

DOMAIN INVARIANCE IN INFINITE-DIMENSIONAL LINEAR SPACES

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ABSTRACT. Let X be an infinite-dimensional locally convex linear space. It is known that X is homeomorphic to a subspace of X which is not open. We prove that every Banach space B contains a dense linear subspace L with the following property: If $U \subseteq L$ is open and if $f: U \rightarrow L$ is continuous and injective, then there exists a dense open $V \subseteq U$ such that (a) $f(V)$ is open in L , and (b) $f: V \rightarrow f(V)$ is a homeomorphism. As a consequence, L and $L \times \mathbf{R}$ are not homeomorphic.

1. Introduction. The Brouwer invariance of domain property for Euclidean spaces implies that, for open $U \subseteq \mathbf{R}^n$, every injective map $g: U \rightarrow \mathbf{R}^n$ is an open imbedding [2]. It is well known that this property does not hold for infinite-dimensional linear spaces. Indeed, for any infinite-dimensional normed linear space Y we have the following examples.

EXAMPLE 1 [1, III, THEOREM 6.3]. There exists a homeomorphism $h: Y \rightarrow h(Y)$ onto a nonopen subset of Y .

PROOF. Let $h_0: S \rightarrow H$ be any homeomorphism from the unit sphere S onto a closed hyperplane H . Then for any $y \in Y \setminus H$, h_0 may be extended to a homeomorphism h of Y onto the nonopen set $(H + (-\infty, 1) \cdot y) \cup \{y\}$ with $h(0) = y$. \square

EXAMPLE 2 (D. W. CURTIS). There exists a bijective map $g: Y \rightarrow Y$ such that $g|Y \setminus K$ is not a homeomorphism for any compact $K \subseteq Y$.

PROOF. Since the unit sphere S is noncompact, there exists a map $\lambda: S \rightarrow (0, 1]$ such that $\inf \lambda(S) = 0$. Define $f: Y \rightarrow Y$ by the formulas

$$\begin{cases} f(y) = \lambda(y/\|y\|) \cdot y, & y \neq 0, \\ f(0) = 0. \end{cases}$$

Clearly, f is a bijective map of Y , but is not a homeomorphism since $0 \notin \text{int } f(B)$ for any bounded set B . Note that $f|Y \setminus \{0\}$ is a homeomorphism.

Using copies of f on a discrete sequence of open balls in Y , we may construct a bijective map $g: Y \rightarrow Y$ such that for any compact $K \subseteq Y$, $g|Y \setminus K$ is not a homeomorphism. In fact, there exists an open $U \subseteq Y \setminus K$ such that for any compact $J \subseteq U$, $g(U \setminus J)$ is nonopen. However, there is a dense open $V \subseteq Y$ such that $g|V$ is an open imbedding. \square

The constructions in this paper show the existence of infinite-dimensional normed linear spaces Y with the property that, for every injective map $g: U \rightarrow Y$ with open

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domain, there is a dense open $V \subseteq U$ such that $g|_V$ is an open imbedding. This property easily implies that Y is not homeomorphic to $Y \times \mathbf{R}$. In fact, if Y is homeomorphic to $Y \times Z$ then Z is a single point.

Our results generalize Pol [7] and answer Question LS12 in Geoghegan [3]. I am greatly indebted to Doug Curtis for many helpful comments. He reworked and significantly simplified my original write-up when presenting the results to a seminar at LSU.

2. Special linear subspaces. Let X be a separable metric linear space and let $\mathcal{K}(X)$ denote the collection of all homeomorphisms $h: K_1 \rightarrow K_2$ between disjoint Cantor sets in X such that $K_1 \cup K_2$ is a linearly independent set.

