COMPOSITIO MATHEMATICA

J. VAN MILL

Topological equivalence of certain function spaces

Compositio Mathematica, tome 63, nº 2 (1987), p. 159-188.
<http://www.numdam.org/item?id=CM_1987__63_2_159_0>

Compositio Mathematica 63: 159–188 (1987)

Topological equivalence of certain function spaces

J. van MILL

Subfaculteit Wiskunde en Informatica, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands; and Mathematisch Instituut, Universiteit van Amsterdam, Roetersstraat 15, 1018 WB Amsterdam, The Netherlands

Received 16 April 1985; accepted in revised form 22 December 1986.

0. Introduction

If X is a space then $C^*(X)$ denotes the set of continuous bounded real-valued functions on X endowed with the topology of uniform convergence. There are classification results for these spaces. For details see [18] and [6]. There is another topology on function spaces that is of interest in abstract functional analysis, namely the topology of point-wise convergence. For any space X, the set of continuous real-valued bounded functions on X endowed with the topology of point-wise convergence shall be denoted by $C_p^*(X)$. For a recent survey on point-wise convergence function spaces, see [16]. So far, no classification results have been obtained for the spaces $C_p^*(X)$. Results in [3], [4] and [17] indicate that a classification result for the spaces $C_p^*(X)$ in the linear world promises to be extremely complicated. Trying to get a topological classification result for the spaces $C_p^*(X)$ seems to be more promising since no obstacles for obtaining topological homeomorphisms are known. The aim of this paper is to prove that if X is any countable, metric space which fails to be locally compact at some point then $C_p^*(X)$ is homeomorphic to the countable infinite product

$$l_f^2 \times l_f^2 \times l_f^2 \times \ldots,$$

where $l_f^2 = \{x \in l^2 \colon x_i = 0 \text{ for almost all } i\}$ and l^2 denotes the separable Hilbert space of course. It is clear that "countable" is essential in this result. In addition, "metric" is essential by results in [14]. I conjecture that "fails to be locally compact at some point" can be replaced by "is not discrete".

I am indebted to the referee for many valuable comments.

1. Preliminaries

Unless otherwise stated, all spaces under discussion are separable metric. All mappings are continuous functions. Let X and Y be spaces. A *homotopy* from X to Y is a map $F: X \times I \to Y$ (I denotes the interval $[0, 1]$ of course) and for $t \in I$ we define $F_t: X \to Y$ by $F_t(x) = F(x, t)$. If $F: X \times I \to Y$ is a homotopy and $\mathcal{U}$ is an open cover of Y, then we say that F is *limited by* $\mathcal{U}$ provided that for each $x \in X$ there exists an element $U \in \mathcal{U}$ containing $F(\{x\} \times I)$. Now let $\mathcal{U}$ be a collection of open subsets of Y. Mappings f, g: $X \to Y$ are called *$\mathcal{U}$-close* if for each $x \in X$ with $f(x) \neq g(x)$ there exists a $U \in \mathcal{U}$ containing both $f(x)$ and $g(x)$ (note that we did not require $\mathcal{U}$ to cover Y). Observe that if f: $Y \to Y$ is $\mathcal{U}$-close to the identity then f is *supported on* $\bigcup\mathcal{U}$, i.e. f restricts to the identity mapping on $Y \setminus \bigcup\mathcal{U}$. The identity mapping on X shall be denoted by 1_X or, if no confusion seems likely, simply by 1. We shall frequently use the following result due to Anderson [1].

1.1. Inductive convergence criterion. *Suppose that a sequence $\{h_n\}_1^\infty$ of homeomorphisms of a compact space X is chosen inductively so that each h_n is sufficiently close to the identity. Then* $\lim_{n\to\infty} h_n \circ \cdots \circ h_1$ *exists and is a homeomorphism of X.*

We consider the Hilbert cube $Q = \Pi_1^\infty[-1, 1]_i$, with metric $d(x, y) = \Sigma_1^\infty 2^{-i}|x_i - y_i|$. A *Hilbert cube* is a space homeomorphic to Q. We shall now define several notions in Q which are all stated in purely topological terms. For this reason we assume that we defined these concepts simultaneously for spaces homeomorphic to Q.

A closed subset F of a space X is a *Z-set* in X if every map f: $Q \to X$ can be approximated by maps g: $Q \to X$ with $g(Q) \cap F = \emptyset$. Note that every *endface* of Q (a subset of the form $\pi_i^{-1}(\{-1\})$ or $\pi_i^{-1}(\{1\})$, where π_i: $Q \to [-1, 1]_i$ is the projection) is a Z-set in Q. We let $\mathcal{Z}(X)$ denote the collection of all Z-sets in X. A countable union of Z-sets is called a *σ-Z-set*. We let $\mathcal{Z}_\sigma(X)$ denote the collection of σ-Z-sets in X. An imbedding h: $X \to Y$ is a *Z-imbedding* if $h(X)$ is a Z-set. A *z-σ*-pair in X is a pair (A, B) with $A \in \mathcal{Z}(X)$, $B \in \mathcal{Z}_\sigma(X)$ and $B \subseteq A$. If (A, B) and (E, F) are two z-σ-pairs then we write $(A, B) \leqslant (E, F)$ whenever $A \subseteq E$ and $F \cap A = B$.

