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A HOMOGENEOUS EXTREMALLY DISCONNECTED COUNTABLY COMPACT SPACE

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It is well known that no infinite homogeneous space is both compact and extremally disconnected. (Since there are infinite compact homogeneous spaces and infinite extremally disconnected homogeneous spaces, it is the combination of compactness and extremal disconnectedness that brings about this result.) The following question then arises naturally: How "close to compact" can a homogeneous, extremally disconnected space be? The aim of this paper is to show that a homogeneous extremally disconnected space can be countably compact. It is shown also, assuming Martin's Axiom, that there exist countably compact, homogeneous, extremally disconnected spaces whose product is not countably compact.

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0. Introduction

All spaces under discussion are Tychonoff spaces, i.e., completely regular, Hausdorff spaces. A space is *extremally disconnected* if the closure of each open set is again open.

In order to orient this paper within the surrounding relevant literature, it is convenient to have in mind the following three known results. Here as usual we say that a subspace Y of a space X is C^**-embedded* in X if every bounded, real-valued continuous function on Y extends continuously over X. The Stone–Čech (= Čech–Stone) compactification of a space X is denoted βX, and X^* abbreviates $\beta X \backslash X$; in particular, ω^* denotes the space of free ultrafilters over the countably infinite discrete space ω.

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0.1. Theorem. *X is extremally disconnected if and only if βX is extremally disconnected.*

0.2. Theorem. *Let X be extremally disconnected and let $Y \subseteq X$. If Y is countable, or open in X, then Y is C^*-embedded in X (and so $\beta Y = \mathrm{cl}_{\beta X}(Y)$).*

0.3. Theorem. *If G is a pseudocompact topological group, then βG is a topological group.*

References: For Theorems 0.1 and 0.2, see Gillman and Jerison [14, (1.H, 6.M, 14.N)]. Theorem 0.3 is from [8]. De Vries [21] gives a proof, set forth also in [5], somewhat simpler than that of [8]; see also Van Douwen [9] for related results.

The Rudin–Keisler (pre-) order $\leqslant$ on ω^* is defined as follows; for p, $q \in \omega^*$ we write $p \leqslant q$ if there is a function $f: \omega \to \omega$ for which the (continuous) Stone extension $\bar{f}: \beta(\omega) \to \beta(\omega)$ satisfies $\bar{f}(q) = p$; and we write $p \sim q$ if there is a homeomorphism f of ω (that is, a permutation) such that $\bar{f}(q) = p$. It is known that the relation $\leqslant$ on ω^* respects the equivalence relation $\sim$ in the sense that if $p \leqslant q$ and $q \leqslant p$ then $p \sim q$.

0.4. Theorem. *Let X be an extremally disconnected space containing a C^*-embedded copy of ω (again denoted ω) and let q and q' be accumulation points of ω in X. [We write q, $q' \in \omega^* \cap X \subseteq \beta X$.] If there is a homeomorphism of X onto X taking q to q', then q and q' are $\leqslant$-comparable in ω^*.*

References: We know of no explicit references in the literature for this result, but the essential features, expanding on an Editor's Note appended by Z. Frolík to related results of Efimov [11, footnote, p. 105], are recorded in [4, § 8] and [5, (2.10)]. Indeed the homeomorphism h extends to a homeomorphism $\bar{h}$ of βX and then, following Frolík [12, 13], either

$$p \in \mathrm{cl}(\omega \cap \mathrm{cl}(\bar{h}^{-1}[\omega]))$$

(in which case $p \leqslant q$) or $p \in \mathrm{cl}(\bar{h}^{-1}[\omega] \cap \mathrm{cl}(\omega))$ (in which case $q \leqslant p$).

The $\leqslant$-minimal elements of ω^*, often called *selective* ultrafilters, were first introduced in another context by Choquet [3] (who called them 'absolute'). There are models of ZFC where none exist [15], but the following result, proved in part in [2] and [18], shows their existence in ZFC+MA.

The reader may consult [7, §§ 9, 10] for a survey of the literature concerning the order $\leqslant$ and for a detailed proof of 0.5 in the system ZFC+CH.