2.1. THEOREM. *Every separable metric linear space X has a linear subspace Y with the following property:*

(*) *for each $h \in \mathcal{K}(X)$, there exists $x \in \text{dom } h$ such that $x \in Y$ but $h(x) \notin Y$.*

PROOF. It is easy to see that the collection $\mathcal{K}(X)$ has size at most \mathfrak{c} . Let $<$ be a well-ordering of $\mathcal{K} = \mathcal{K}(X)$ such that for each $h \in \mathcal{K}$, the section $\{g \in \mathcal{K} : g < h\}$ has size less than \mathfrak{c} .

We show by transfinite induction that for all $h \in \mathcal{K}$, there exist linear subspaces Y_h and Z_h in X satisfying the following conditions.

- (1) $Y_h \cap Z_h = \{0\}$;
- (2) for $g < h$, $Y_g \subseteq Y_h$ and $Z_g \subseteq Z_h$;
- (3) all Y_h and Z_h have algebraic dimension less than or equal to the cardinality of the section $\{g \in \mathcal{K} : g < h\}$; and
- (4) there exists $x \in \text{dom } h$ such that $x \in Y_h$ and $h(x) \in Z_h$.

Then $Y = \bigcup \{Y_h : h \in \mathcal{K}\}$ is a linear subspace with the required property (*).

For the first element $f \in \mathcal{K}$, we may take $Y_f = \text{span}\{x\}$ and $Z_f = \text{span}\{f(x)\}$ for any $x \in \text{dom } f$. For $h \in \mathcal{K}$, suppose the spaces Y_g and Z_g have been constructed for $g < h$. Let $Y^h = \bigcup \{Y_g : g < h\}$ and $Z^h = \bigcup \{Z_g : g < h\}$. Then $Y^h \cap Z^h = \{0\}$, and $\text{span}(Y^h \cup Z^h)$ has algebraic dimension less than \mathfrak{c} . Consider

$$H = \{x \in \text{dom } h : \text{span}(\{x\} \cup Y^h) \cap \text{span}(\{h(x)\} \cup Z^h) \neq \{0\}\}.$$

We claim that H has size less than \mathfrak{c} . For each $x \in H$, there exist scalars $\lambda(x)$ and $\mu(x)$ such that

$$w(x) = \lambda(x) \cdot x + \mu(x) \cdot h(x) \in \text{span}(Y^h \cup Z^h) \setminus \{0\}.$$

Thus either $\lambda(x) \neq 0$ or $\mu(x) \neq 0$. To establish the claim on H , we show that the correspondence of $x \rightarrow w(x)$ is injective and has linearly independent range $\{w(x) : x \in H\}$. Consider any finite, faithfully indexed subset $\{x_1, \dots, x_n\}$ of H . Let $I = \{i : \lambda(x_i) = 0\}$ and $J = \{i : \lambda(x_i) \neq 0\}$. Then $\text{span}\{x_i, h(x_i)\} = \text{span}\{x_i, w(x_i)\}$ for $i \in I$, and $\text{span}\{x_i, h(x_i)\} = \text{span}\{h(x_i), w(x_i)\}$ for $i \in J$. Hence the set $\{x_i : i \in I\} \cup \{h(x_i) : i \in J\} \cup \{w(x_i) : 1 \leq i \leq n\}$ has the same span as the set $\{x_i : 1 \leq i \leq n\} \cup \{h(x_i) : 1 \leq i \leq n\}$. The stipulated properties of $h \in \mathcal{K}$ imply that the latter set has size $2n$ and is linearly independent. It follows that $\{w(x_i) : 1 \leq i \leq n\}$ has size n and is linearly independent. Since $\{x_i : 1 \leq i \leq n\} \subseteq H$ was arbitrary, this completes the argument for the claim on H .