Observe that "$\leqslant$" is a partial order on the family of all z-σ-pairs in X. For a discussion of Z-sets see [6] and [9]. We shall now define two important concepts. A *skeleton* in (a topological copy of) Q is an increasing sequence $A_1 \subseteq A_2 \subseteq \ldots$ of Z-sets in Q having the following absorption property:

For every $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *and for every* $Z \in \mathscr{Z}(Q)$ *there exist a homeomorphism* $h: Q \to Q$ *and an* $m \in \mathbb{N}$ *such that* $d(h, 1) < \varepsilon$, $h|A_n = 1$, *and* $h(Z) \subseteq A_m$.

The idea of a skeleton for a given class of compacta is due independently to Anderson [2] and Bessaga-Pelczyński [5].

1.2. THEOREM. *For every* $n \in \mathbb{N}$ *let* $A_n = \Pi_{i=1}^{\infty} [-1 + 1/n, 1 - 1/n]_i$. *Then the sequence* $\{A_n\}_1^{\infty}$ *is a skeleton in* Q.

For a proof of this basic result due to Anderson and Bessaga-Pelczyński, see [9], chapter 5.

If the sequence $\{A_n\}_1^{\infty}$ is a skeleton in Q then $A = \bigcup_1^{\infty} A_n$ is called a *skeletoid*.

1.3. THEOREM. *If* A *and* B *are skeletoids in* Q *and* $K \in \mathscr{Z}_\sigma(Q)$ *then for every collection* $\mathscr{U}$ *of open subsets of* Q *there is a homeomorphism* $h: Q \to Q$ *that is* $\mathscr{U}$*-close to* 1 *while moreover* $h((A \cup K) \cap \bigcup\mathscr{U}) = B \cap \bigcup\mathscr{U}$.

For details see e.g. [6], pp. 124, 129 and 131.

Let X be a compact space. An *isotopy* on X is a homotopy $H: X \times I \to X$ such that for each $t \in I$ the mapping $H_t: X \to X$ is a homeomorphism. We shall also need the following consequence of a result due to Anderson-Chapman. For details see [9], §19.

1.4. THEOREM. *Let* X *be a compact space and let* $H: X \times I \to Q$ *be a homotopy such that* H_0 *and* H_1 *are* $\mathscr{Z}$*-imbeddings. Then for every open covering* $\mathscr{U}$ *of* Q *such that* H *is limited by* $\mathscr{U}$ *there is an isotopy* $F: Q \times I \to Q$ *having the following properties:*
1. F *is limited by* $\mathscr{U}$,
2. $F_0 = 1$ *and* $F_1 \circ H_0 = H_1$.

Finally, we shall need the following result, [9], 11.2.

1.5. THEOREM. *If* (A, A_0) *is a compact pair and* $f: A \to Q$ *is a map such that* $f|A_0$ *is a Z-imbedding, then there is a Z-imbedding* $g: A \to Q$ *which agrees with* f *on* A_0. *Moreover,* g *can be constructed arbitrarily close to* f *and such that* $g(A \backslash A_0)$ *misses a pre-given* σ*-Z-set in* Q.

2. $\mathbb{N}$-skeletons

In this section we shall define the concept of an $\mathbb{N}$-skeleton, which is roughly speaking a decreasing sequence of skeletons with some absorption property. In addition, we prove that $\mathbb{N}$-skeletons are "unique" up to homeomorphism.

Let X be a compact space. *A $\mathscr{Z}$-matrix* in X is a collection $\mathscr{A} = \{A_m^n: n, m \in \mathbb{N}\}$ of Z-sets in X having the following properties:
1. $A_1^n = 0$ for every $n \in \mathbb{N}$,
2. $A_m^n \subseteq A_{m+1}^n$ for all $n, m \in \mathbb{N}$, and
3. $A_m^{n+1} \subseteq A_m^n$ for all $n, m \in \mathbb{N}$.

If $\mathscr{A} = \{A_m^n: n, m \in \mathbb{N}\}$ is a $\mathscr{Z}$-matrix then for each $n \in \mathbb{N}$ the union of the nth row of $\mathscr{A}$ shall be denoted by $\mathscr{A}(n)$, i.e.

$$\mathscr{A}(n) = \bigcup_{m=1}^{\infty} A_m^n.$$

Also, for all $n, m \in \mathbb{N}$ put

$$\mathscr{A}_m^n = A_m^n \cap \mathscr{A}(n+1).$$

Observe that $(A_m^n, \mathscr{A}_m^n)$ is a z-σ-pair. A decreasing collection

$$Z_1 \supseteq S_1 \supseteq Z_2 \supseteq S_2 \supseteq \cdots \supseteq Z_n \supseteq S_n$$

of subsets of X is called an *n-sieve* if
1. $Z_i \in \mathscr{Z}(X)$ for $i \leqslant n$, and
2. $S_i \in \mathscr{Z}_\sigma(X)$ for $i \leqslant n$.

