Y_i$,

(4) $\xi$ restricted to $T \setminus \theta(X_i)$ is a homeomorphism from $T \setminus \theta(X_i)$ onto $Z \setminus Y_i$.

Now define $s\colon Z \to Y_i$ by

$$s(y) = \begin{cases} y & (y \in Y_i), \\ \xi r \xi^{-1}(y) & (y \in Z \setminus Y_i). \end{cases}$$

An easy check shows that $s$ is a retraction.

2.2. LEMMA. *Let $n \in \mathbf{N} \cup \{\infty\}$. If every cell-like image of $I^{2n+1}$ is contractible then no cell-like map of an at most $n$-dimensional compactum raises dimension.*

PROOF. Let $X$ be a compact space with $\dim X \leqslant n$ and let $f\colon X \to Y$ be cell-like. If $n = \infty$ then there is nothing to prove, so assume that $n < \infty$. We shall prove that $\dim Y \leqslant 2n + 1$ from which follows that $\dim Y \leqslant \dim X$ since the Vietoris Theorem implies that the cohomological dimension of $Y$ is less than or equal to dim $X$. By [**4**, 1.11.4] we may assume that $X \subseteq I^{2n+1}$. Let $A \subseteq Y$ be closed. Put $B = f^{-1}(A)$ and let $g = f \mid B$. Let $\xi$: $A \to S^{2n+1}$ be continuous. We shall prove that $\xi$ is null homotopic. To this end, let $Z = I^{2n+1} \cup_g A$. By assumption, $Z$ is contractible. Also, observe that $\dim Z \setminus A \leqslant 2n + 1$. Consequently, the function $\xi$ can be extended to a continuous function $\eta$: $Z \to S^{2n+1}$ [**4**, 1.9.2]. Since $Z$ is contractible, $\eta$ is null homotopic, so $\xi$ is null homotopic.

By the Borsuk Homotopy Extension Theorem [**4**, 1.9.7], we can now extend $\xi$ to a continuous function $\bar{\xi}$: $Y \to S^{2n+1}$. Since $A$ was arbitrarily chosen, it now follows from [**4**, 1.9.3] that $\dim Y \leqslant 2n + 1$, as required.

It is easily seen that by Lemmas 2.1 and 2.2 the proof of our theorem is completed.

We shall now construct the required example. Keesling [**5**] used the Taylor Example [**7**] to get a cell-like map $f$: $I^\infty \to X$ such that $X$ is not movable. In particular, $X$ cannot have the shape of an AR and is therefore not contractible. By Lemma 2.1 there consequently exists a cell-like image of $I^\infty$ such that no nonempty open subset is contractible in the space. Alternatively, one can get such an example via the following procedure. For any space $Y$ let $Y^\infty$ denote the countable infinite product of copies of $Y$. Let $f$ and $X$ be such as above. Define $F$: $(I^\infty)^\infty \to X^\infty$ by $F = f \times f \times f \times \cdots$. Then $F$ is clearly cell-like and no nonempty open subset of $X^\infty$ is contractible in $X^\infty$ since $X$ is not contractible.

Added in proof. J. Mogilski kindly informed me that Lemma 2.2 was also obtained independently by S. Nowak, *Some extension and classification theorems of movable spaces*, to appear in Fund. Math.

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