Since H has size less than the size of the Cantor set $\text{dom } h$, there exists $x \in \text{dom } h \setminus H$, i.e.

$$\text{span}(\{x\} \cup Y^h) \cap \text{span}(\{h(x)\} \cup Z^h) = \{0\}.$$

The induction step is completed by setting $Y_h = \text{span}(\{x\} \cup Y^h)$ and $Z_h = \text{span}(\{h(x)\} \cup Z^h)$. \square

3. Countable type for maps. A function $f: Y \rightarrow Y$ on linear space has *countable type* if there exists a countable set Z in Y such that for each $y \in Y$, $f(y) \in \text{span}(\{y\} \cup Z)$.

If each map $f: Y \rightarrow Y$ on a topological linear space has countable type, we say that Y has *countable type for maps*.

Observe that if Y is \aleph_0 -dimensional, i.e. Y has a Hamel basis of cardinality at most \aleph_0 , then Y has countable type for maps. The aim of this section is to show that there exist normed linear spaces having countable type for maps which are not \aleph_0 -dimensional.

3.1. DEFINITION. Let X be a linear space, and n a positive integer. A finite subset $\{x_1, \dots, x_k\}$ of X is said to be *n-linearly essential* if $\sum_1^k \lambda_i x_i \neq 0$ for all λ_i such that $1/n \leq |\lambda_i| \leq n$ for each i .

The following lemma is trivial; nonetheless it will be quite useful for establishing linear independence of a set which is constructed by “approximation” to a sequence of linearly independent sets.

3.2. LEMMA. (1) *A subset K of a linear space is linearly independent if and only if each finite subset of K is n-linearly essential for each n .*

(2) *If $\{x_1, \dots, x_k\}$ is an n-linearly essential subset of a metric linear space, there exists $\varepsilon > 0$ such that, if $d(x_i, y_i) < \varepsilon$ for each i , then the set $\{y_1, \dots, y_k\}$ is also n-linearly essential.* \square

3.3. DEFINITION. Let X be a linear space, and $g: A \rightarrow X$ a function defined on a subset of X . A subset P of A is said to be *g-independent* if the following conditions are satisfied:

- (i) $g|P$ is injective;
- (ii) $P \cap g(P) = \emptyset$; and
- (iii) $P \cup g(P)$ is linearly independent.

3.4. PROPOSITION. *Let X be a metric linear space, and $g: A \rightarrow X$ a map defined on a separable, topologically complete subset. If A contains an uncountable g-independent set, then A contains a g-independent Cantor set.*

PROOF. Let d be a metric on X , and choose a complete metric ρ on A . For each $x \in A$ and $\varepsilon > 0$, let $B(x, \varepsilon) = \{a \in A: \rho(a, x) \leq \varepsilon\}$. Since each separable metric space is the union of a countable set and a perfect set (each point a limit point), the hypothesis implies that A contains a perfect g -independent set P . Using finite disjoint unions of balls about points of P , we may construct a Cantor set K in the complete space A by the standard procedure; a little extra care will ensure that K is g -independent. It suffices to describe the first two steps in the inductive construction.

Pick any $p_1 \in P$. Since $g(p_1) \neq p_1$, there exists $0 < \varepsilon_1 < 1$ such that $B(p_1, \varepsilon_1) \cap g(B(p_1, \varepsilon_1)) = \emptyset$. Let $B_1 = B(p_1, \varepsilon_1)$. Since the set $\{p_1, g(p_1)\}$ is

linearly independent, we may assume by Lemma 3.2(2) that ε_1 is sufficiently small so that, for any $F \subseteq B_1 \cup g(B_1)$ such that F contains at most a single point from each of B_1 and $g(B_1)$, F is 1-linearly essential. Let $K_1 = B_1$.