We are now in a position to define the concept of an $\mathbb{N}$-skeleton. An *$\mathbb{N}$-skeleton* in X is a $\mathscr{Z}$-matrix $\mathscr{A} = \{A_m^n: n, m \in \mathbb{N}\}$ having the following absorption property:

For every $\varepsilon > 0$, for every $n \in \mathbb{N}$, for every decreasing sequence $i_1 \geqslant \cdots \geqslant i_n$ of natural numbers and for every n-sieve $Z_1 \supseteq S_1 \supseteq \cdots \supseteq Z_n \supseteq S_n$ in X such that for every $k \leqslant n$,

$$(A_{i_k}^k, \mathscr{A}_{i_k}^k) \leqslant (Z_k, S_k),$$

there exist a homeomorphism $h: X \to X$ and a decreasing sequence $j_1 \geqslant \cdots \geqslant j_n$ of natural numbers such that
1. $d(h, 1) < \varepsilon$,
2. $\forall k \leqslant n \; \forall l \leqslant i_k: h(A_l^k) = A_l^k$,
3. $\forall k \leqslant n: (h(Z_k), h(S_k)) \leqslant (A_{j_k}^k, \mathscr{A}_{j_k}^k)$.

Observe that by 2 many elements of the $\mathbb{N}$-skeleton are kept invariant under h, but that h is nowhere required to be the identity.

2.1. LEMMA. *Let X be a compact space and let $\mathscr{A} = \{A^n_m\colon n, m \in \mathbb{N}\}$ be an $\mathbb{N}$-skeleton in X. If $h\colon X \to X$ is homeomorphism then $h(\mathscr{A}) = \{h(A^n_m)\colon n, m \in \mathbb{N}\}$ is also an $\mathbb{N}$-skeleton.*

Proof. Use that h is uniformly continuous and that $\mathscr{Z}(X)$ is invariant under homeomorphisms of X. □

We now come to the main result in this section which we prove be a complexification of a standard back and forth technique, cf. [6], chapter 4.

2.2. THEOREM. *Let X be a compact space and let $\mathscr{A} = \{A^n_m\colon n, m \in \mathbb{N}\}$ and $\mathscr{B} = \{B^n_m\colon n\ m \in \mathbb{N}\}$ be $\mathbb{N}$-skeletons in X. Then for every $\varepsilon > 0$ there is a homeomorphism $h\colon X \to X$ having the following properties:*
1. $d(h, 1) < \varepsilon$,
2. $\forall n \in \mathbb{N}\colon h(\mathscr{A}(n)) = \mathscr{B}(n)$.

Proof. The homeomorphism h will have the form $\lim_{m\to\infty} h_m \circ \cdots \circ h_1$ for some inductively constructed sequence $\{h_m\}_{m=1}^{\infty}$ of homeomorphisms of X. From the construction it will be clear that each h_m can be chosen arbitrarily close to the identity. So without further mentioning it is understood that each h_m is chosen in accordance with the Inductive convergence criterion 1.1. If in addition we also make sure that $\Sigma_{m=1}^{\infty} d(h_m, 1) < \varepsilon$, then in will follow that $d(h, 1) < \varepsilon$.

We shall prove that there exist

a) for each $k \in \mathbb{N}$ a strictly increasing function $\xi_k\colon \mathbb{N} \to \mathbb{N}$ having the following properties:

A) $\xi_k(1) = 1$ and $\xi_k(2) = 2$,

B) $\forall m \in \mathbb{N}\colon \xi_1(m) \geqslant \xi_2(m-1) \geqslant \cdots \geqslant \xi_m(1)$,

b) for each $k \in \mathbb{N}$ a strictly increasing function $\eta_k\colon \mathbb{N}\backslash\{1\} \to \mathbb{N}$ having the following properties:

A) $\eta_k(2) = 1$,

B) $\forall m \geqslant 2\colon \eta_1(m) \geqslant \eta_2(m-1) \geqslant \cdots \geqslant \eta_{m-1}(2)$,

c) a sequence $\{h_m\}_1^{\infty}$ of homeomorphisms of X with $h_1 = 1$ such that for every $m \in \mathbb{N}$ if we put $f_m = h_m \circ \cdots \circ h$, then
d) $\forall k \leqslant m\ \forall 2 \leqslant l \leqslant m-(k-2)\colon f_m(A^k_{\xi_k(l-1)}) \subseteq B^k_{\eta_k(l)} \subseteq f_m(A^k_{\xi_k(l)})$,
e) $\forall k \leqslant m\colon (B^k_{\eta_k(m-(k-2))}, \mathscr{B}^k_{\eta_k(m-(k-2))}) \leqslant (f_m(A^k_{\xi_k(m-(k-2))}), f_m(\mathscr{A}^k_{\xi_k(m-(k-2))})$.

Suppose for a moment that we proved statements a) through e). Put $h = \lim_{m\to\infty} h_m \circ \cdots \circ h_1$. By the above remarks, h is a homeomorphism of X onto itself such that $d(h, 1) < \varepsilon$. Fix $k \geqslant 1$ and $k \geqslant 2$. By d), for every $m \geqslant l + (k - 2)$ we have

$$f_m(A^k_{\xi_k(l-1)}) \subseteq B^k_{\eta_k(l)} \subseteq f_m(A^k_{\xi_k(l)}).$$

The continuity of h and the compactness of X now easily imply that

$$h(A^k_{\xi_k(l-1)}) \subseteq B^k_{\eta_k(l)} \subseteq h(A^k_{\xi_k(l)})$$

(notice that k and l are fixed). Since ξ and η are strictly increasing, from this it now directly follows that

$$h(\mathscr{A}(k)) = \mathscr{B}(k) \quad \text{for every} \quad k \in \mathbb{N},$$

i.e. h is as required.