0.5. Theorem [ZFC+MA]. *There is $S \subseteq \omega^*$ such that $|S| = 2^{\mathfrak{c}}$, every two elements of S are $\leqslant$-inequivalent, and each element of S is $\leqslant$-minimal in ω^*.*

0.6. Discussion. It is a question raised nearly 20 years ago by Arhangel'skiĭ [1] whether there exists a non-discrete extremally disconnected topological group. To the best of our knowledge, this remains open as a question in ZFC. Solutions are known, however, when ZFC is augmented by suitable additional (consistent) axioms. First, Sirota [13] gave an example based on the existence of a so-called k-ultrafilter over the countably infinite discrete space ω; he showed further that a k-ultrafilter exists if the continuum hypothesis is assumed. Next Louveau [10], using an ultrafilter over ω which is absolute in the sense of Choquet [2], simplified Sirota's construction. More recently Malyhin [17] has proved the existence of another non-discrete extremally disconnected group on the basis of a combinatorial principle usually called $P(\mathfrak{c})$.

It is natural to wonder whether the groups of Sirota-Louveau-Malyhin may be constructed so as to possess in addition some of the familiar topological properties of compactness type. The groups are already countable, hence σ-compact. Theorem 0.7 thwarts improvement in the direction of countable compactness.

0.7. Theorem. *A non-discrete, extremally disconnected topological group cannot be pseudocompact (hence, cannot be countably compact).*

Proof. If a counterexample exists, it may by Theorems 0.1 and 0.3 be chosen to be compact. Then by Theorems 0.2 and 0.4, the Rudin-Keisler order $\leqslant$ on ω^* is a linear order. This contradicts a theorem (proved in ZFC) of Kunen [15] (see also [7, (10.4)] for a proof). $\square$

Our goal is to show that there does exist an extremally disconnected, countably compact space which is homogeneous. Since our space is based on a construction due to Louveau [15] we shall describe this construction in some detail.

For a space X we write

$$\mathscr{A}(X)=\{U\subseteq X\colon U \text{ is open-and-closed in } X\}.$$

The elements of $\mathscr{A}(X)$ are called clopen (in X).

1. Louveau's construction

For a set X we denote by $[X]^{<\omega}$ the set of all finite subsets of X. The set $[X]^{<\omega}$ is a group under the symmetric difference operation Δ defined by

$$E\Delta F=(E\setminus F)\cup(F\setminus E) \quad \text{for } E, F\in[X]^{<\omega}.$$

As usual, for $D\in[X]^{<\omega}$ and $\mathscr{A}$, $\mathscr{B}\subseteq[X]^{<\omega}$ we write

$$D\Delta\mathscr{A}=\{D\Delta A : A\in\mathscr{A}\} \quad \text{and} \quad \mathscr{A}\Delta\mathscr{B}=\{A\Delta B : A\in\mathscr{A}, B\in\mathscr{B}\}.$$

Now let p be a free ultrafilter on ω. A subset $\mathscr{A}$ of $[\omega]^{<\omega}$ is said to be *p-stable* if for each $A \in \mathscr{A}$ we have

$$\{n \in \omega: A\Delta\{n\} \in \mathscr{A}\} \in p.$$

Louveau [16] proved that the collection of all p-stable sets forms a basis for a topology on $[\omega]^{<\omega}$ which is extremally disconnected and non-discrete, and which moreover has the property that the group operation $\Delta: [\omega]^{<\omega} \times [\omega]^{<\omega} \to [\omega]^{<\omega}$ is continuous in each variable separately. In what follows we denote by $G(p)$ the group $[\omega]^{<\omega}$ with the topology just defined.

Observe that $G(p)$ is countable, non-discrete, extremally disconnected and homogeneous. However, $G(p)$ with the operation Δ need not be a topological group: Louveau [16] proved that $(G(p), \Delta)$ is a topological group if and only if p is a selective ultrafilter.

Now fix a free ultrafilter p on ω and put $G = G(p)$. The space we will define in Section 2 will lie between G and βG. The following interesting lemma due to Van Douwen [10] will be helpful; for the reader's convenience we shall include a sketch of its proof.