Since P is perfect, there exist distinct points $p_{1,0}$ and $p_{1,1}$ in $P \cap B_1$. Choose $0 < \varepsilon_2 < \frac{1}{2}$ such that, for $B_{1,0} = B(p_{1,0}, \varepsilon_2)$ and $B_{1,1} = B(p_{1,1}, \varepsilon_2)$, we have $B_{1,0} \cup B_{1,1} \subseteq B_1$, $B_{1,0} \cap B_{1,1} = \emptyset$, and $g(B_{1,0}) \cap g(B_{1,1}) = \emptyset$. Since the set $\{p_{1,0}, p_{1,1}, g(p_{1,0}), g(p_{1,1})\}$ is linearly independent and has size 4, we may also assume that ε_2 is small enough so that, for any $F \subseteq B_{1,0} \cup B_{1,1} \cup g(B_{1,0}) \cup g(B_{1,1})$ such that F contains at most a single point from each of $B_{1,0}, B_{1,1}, g(B_{1,0})$, and $g(B_{1,1})$, F is 2-linearly essential. Set $K_2 = B_{1,0} \cup B_{1,1}$.

Continuing with this procedure in the standard manner, we obtain a nested sequence (K_n) of closed sets in A . Let $K = \bigcap_1^\infty K_n$. The requirements of the type $B_{1,0} \cup B_{1,1} \subseteq B_1$ and $B_{1,0} \cap B_{1,1} = \emptyset$, together with the requirement that $\varepsilon_n \rightarrow 0$ and the fact that ρ is a complete metric, show that K is a Cantor set. The requirements of the type $g(B_{1,0}) \cap g(B_{1,1}) = \emptyset$ show that $g|_K$ is injective. Since $g(K) \subseteq g(B_1)$ and $K \subseteq B_1$, $K \cap g(K) = \emptyset$. And finally, the n -linearly essential requirement at the n th-stage of construction ensures that each finite subset of $K \cup g(K)$ is n -linearly essential for each n , hence $K \cup g(K)$ is linearly independent by Lemma 3.2(1). Thus K is g -independent. \square

REMARK. If we assume only that A contains an uncountable linearly independent set, the above construction shows that A contains a linearly independent Cantor set.

3.5. PROPOSITION. *A function $f: Y \rightarrow Y$ on a linear space has countable type if and only if it satisfies the following conditions:*

- (i) every f -independent set is countable; and
- (ii) for every countable set $P \subseteq Y$, there exists a countable set $\hat{P} \subseteq Y$ such that $f(\text{span } P) \subseteq \text{span } \hat{P}$.

PROOF. Suppose first that f has countable type; let Z be a countable subset of Y such that $f(y) \in \text{span}(\{y\} \cup Z)$ for each y . Let $T = \{y \in Y: f(y) \notin \text{span}\{y\}\}$. Then for each $y \in T$, $\text{span } Z \cap \text{span}\{y, f(y)\} \neq \{0\}$. Consider any f -independent set A . We have $A \subseteq T$, and for each $a \in A$ we may choose $s_a \in (\text{span } Z \cap \text{span}\{a, f(a)\}) \setminus \{0\}$. Since A is f -independent, the set $\{s_a: a \in A\}$ is a linearly independent subset of $\text{span } Z$, and is therefore countable. It follows that A is countable, and condition (i) is satisfied. Condition (ii) is clear, since $f(\text{span } P) \subseteq \text{span}(P \cup Z)$ for every $P \subseteq Y$.

Conversely, assume conditions (i) and (ii). It is easily seen that in the collection of all f -independent subsets, partially ordered by inclusion, every chain has an upper bound. Thus there exists a maximal f -independent set Q , which by hypothesis is countable. If $Q = \emptyset$, then $f(y) \in \text{span}\{y\}$ for each $y \neq 0$, and f obviously has countable type. Otherwise, construct a tower (P_i) of countable sets by taking $P_1 = Q$, and $P_{n+1} = P_n \cup \hat{P}_n$ for $n \geq 1$. Take $Z = \bigcup_1^\infty P_n$. Then Z is countable, $Q \subseteq Z$ and $f(\text{span } Z) \subseteq \text{span } Z$. We claim that $f(y) \in \text{span}(\{y\} \cup Z)$ for each y . This is clear for $y \in \text{span } Z$ so consider $y \notin \text{span } Z$. The set $Q \cup \{y\}$ cannot be f -independent, and one of the following occurs:

- (1) $f(y) = f(q)$ for some $q \in Q$. Then $f(y) \in \text{span } Z$.
- (2) $y = f(q)$ for some $q \in Q$. Then $y \in \text{span } Z$.