It remains to give a proof of the statements a) through e). We already know that

$$\xi_1(1) = 1, \quad \xi_1(2) = 2, \quad \eta_1(2) = 1 \quad \text{and} \quad h_1 = 1.$$

Since $A^1_1 = B^1_1 = 0$, we also have

$$h_1(A^1_{\xi_1(1)}) \subseteq B^1_{\eta_1(2)} \subseteq h_1(A^1_{\xi_1(2)}),$$

and

$$(B^1_{\eta_1(2)}, \mathscr{B}^1_{\eta_1(2)}) \leqslant (h_1(A^1_{\xi_1(2)}), h_1(\mathscr{A}^1_{\xi_1(2)})).$$

So now it is clear how to proceed inductively. Suppose that for certain $m \in \mathbb{N}$ we defined

f) $\forall k \leqslant m$ the function ξ_k on the set $\{1, 2, \ldots, m - (k - 2)\}$ such that for every $i \leqslant m + 1$ we have

$$\xi_1(i) \geqslant \xi_2(i - 1) \geqslant \cdots \geqslant \xi_i(1),$$

g) $\forall k \leqslant m$ the function η_k on the set $\{2, 3, \ldots, m - (k - 2)\}$ such that for every $2 \leqslant i \leqslant m + 1$ we have

$$\eta_1(i) \geqslant \eta_2(i - 1) \geqslant \cdots \geqslant \eta_{i-1}(2),$$

h) the homeomorphisms $\{h_1, \ldots, h_m\}$ such that if $f_m = h_m \circ \cdots \circ h_1$ then d) and e) hold.

Some remarks seem necessary. Among other things, we wish to construct a strictly increasing function $\xi_1: \mathbb{N} \to \mathbb{N}$. By the statement "$\xi_1$ on the set $\{1, 2, \ldots, m + 1\}$" we mean that we already know the values of the function we are constructing on the set $\{1, 2, \ldots, m + 1\}$. So in fact we are dealing with a partial function, the domain of which is $\{1, 2, \ldots, m + 1\}$, which, for convenience, we shall also denote by ξ_1. At stage $m + 1$ of the construction we shall enlarge the domain of the partial function ξ_1. So if we discuss ξ_1 later on, the domain of ξ_1 should be taken into consideration. The same remark applies to all of the functions ξ_k and η_k.

By f) we have

$$\xi_1(m + 1) \geqslant \xi_2(m) \geqslant \cdots \geqslant \xi_m(2)$$

which implies, by the definition of an $\mathbb{N}$-skeleton, that

$$A^1_{\xi_1(m+1)} \supseteq \mathscr{A}^1_{\xi_1(m+1)} \supseteq \cdots \supseteq A^m_{\xi_m(2)} \supseteq \mathscr{A}^m_{\xi_m(2)}.$$

Consequently,

$$f_m(A^1_{\xi_1(m+1)}) \supseteq f_m(\mathscr{A}^1_{\xi_1(m+1)}) \supseteq \cdots \supseteq f_m(A^m_{\xi_m(2)}) \supseteq f_m(\mathscr{A}^m_{\xi_m(2)})$$

is an m-sieve. From h) it follows that this m-sieve and the natural numbers $\eta_1(m + 1), \ldots, \eta_m(2)$ are "admissible" as input for the $\mathbb{N}$-skeleton $\mathscr{B}$. So by the definition of an $\mathbb{N}$-skeleton there is a (small) homeomorphism α: $X \to X$ and a sequence $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_m$ of natural numbers such that
i) $\forall k \leqslant m \, \forall l \leqslant \eta_k(m - (k - 2))$: $\alpha(B^k_l) = B^k_l$,
j) $\forall k \leqslant m$: $(\alpha f_m(A^k_{\xi_k(m-(k-2))}), \alpha f_m(\mathscr{A}^k_{\xi_k(m-(k-2))})) \leqslant (B^k_{i_k}, \mathscr{B}^k_{i_k})$.
It is clear that we may assume that for every $k \leqslant m$ we have

$$\eta_k(m - (k - 2)) < i_k.$$

We are now in a position to define for each $k \leqslant m + 1$ the function η_k in the point $(m + 1) - (k - 2)$, namely, we put

$$\eta_k((m + 1) - (k - 2)) = i_k \quad (k \leqslant m),$$

$$\eta_{m+1}(2) = 1.$$

Now consider the natural numbers $\xi_1(m + 1) \geqslant \cdots \geqslant \xi_m(2)$ and the m-sieve

$$B^1_{\eta_1(m+2)} \supseteq \mathscr{B}^1_{\eta_1(m+2)} \supseteq \cdots \supseteq B^m_{\eta_m(3)} \supseteq \mathscr{B}^m_{\eta_m(3)}.$$