1.1. Lemma [10]. *Let X be a countable, dense in itself, homogeneous space. Then all non-empty open subsets of X are homeomorphic.*

Proof. Let U and V be non-empty, open subsets of X, write $U = \{x_n: n < \omega\}$ and $V = \{y_m: m < \omega\}$, and set $n_0 = m_0 = 0$. Since $\mathscr{A}(X)$ is a base for X there is a triple $\langle h_0, U_0, V_0\rangle$ with $x_{n_0} \in U_0 \in \mathscr{A}(X)$, $y_{m_0} \in V_0 \in \mathscr{A}(X)$, h_0 a homeomorphism of U_0 onto V_0, and $|U \setminus U_0| = |V \setminus V_0| = \omega$. At stage $k < \omega$ let n_k and m_k be the least integers n and m such that $x_n \notin \bigcup_{i<k} U_i$ and $y_m \notin \bigcup_{i<k} V_i$, and choose $\langle h_k, U_k, V_k\rangle$ so that $x_{n_k} \in U_k \in \mathscr{A}(X)$, $y_{m_k} \in V_k \in \mathscr{A}(X)$, U_k and V_k are disjoint from $\bigcup_{i<k} U_i$ and $\bigcup_{i<k} V_i$, respectively, h_k is a homeomorphism of U_k onto V_k, and $|U \setminus \bigcup_{i\leq k} U_i| = |V \setminus \bigcup_{i\leq k} V_i| = \omega$. Then $h = \bigcup_{k<\omega} h_k$ is a homeomorphism of U onto V. □

1.2. Corollary. *All non-empty elements of $\mathscr{A}(\beta G)$ are homeomorphic.*

Proof. Take $E \in \mathscr{A}(\beta G) \setminus \{\emptyset\}$ arbitrarily. By Lemma 1.1 there is a homeomorphism h from $E \cap G$ onto G. Since $E \cap G$ is dense in E, we have $\beta(E \cap G) = E$ from Theorems 0.1 and 0.2. Consequently, f can be extended to a homeomorphism from E onto βG.

1.3. Lemma. *Let $D = \{\{n\}: n < \omega\}$. Then*

(a) *D is discrete (in G), and*

(b) $\emptyset \in \bar{D}^G$.

Proof. (a) For $n < \omega$ we have that $\mathscr{A} = \{n\}\, \Delta[\omega \setminus \{n\}]^{<\omega}$ is p-stable and $\mathscr{A} \cap D = \{n\}$.

(b) Let $\mathscr{A} \subseteq G$ be p-stable such that $\emptyset \in \mathscr{A}$. Then $\{n < \omega: \{n\} \in \mathscr{A}\} = \{n < \omega: \emptyset\Delta\{n\} \in \mathscr{A}\} \in p$. Consequently, $\mathscr{A} \cap D \neq \emptyset$. □

2. The example

For a space X we write

$$\mathcal{H}(X) = \{h : h \text{ is a homeomorphism of } X \text{ onto } X\}.$$

2.1. Theorem. *If* $X = \{h(\emptyset) : h \in \mathcal{H}(\beta G)\}$, *then* X *is an extremally disconnected, homogeneous, countably compact space.*

Proof. As with any space, each element of $\mathcal{H}(G)$ extends to an element of $\mathcal{H}(\beta G)$. Hence $G \subseteq X \subseteq \beta G$ so from 0.1 we have that X is extremally disconnected. That X is homogeneous is clear, so it remains to show that every faithfully indexed (countable) discrete subset

$$C = \{x_n : n < \omega\}$$

of X has an accumulation point in X.

Let $D = \{\{n\} : n < \omega\}$ be as in Lemma 1.3 and for $n < \omega$ choose $h_n \in \mathcal{H}(\beta G)$ such that $h_n(\{n\}) = x_n$. It is now easy to define by induction pairwise disjoint families

$$\mathcal{U} = \{U_n : n < \omega\} \subseteq \mathcal{A}(\beta G) \quad \text{and} \quad \mathcal{V} = \{V_n : n < \omega\} \subseteq \mathcal{A}(\beta G)$$

such that $\{n\} \in U_n$, $x_n \in V_n$, and $h_n[U_n] = V_n$. We assume also without loss of generality, replacing (say) U_0 or V_0 by a suitable proper clopen subset, that neither $\bigcup \mathcal{U}$ nor $\bigcup \mathcal{V}$ is dense in βG. Let $E = \beta G \setminus \overline{\bigcup \mathcal{U}}^{\beta G}$ and $F = \beta G \setminus \overline{\bigcup \mathcal{V}}^{\beta G}$.