(3) $\{y, f(y)\} \cup Q \cup f(Q)$ is not linearly independent. in this case, either

$$f(y) \in \text{span}(\{y\} \cup Q \cup f(Q)) \subseteq \text{span}(\{y\} \cup Z),$$

or

$$y \in \text{span}(Q \cup f(Q)) \subseteq \text{span } Z.$$

Thus f has countable type. \square

We now come to the main result in this section.

3.6. THEOREM. *Let X be an infinite-dimensional separable, complete metric linear space, and Y a linear subspace with the property (*) of Theorem 2.1. Then*

- (1) Y is dense in X ;
- (2) Y is Baire space; and
- (3) Y has countable type for maps.

PROOF. We show first that Y intersects every dense G_δ -subset of X . Every such set A must contain an uncountable linearly independent set, since otherwise $\text{span } A$ is an \aleph_0 -dimensional linear subspace, and $X = \text{span } A \cup (X \setminus A)$ would be first category. Then A contains a linearly independent Cantor set K (see the remark at the end of proof of Proposition 3.4; see also [6]). Let $h: K_1 \rightarrow K_2$ be any homeomorphism between disjoint Cantor subsets of K . Then $h \in \mathcal{K}(X)$, so $Y \cap A \supseteq Y \cap K_1 \neq \emptyset$.

In particular, \bar{Y} must have nonempty interior in X , which implies that $\bar{Y} = X$. Further, the fact that Y intersects every dense G_δ -subset of X means that Y is second category, which implies that Y is a Baire space.

We now show that every map $f: Y \rightarrow Y$ has countable type. Since X is complete, f extends to a map $g: A \rightarrow X$ for some G_δ -subset A of X . Suppose that A contains an uncountable g -independent set. By Proposition 3.4, A contains a g -independent Cantor set K . Then $g|K$ is a member of the collection $\mathcal{K}(X)$, and by hypothesis there exists $x \in K \cap Y$ such that $g(x) \notin Y$. But this contradicts the fact that $g(x) = f(x) \in Y$. Thus every g -independent subset of A is countable, and in particular, every f -independent set is countable. Therefore, condition (i) of Proposition 3.5 is satisfied.

It remains to verify condition (ii). For any countable set $P \subseteq Y$, $\text{span } P$ is σ -compact, and $f(\text{span } P)$ is σ -compact. If some compactum in Y contains an uncountable linearly independent set, then Y contains a linearly independent Cantor set, but this contradicts property (*). (If $K \subseteq Y$ is a linearly independent Cantor set, let $h: K_1 \rightarrow K_2$ be any homeomorphism between Cantor subsets of K ; by assumption there exists $x \in K_1$ such that $h(x) \in Y$, which is a contradiction.) Thus each compactum in Y lies in an \aleph_0 -dimensional linear subspace, and $f(\text{span } P) \subseteq \text{span } \hat{P}$ for some countable set \hat{P} . This completes the proof that Y has countable type for maps, Proposition 3.5. \square

4. Domain invariance. Let Y be a space. Suppose that, for every injective map $g: U \rightarrow Y$ with domain an open subset of Y , there exists a nonempty open $V \subseteq U$ such that $g|V$ is an open imbedding in Y . Then we say that Y has *restricted domain invariance*.

We may suppose that V is dense in U , since the condition can be applied to every restriction $g|W$ to an open $W \subseteq U$.

For Y a normed linear space, it suffices to verify the condition for every injective map $f: Y \rightarrow Y$, since there is an open imbedding of Y into every nonempty open subset.