By j), these are admissible as input for the $\mathbb{N}$-skeleton

$$\alpha f_m(\mathscr{A}) = \{\alpha f_m(A_j^i): i, j \in \mathbb{N}\}$$

(LEMMA 2.1). Consequently, there exist a (small) homeomorphism $\beta: X \to X$ and a sequence of natural numbers $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_m$ such that
k) $\forall k \leqslant m\ \forall l \leqslant \xi_k(m - (k - 2))$: $\beta\alpha f_m(A_l^k) = \alpha f_m(A_l^k)$,
l) $\forall k \leqslant m$: $(\beta(B^k_{\eta_k(m-(k-3))}), \beta(\mathscr{B}^k_{\eta_k(m-(k-3))})) \leqslant (\alpha f_m(A^k_{\varepsilon_k}), \alpha f_m(\mathscr{A}^k_{\varepsilon_k}))$.

It is clear that for every $k \leqslant m$ we may assume that

$$\xi_k(m - (k - 2)) < \varepsilon_k.$$

We are now in a position to define for each $k \leqslant m + 1$ the function ξ_k in the point $(m + 1) - (k - 2)$, namely, we put

$$\xi_k((m + 1) - (k - 2)) = \varepsilon_k \quad (k \leqslant m),$$

$$\xi_{m+1}(2) = 2.$$

$$\xi_{m+1}(1) = 1.$$

Also, define $h_{m+1} = \beta^{-1} \circ \alpha$. We claim that our choices are as required.

Claim 1. $\forall k \leqslant m + 1\ \forall 2 \leqslant l \leqslant (m + 1) - (k - 2)$: $f_{m+1}(A^k_{\xi_k(l-1)}) \subseteq B^k_{\eta_k(l)} \subseteq f_{m+1}(A^k_{\eta_k(l)})$.

Indeed, fix $k \leqslant m + 1$ and $2 \leqslant l \leqslant (m + 1) - (k - 2)$, arbitrarily.

Case 1. $k \leqslant m$ and $2 \leqslant l \leqslant m - (k - 2)$.

If $l = 2$ then $f_{m+1}(A^k_{\xi_k(l-1)}) = B^k_{\eta_k(l)} = \varnothing$, so then there is nothing to prove. So assume that $l > 2$. Then

$$f_{m+1}(A^k_{\xi_k(l-1)}) = \beta^{-1}\alpha f_m(A^k_{\xi_k(l-1)}) \overset{(k)}{=} \alpha f_m(A^k_{\xi_k(l-1)}) \overset{(h)}{\subseteq} \alpha(B^k_{\eta_k(l)}) \overset{(i)}{=} B^k_{\eta_k(l)},$$

and

$$B^k_{\eta_k(l)} \overset{(i)}{=} \alpha(B^k_{\eta_k(l)}) \overset{(h)}{\subseteq} \alpha f_m(A^k_{\xi_k(l)}) \overset{(k)}{=} \beta^{-1}\alpha f_m(A^k_{\xi_k(l)}) = f_{m+1}(A^k_{\xi_k(l)}).$$

Case 2. $k = m + 1$.

Then $l = 2$. Since $A^{m+1}_{\xi_{m+1}(1)} = A^{m+1}_1 = \varnothing$ and $B^{m+1}_{\eta_{m+1}(2)} = \varnothing$, there is nothing to prove.

Case 3. $k \leqslant m$ and $l = (m + 1) - (k - 2) = m - k + 3$.

Then $f_{m+1}(A^k_{\xi_k(m-k+2)}) = \beta^{-1}\alpha f_m(A^k_{\xi_k(m-k+2)}) \overset{(k)}{=} \alpha f_m(A^k_{\xi_k(m-k+2)}) \overset{(j)}{\subseteq} B^k_{\eta_k(m-k+3)}$, and by (l), $B^k_{\eta_k(m-k+3)} \subseteq \beta^{-1}\alpha f_m(A^k_{\xi_k(m-k+3)}) = f_{m+1}(A^k_{\xi_k(m-k+3)})$.

Claim 2. $\forall k \leqslant m + 1$: $(B^k_{\eta_k(m-k+3)}, \mathscr{B}^k_{\eta_k(m-k+3)}) \leqslant (f_{m+1}(A^k_{\xi_k(m-k+3)}), f_{m+1}(\mathscr{A}^k_{\xi_k(m-k+3)}))$.

Indeed, if $k = m + 1$ then $B^k_{\eta_k(m-k+3)} = 0$, so assume that $k \leqslant m$. Now the claim follows directly from (l).

This completes the inductive construction and therefore also the proof of the theorem. □

Let $\mathscr{A} = \{A^n_m : n, m \in \mathbb{N}\}$ be a $\mathscr{Z}$-matrix in the compact space X. Define the kernel, *ker* ($\mathscr{A}$), of $\mathscr{A}$ by

$$\ker(\mathscr{A}) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A^n_m.$$

2.3. COROLLARY. *Let X be a compact space and let $\mathscr{A}$ and $\mathscr{B}$ be $\mathbb{N}$-skeletons in X. Then for each $\varepsilon > 0$ there is a homeomorphism $h: X \to X$ such that*
1) $d(h, 1) < \varepsilon$,
2) $h(\ker(\mathscr{A})) = \ker(\mathscr{B})$. □

Observe that nowhere in this section we used specific properties of Z-sets, except that $\mathscr{Z}(X)$ is a topological class of compacta, i.e. if $h: X \to X$ is any homeomorphism then $h(\mathscr{Z}(X)) = \mathscr{Z}(X)$. So we could have stated our results in a more general form.