By Corollary 1.2 there is a homeomorphism $f : E \to F$. We define a homeomorphism

$$h : (\bigcup \mathcal{U}) \cup E \to (\bigcup \mathcal{V}) \cup F$$

by $h|U_n = h_n$, $h|E = f$ and, using Theorem 0.2 and the fact that both the domain and the range of h are dense in βG, we extend h to $\bar{h} \in \mathcal{H}(\beta G)$. From

$$\emptyset \in \bar{D}^G \setminus D \quad \text{(Lemma 1.3(b))}$$

then follows $\bar{h}(\emptyset) \in \bar{C}^X \setminus C$, as required. $\square$

3. Remarks on cardinality

Let us note first that the space X constructed in Theorem 2.1 satisfies $|X| \geq \mathfrak{c}$.

3.1. Proposition. *Let* Y *be an infinite, countably compact space in which each countable, discrete subset is* C^**-embedded. Then* $|Y| \geq \mathfrak{c}$.

Proof. It is a well-known result of Sierpiński [19] that there is a faithfully indexed family $\mathcal{F} = \{A_\xi : \xi < \mathfrak{c}\}$ of infinite subsets of ω such that

$$|A_\xi \cap A_{\xi'}| < \omega \quad \text{for } \xi < \xi' < \mathfrak{c}.$$

Now identify ω with a (countably infinite, discrete) subset of Y. Then with $\mathcal{F}$ as above and p_ξ an accumulation point in Y of A_ξ, it is clear from the C^*-embedded hypothesis that the function $\xi \to p_\xi$ is one-to-one from $\mathfrak{c}$ into Y. □

While we have been unable to compute the cardinality of the space X constructed in Theorem 2.1, we can show by a standard argument that its essential features are retained by a suitably chosen subspace of minimal cardinality.

3.2. Theorem. *There is an extremally disconnected, homogeneous, countably compact space Y such that $|Y| = \mathfrak{c}$.*

Proof. Let X be as in Theorem 2.1. Continuing the notation used there, we use transfinite induction to find for $\xi \leq \omega_1$ sets $Y_\xi \subseteq X$ and subgroups $\mathcal{H}_\xi$ of $\mathcal{H}(\beta G)$ such that

(a) $|Y_\xi| \leq \mathfrak{c}$ and $|\mathcal{H}_\xi| \leq \mathfrak{c}$;
(b) $G \subseteq Y_\zeta \subseteq Y_\xi$ and $\mathcal{H}_\zeta \subseteq \mathcal{H}_\xi$ for $\zeta < \xi \leq \omega_1$;
(c) every $h \in \mathcal{H}_\xi$ has $h[Y_\xi] = Y_\xi$;
(d) for every $y \in Y_\xi$ there is $h \in \mathcal{H}_\xi$ such that $h(\emptyset) = y$; and,
(e) every countable, discrete subset of Y_ξ has an accumulation point in $Y_{\xi+1}$.

To do this, set $Y_0 = G$ and, using the fact that $|Y_0| = \omega$ and the fact that each element of $\mathcal{H}(Y_0)$ extends to an element of $\mathcal{H}(\beta G)$, choose $\mathcal{H}_0 \subseteq \mathcal{H}(\beta G)$ so that (a), (c) and (d) are satisfied (with $\xi = 0$). Suppose now that $0 < \xi \leq \omega_1$ and that Y_ζ, $\mathcal{H}_\zeta$ have been defined for $0 \leq \zeta < \xi$.

Case 1. $\xi = \zeta + 1$. Since X is countably compact and the number of countable subsets of Y_ζ is at most

$$|Y_\zeta|^\omega \leq \mathfrak{c}^\omega = \mathfrak{c},$$

there is S such that $Y_\zeta \subseteq S$ and $|S| \leq \mathfrak{c}$ and every countable discrete subset of Y_ζ has an accumulation point in S. For $s \in S$ we choose $h_s \in \mathcal{H}(\beta G)$ such that $h_s(\emptyset) = s$, we let $\mathcal{H}_\xi$ be the subgroup of $\mathcal{H}(\beta G)$ generated by

$$\mathcal{H}_\zeta \cup \{h_s : s \in S\},$$

and we define

$$Y_\xi = \bigcup\{h[S] : h \in \mathcal{H}_\xi\}.$$

Case 2. ξ is a limit ordinal. We set

$$\mathcal{H}_\xi = \textstyle\bigcup_{\zeta<\xi} \mathcal{H}_\zeta \quad \text{and} \quad Y_\xi = \bigcup_{\zeta<\xi} Y_\zeta.$$