The aim of this section to prove that every normed linear space with the Baire property and with countable type for maps has restricted domain invariance.

4.1. PROPOSITION. *Let Y be a normed linear space, F a finite-dimensional linear subspace, and $V \subseteq Y \setminus F$ an open subset. Let $\lambda: V \rightarrow (0, \infty)$ and $\alpha: V \rightarrow F$ be maps such that the map $f: V \rightarrow Y$ defined by $f(v) = \lambda(v) \cdot v + \alpha(v)$ is injective. Then f is an open imbedding.*

PROOF. Since the restriction $f|_W$ to any open $W \subseteq V$ is a map with the same properties, it suffices to show that $f(V)$ is open.

Consider an arbitrary point $p \in V$. We may assume that $\lambda(p) = 1$ and $\alpha(p) = 0$, thus $f(p) = p$. Let $E = \text{span}(\{p\} \cup F)$. Note that $f(E \cap V) \subseteq E$. Choose $\delta > 0$ and a compact neighborhood B of 0 in F such that $D = [1 - \delta, 1 + \delta] \cdot p + B \subseteq V$. Then D is a compact neighborhood of p in E . Since f is injective and E is finite dimensional, $f|_D: D \rightarrow E$ is an imbedding and $f(D)$ is a neighborhood of $f(p) = p$ in E [2]. Thus p is a stable point of $f(D)$, and any map $\tilde{f}: D \rightarrow E$ which is sufficiently close to $f|_D$ must cover p .

For each $v \in V$, let $E_v = \text{span}(\{v\} \cup F)$, and define the homeomorphism $h_v: E \rightarrow E_v$ by

$$h_v(t \cdot p + s) = t \cdot v + s$$

for $t \in (-\infty, \infty)$ and $s \in F$. Let $D_v = h_v(D) = [1 - \delta, 1 + \delta] \cdot v + B$. Then for all v near p , $D_v \subseteq V$ and $f(D_v) \subseteq E_v$. Define a map $\tilde{f}_v: D \rightarrow E$ by $\tilde{f}_v = h_v^{-1} \circ f \circ h_v|_D$. Then $\tilde{f}_v \rightarrow f|_D$ as $v \rightarrow p$. Hence for all v sufficiently close to p , $p \in \tilde{f}_v(D)$, which means that $v \in f(D_v)$. Thus $f(V)$ is a neighborhood of $p = f(p)$. \square

We now come to the main result in this section.

4.2. THEOREM. *Let Y be a normed linear space with the Baire property and with countable type for maps. Then Y has restricted domain invariance.*

PROOF. We may assume Y is infinite dimensional [2]. By the remarks at the beginning of this section, we need only consider an injective map $f: Y \rightarrow Y$. Let $Z \subseteq Y$ be a countable set such that $f(y) \in \text{span}(\{y\} \cup Z)$ for each y . There is a tower of compacta (A_n) such that $\text{span } A_n$ is finite dimensional for each n , and $\bigcup_1^\infty A_n = \text{span } Z$. For each n , set

$$Y_n = \{y \in Y : \text{for some } \lambda \in [-n, n], f(y) - \lambda \cdot y \in A_n\}.$$

By compactness of $[-n, n]$ and A_n , each Y_n is closed in Y . Since $\bigcup_1^\infty Y_n = Y$, and Y is a Baire space, some Y_n has nonempty interior. Since $\text{span } A_n$ is finite dimensional, there exists a nonempty open set $W \subseteq Y_n \setminus \text{span } A_n$. Then for each $w \in W$, there is a unique $\lambda(w) \in [-n, n]$ such that $f(w) - \lambda(w) \cdot w \in A_n$ (otherwise, $W \cap \text{span } A_n \neq \emptyset$). Furthermore, the compactness of $[-n, n]$ and A_n , and the continuity of f , show that the assignment $w \rightarrow \lambda(w)$ is continuous. Let $\lambda: W \rightarrow [-n, n]$ denote this map, and let $\alpha: W \rightarrow A_n$ denote the map defined by $\alpha(w) = f(w) - \lambda(w) \cdot w$. If $\lambda(W) = \{0\}$, we have $f(W) \subseteq A_n$. But this is impossible, since f is injective, the open set W contains cells of all dimensions, and A_n is finite dimensional. Thus, there exists a nonempty open $V \subseteq W$ with either $\lambda(V) \subseteq [-n, 0)$ or $\lambda(V) \subseteq (0, n]$. In the latter