3. $\mathscr{Q}$-matrices

In this section we shall define the comcept of a $\mathscr{Q}$-matrix in Q and we shall prove that each $\mathscr{Q}$-matrix is an $\mathbb{N}$-skeleton.

Let $\mathscr{A} = \{A^n_m : n, m \in \mathbb{N}\}$ be a $\mathscr{Z}$-matrix in some compact space X.

By a *T-set* for $\mathscr{A}$ we shall mean a Z-set $B \subseteq X$ having the following property:

$\forall n_1 < n_2 < \cdots < n_m \in \mathbb{N}\ \forall i_1, i_2, \ldots, i_m \in \mathbb{N}$ we have

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \nsubseteq B \Rightarrow \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap B \in \mathscr{Z}\left(\bigcap_{k=1}^{m} A_{i_k}^{n_k}\right).$$

Let $\mathscr{T}(\mathscr{A})$ denote the collection of all T-sets for $\mathscr{A}$.

The proof of the following lemma is straightforward and is left to the reader.

3.1. Lemma. *Let X be a compact space and let $\mathscr{A} = \{A_m^n\colon n, m \in \mathbb{N}\}$ be a $\mathscr{Z}$-matrix in X. Then $\emptyset \in \mathscr{T}(\mathscr{A})$ and*
1. *if $E, F \in \mathscr{T}(\mathscr{A})$ then $E \cap F \in \mathscr{T}(\mathscr{A})$,*
2. *if $E, F \in \mathscr{T}(\mathscr{A})$ then $E \cup F \in \mathscr{T}(\mathscr{A})$.*

A *$\mathscr{Q}$-matrix* in Q is a $\mathscr{Z}$-matrix $\mathscr{A} = \{A_m^n\colon n, m \in \mathbb{N}\}$ having the following properties:
1. $\forall n \in \mathbb{N}$: $\{A_m^n\}_{m>1}$ is a skeleton,
2. $\forall n_1 < \cdots < n_m \in \mathbb{N}\ \forall i_1, \ldots, i_m \in \mathbb{N}\backslash\{1\}$; $\bigcap_{k=1}^m A_{i_k}^{n_k} \approx Q$,
3. $\forall n_1 < \cdots < n_m \in \mathbb{N}\ \forall i_1, \ldots, i_m \in \mathbb{N}\backslash\{1\}\ \forall p \in \mathbb{N}$: $\{\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_i^{n_m+p}\}_{i>1}$ is a skeleton in $\bigcap_{k=1}^m A_{i_k}^{n_k}$,
4. $\forall n, m \in \mathbb{N}$: $A_m^n \in \mathscr{T}(\mathscr{A})$.

In the remaining part of this section we shall prove that every $\mathscr{Q}$-matrix is an $\mathbb{N}$-skeleton. This result is our main tool in recognizing $\mathbb{N}$-skeletons.

3.2. Lemma. *Let $\mathscr{A} = \{A_m^n\colon n, m \in \mathbb{N}\}$ be a $\mathscr{Q}$-matrix in Q. Then*
1. $\forall n_1 < \cdots < n_m \in \mathbb{N}\ \ \forall i_1, \ldots i_m \in \mathbb{N}\backslash\{1\}$: $\mathscr{B} = \{\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_j^{n_m+i}\colon i, j \in \mathbb{N}\}$ *is a $\mathscr{Q}$-matrix in* $\bigcap_{k=1}^m A_{i_k}^{n_k}$.
2. $\forall p \geqslant 0$: $\{A_m^{n+p}\colon n, m \in \mathbb{N}\}$ *is a $\mathscr{Q}$-matrix.*

Proof. Observe that 2 is a triviality and 1 follows from the definitions and lemma 3.1. □

3.3. Lemma. *Let $\mathscr{A} = \{A_m^n\colon n, m \in \mathbb{N}\}$ be a $\mathscr{Q}$-matrix in Q. Then $\forall \varepsilon > 0$ $\forall 1$-sieve $Z_1 \supseteq S_1$ $\forall T \in \mathscr{Z}(Q)$ $\forall m \in \mathbb{N}$ such that*
1. $\exists p \in \mathbb{N}$ *such that* $A_q^1 \cap T \in \mathscr{Z}(A_q^1)$ *for every* $q \geqslant p$,
2. $(Z_1 \cap T, S_1 \cap T) \leqslant (A_m^1, \mathscr{A}_m^1)$,

there exist a homeomorphism $h\colon Q \to Q$ and an element $\bar{m} \in \mathbb{N}$ such that
3. $d(h, 1) < \varepsilon$ *and* $h|T = 1$,
4. $(h(Z_1), h(S_1)) \leqslant (A_{\bar{m}}^1, \mathscr{A}_{\bar{m}}^1)$.