It is clear that the families $\{Y_\xi : \xi \leq \omega_1\}$ and $\{\mathcal{H}_\xi : \xi \leq \omega_1\}$ satisfy conditions (a) through (e), and that the space $Y = Y_{\omega_1}$ is as required. □

4. Concerning products

In earlier work [6] we showed in ZFC that there are pseudocompact, homogeneous spaces X_0, X_1 such that $X_0 \times X_1$ is not pseudocompact; if in addition MA is assumed, the spaces X_i may be chosen countably compact. In this section we use our present methods and construction to show (again in ZFC + MA) there are countably compact, homogeneous, extremally disconnected spaces whose product is not countably compact.

In what follows, the symbol S denotes a set with the properties described in Theorem 0.5, and for $p \in \omega^*$ we write

$$E(p) = \{x \in \omega^* \colon p \preccurlyeq x\}.$$

4.1. Lemma. *Let $A \subseteq \omega^*$ satisfy $|A| < 2^{\mathfrak{c}}$. Then there is $q \in S$ such that $A \cap E(q) = \emptyset$.*

Proof. If not, then for every $q \in S$ there are $x(q) \in A$ and $f_q \in \omega^\omega$ such $\overline{f_q}(x(q)) = q$. From $|S| = 2^{\mathfrak{c}}$ and $|A \times \omega^\omega| < 2^{\mathfrak{c}}$ it follows that there are distinct q, $q' \in S$ such that $x(q) = x(q')$ and $f_q = f_{q'}$; hence

$$q = \overline{f_q}(x(q)) = \overline{f_{q'}}(x(q')) = q',$$

a contradiction.

4.2. Theorem [ZFC + MA]. *There are extremally disconnected, homogeneous, countably compact spaces Y and Z such that $Y \times Z$ is not countably compact.*

Proof. Let $p \in \omega^*$, let G be the Louveau group $G = G(p)$, and define X and Y as in Theorems 2.1 and 3.2. Then $|Y| = \mathfrak{c}$, Y is extremally disconnected, homogeneous and countably compact, and $D = \{\{n\} \colon n < \omega\}$ is C^*-embedded in Y. For notational simplicity we identify D with ω and we set $A = Y \cap \omega^*$.

From MA there is S as in Theorem 0.5, and Lemma 4.1 yields $q \in S$ such that $A \cap E(q) = \emptyset$. We denote by H the Louveau group $H = G(q)$ and again we use Theorem 2.1 to find an extremally disconnected, homogeneous, countably compact space Z such that $H \subseteq Z \subseteq \beta H$ and $D = \{\{n\} \colon n < \omega\}$ is C^*-embedded in Z. To see that $Y \times Z$ is not countably compact it is enough to note that D in Y (identified with ω) and D in Z (also identified with ω) satisfy $\bar{\omega}^Y \cap \bar{\omega}^Z = \omega$. Our identification of D in Z with ω identifies $\emptyset$ with q. Thus if $x \in \bar{\omega}^Z \backslash \omega$ then from Theorem 0.4 we have $x \in E(q)$ or $q \in E(x)$; since q is $\preccurlyeq$-minimal we have $x \in E(q)$ and hence $x \notin A = \bar{\omega}^Y \backslash \omega$, as required.

5. Questions

In our opinion the most interesting unsolved problems closely related to the topic of this paper are as follows.

5.1. Is there, in ZFC alone without additional axioms, a non-discrete, extremally disconnected, topological group?

5.2. Is the result of Theorem 4.2 available in ZFC alone? If so, can Y and Z be chosen so that $Y \times Z$ is not pseudocompact? Are there, in ZFC or in ZFC + MA, extremally disconnected, pseudocompact, homogeneous spaces whose product is not pseudocompact?

5.3. Is there an extremally disconnected, homogeneous, countably compact space X such that $|X| > \mathfrak{c}$?

Acknowledgement

In an early version of this paper we proved our principal result (that is: there exists a countably compact, extremally disconnected, homogeneous space) in the axiom system ZFC+MA. It was brought to our attention subsequently by Eric van Douwen that his homogeneity lemma (Lemma 1.1) makes the appeal to Martin's Axiom unnecessary. We are indebted to him for this helpful observation.

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