case an, application of Proposition 4.1, with $F = \text{span } A_n$, shows that $f|V$ is an open imbedding. And in the former case, the map $(-1) \cdot f|V$ is an open imbedding, hence so is $f|V$. This completes the proof of restricted domain invariance for Y . \square

4.3. COROLLARY. *Every separable Banach space contains a dense linear subspace Y such that:*

- (1) Y is a Baire space; and
- (2) Y has restricted domain invariance.

PROOF. Combine Theorems 2.1, 3.6, and 4.2. \square

4.4. COROLLARY TO COROLLARY. *Every separable Banach space contains a dense linear subspace Y such that for all closed linear subspaces E and F of Y of finite codimension the following statements are equivalent:*

- (1) E and F are homeomorphic;
- (2) E and F are linearly homeomorphic; and
- (3) $\text{codim } E = \text{codim } F$.

PROOF. Let Y be as in Corollary 4.3 and let E and F be closed linear subspaces of Y of finite codimension.

That (3) implies (2) follows from [1, III, Proposition 1.5]. Clearly, (2) implies (1). Suppose that E and F are homeomorphic and that $n = \text{codim } E - \text{codim } F > 0$. By [1, III, Proposition 1.5], $E \times \mathbf{R}^n$ and F are linearly homeomorphic. Consequently, E and $E \times \mathbf{R}^n$ are homeomorphic. Since Y has countable type for maps, every closed linear subspace of Y has the same property. From Theorem 4.2 we therefore conclude that E has restricted domain invariance, which contradicts E being homeomorphic to $E \times \mathbf{R}^n$. \square

5. Further applications and remarks. By Corollary 4.3, there exists an infinite-dimensional pre-Hilbert space L with restricted domain invariance. Consequently, $L \neq L \times E$ for any nondegenerate space E . This result generalizes Pol [7] and answers Question LS12 in Geoghegan [3].

In fact, there exists an infinite-dimensional pre-Hilbert space L such that $L \times F \not\approx L \times G$ for any linear spaces F and G with different dimensions (finite or ∞). Let Y be an infinite-dimensional pre-Hilbert space which is a Baire space and which has countable type for maps. It is easily seen that every closed linear subspace in Y inherits these properties, and therefore has restricted domain invariance. Take L to be an infinite-dimensional closed linear subspace of Y with infinite codimension. Suppose $L \times F \approx L \times G$ for certain linear spaces F and G with $0 \leq \dim F < \dim G \leq \infty$. Consider a closed linear subspace M in Y containing L , such that the codimension of L in M equals $\dim F$. Since M is closed in Y , M has restricted domain invariance. We have $M \approx L \times F$, and $G \approx F \times E$ for some nondegenerate space E . Then the supposition that $L \times F \approx L \times G$ implies $M \approx M \times E$, a contradiction.

A linear space Y is said to have *few operators* if every bounded linear operator $A: Y \rightarrow Y$ has the form $A = \lambda I + B$ for some scalar λ and some operator B with finite-dimensional range. In [5], I shall prove that every Baire linear space with countable type for maps has few operators.

The method used in the proofs of Theorems 2.1 and 3.6 was inspired by Sierpiński [8]. It improves the method used in [4] and can be used to construct a topological

group G such that every continuous function $f: G \rightarrow G$ is either constant or a translation. This answers a question in [4].

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