Proof. Since $\{A_p^1: p \in \mathbb{N}\}$ is a skeleton in Q there exist $t \geqslant m + p$ and an imbedding $u: Z_1 \to A_t^1$ such that $u|Z_1 \cap A_m^1 = 1$ and $d(u, 1) < \varepsilon/3$. Since $\mathscr{A}_t^1 \cup (T \cap A_t^1)$ is a σ-Z-set in A_t^1 we may use Theorem 1.5 to get an imbedding $v: Z_1 \to A_t^1$ such that $d(v, u) < \varepsilon/3, v|T \cap Z_1 = u|T \cap Z_1 (=1)$ and $v(Z_1 \backslash T) \subseteq A_t^1 \backslash \mathscr{A}_t^1 \backslash T$. By Theorem 1.4 we may extend $v \cup 1_T$ to a homeomorphism $\bar{v}$ of Q such that $d(\bar{v}, 1) < 2\varepsilon/3$. Since $\mathscr{A}_t^1$ is a skeletoid in A_t^1 there exists a homeomorphism w of A_t^1 such that $w|A_t^1 \cap T = 1$ and $w((\mathscr{A}_t^1 \cup v(S_1))\backslash T) = \mathscr{A}_t^1 \backslash T$; we may require w to be so close to 1 as to assure the existence of a homeomorphism $\bar{w}$ of Q that extends $w \cup 1_T$ and satisfies $d(\bar{w}, 1) < \varepsilon/3$, see Theorems 1.3 and 1.4. We put $h = \bar{w} \circ \bar{v}$; then by abuse of notation,

$$d(h, 1) < \varepsilon, (h(Z_1 \backslash T), h(S_1 \backslash T)) \leqslant (A_t^1, \mathscr{A}_t^1) \quad \text{and} \quad h|T = 1.$$

Thus 2 and $t \geqslant m$ show that h and $\bar{m} = t$ have the desired properties. □

This Lemma has some nontrivial consequences.

3.4. Lemma. *Let $\mathscr{A} = \{A_m^n: n, m \in \mathbb{N}\}$ be a $\mathscr{Q}$-matrix in Q. Then $\forall \varepsilon > 0$ $\forall m \in \mathbb{N}\ \exists \delta > 0\ \forall 1$-sieve $Z_1 \supseteq S_1$ with*

$$(A_m^1, \mathscr{A}_m^1) \leqslant (Z_1, S_1), \tag{1}$$

$\forall T \in \mathscr{T}(\mathscr{A})$ $\forall$homeomorphism $h: T \to T$ $\forall p \in \mathbb{N}$ such that

$$\forall i \leqslant m: h(A_i^1 \cap T) = A_i^1 \cap T \quad \text{and} \quad h(\mathscr{A}_i^1 \cap T) = \mathscr{A}_i^1 \cap T, \tag{2}$$

$$d(h, 1) < \delta, \tag{3}$$

$$((h(Z_1 \cap T), h(S_1 \cap T)) \leqslant (A_p^1, \mathscr{A}_p^1), \tag{4}$$

there exist a homeomorphism $H: Q \to Q$ and an $\bar{m} \in \mathbb{N}$ such that

$$\forall i \leqslant m: H(A_i^1) = A_i^1 \quad \text{and} \quad H(\mathscr{A}_i^1) = \mathscr{A}_i^1, \tag{5}$$

$$d(H, 1) < \varepsilon, \tag{6}$$

$$(H(Z_1), H(S_1)) \leqslant (A_{\bar{m}}^1, \mathscr{A}_{\bar{m}}^1), \tag{7}$$

$$H|T = h. \tag{8}$$

Proof. Let $\varepsilon > 0$. If $m = 1$ put $\delta = \varepsilon$. If $m > 1$ let $0 < \delta = \delta_2 < \delta_3 < \cdots < \delta_{m+1} = \varepsilon$ be so small that

(∗) $\forall 2 \leqslant i \leqslant m$ $\forall Z, Z' \in \mathscr{Z}(A_i^1)$ $\forall$homeomorphism $f\colon Z \to Z'$ with $d(f, 1) < \delta_i$ there exists a homeomorphism $\bar{f}\colon A_i^1 \to A_i^1$ extending f with $d(\bar{f}, 1) < \frac{1}{2}\delta_{i+1}$.

That these δ's exist follows directly from Theorem 1.4. We claim that δ is as required. Let $Z_1 \supseteq S_1$ be a 1-sieve and choose an element $T \in \mathscr{T}(\mathscr{A})$, a homeomorphism $h\colon T \to T$ and a natural number p satisfying (1) through (4). By induction we shall prove that for every $i \leqslant m$ there exists a homeomorphism $h_i\colon A_i^1 \cup T \to A_i^1 \cup T$ such that

$$d(h_i, 1) < \delta_{i+1}, \tag{9}$$

$$\forall j \leqslant i\colon h_i(A_j^1) = A_j^1 \quad \text{and} \quad h_i(\mathscr{A}_j^1) = \mathscr{A}_j^1, \tag{10}$$

$$h_i|T = h \quad \text{and} \quad h_i(x) = h_{i-1}(x) \quad \text{for} \quad x \in A_{i-1}^1. \tag{11}$$

Put $h_1 = h$ (if $m = 1$ we are done already). Suppose that for $i - 1 < m$ the homeomorphism h_{i-1} has been constructed. If $A_i^1 \subseteq T \cup A_{i-1}^1$ then define $h_i = h_{i-1}$. If $A_i^1 \nsubseteq T \cup A_{i-1}^1$ then $A_i^1 \cap (T \cup A_{i-1}^1) \in \mathscr{Z}(A_i^1)$ (Lemma 3.1) from which follows that

$$E = A_{i-1}^1 \cup (A_i^1 \cap T) \in \mathscr{Z}(A_i^1).$$

By (2) and (10) we have $h_{i-1}(E) = E$ and since $d(h_{i-1}, 1) < \delta_i$, by (∗) we can extend $h_{i-1}|E$ to a homeomorphism $\alpha\colon A_i^1 \to A_i^1$ with $d(\alpha, 1) < \frac{1}{2}\delta_{i+1}$. Since $\alpha(\mathscr{A}_i^1)$ and $\mathscr{A}_i^1$ are both skeletoids for A_i^1, by Theorem 1.3 there is a homeomorphism $\beta\colon A_i^1 \to A_i^1$ such that

$$d(\beta, 1) < \tfrac{1}{2}\delta_{i+1}, \tag{12}$$

$$\beta|E = 1, \tag{13}$$

$$\beta\alpha(\mathscr{A}_i^1)\setminus E = \mathscr{A}_i^1\setminus E. \tag{14}$$

Define $h_i\colon A_i^1 \cup T \to A_i^1 \cup T$ by

$$\begin{cases} h_i(x) = h(x) & (x \in T), \\ h_i(x) = \beta\alpha(x) & (x \in A_i^1). \end{cases}$$

Then h_i is a homeomorphism such that

$$_i d(h_i, 1) < \delta_{i+1}, \tag{15}$$

$$_i\forall j \leqslant i: h_i(A_j^1) = A_j^1 \quad \text{and} \quad h_i(\mathscr{A}_j^1) = \mathscr{A}_j^1, \tag{16}$$

$$_i h_i | T = h. \tag{17}$$

It is clear that for this we only need to verify that $h_i(\mathscr{A}_i^1) = \mathscr{A}_i^1$. By (14) this follows from $h_i(\mathscr{A}_i^1 \cap E) = \mathscr{A}_i^1 \cap E$, which is a consequence of (10), (2) and (11).

Let $F = h_m$; Then $d(F, 1) < \varepsilon$ and by Theorem 1.4 we may extend F to a homeomorphism $\bar{F}: Q \to Q$ with $d(\bar{F}, 1) < \varepsilon$ (the straight-line homotopy in Q between $1_{T \cup A_m^1}$ and F is "ε-small", so the existence of $\bar{F}$ follows directly from Theorem 1.4).

Let $\bar{Z}_1 = \bar{F}(Z_1)$ and $\bar{S}_1 = \bar{F}(S_1)$, respectively. Then

$$(A_m^1, \mathscr{A}_m^1) \leqslant (\bar{Z}_1, \bar{S}_1), \tag{18}$$

and

$$(\bar{Z}_1 \cap T, \bar{S}_1 \cap T) \leqslant (A_p^1, \mathscr{A}_p^1). \tag{19}$$

Observe that (18) follows directly from (1) and $(16)_m$. Since $\bar{F}$ extends F and by $(17)_m$ F extends h, (19) follows from (4). Now put $\bar{T} = T \cup A_m^1$. We shall prove that for $q = \max(p, m)$ we have

$$(\bar{Z}_1 \cap \bar{T}, \bar{S}_1 \cap \bar{T}) \leqslant (A_q^1, \mathscr{A}_q^1). \tag{20}$$

This is a triviality since by (18) and (19), $\bar{Z}_1 \cap \bar{T} = \bar{Z}_1 \cap (T \cup A_m^1) = (\bar{Z}_1 \cap T) \cup A_m^1 \subseteq A_q^1$. Moreover,

$$\begin{aligned} \mathscr{A}_q^1 \cap \bar{Z}_1 \cap \bar{T} &= (\mathscr{A}_q^1 \cap (\bar{Z}_1 \cap T)) \cup (\mathscr{A}_q^1 \cap A_m^1) \\ &= (\bar{S}_1 \cap T) \cup \mathscr{A}_m^1 \\ &= (\bar{S}_1 \cap T) \cup (\bar{S}_1 \cap A_m^1) \\ &= \bar{S}_1 \cap \bar{T}. \end{aligned}$$

By Lemma 3.3, there are a homeomorphism $\alpha: Q \to Q$ and an element

$\bar{m} \in \mathbb{N}$ such that

$$d(\alpha, 1) < \varepsilon - d(\bar{F}, 1) \quad \text{and} \quad \alpha|\bar{T} = 1, \tag{21}$$

$$(\alpha(\bar{Z}_1), \alpha(\bar{S}_1)) \leqslant (A^1_{\bar{m}}, \mathscr{A}^1_{\bar{m}}). \tag{22}$$

It is easy to see that $H = \alpha \circ \bar{F}$ is as required. □

Let us now consider the following statement, where $\mathscr{A}$ is a $\mathscr{Q}$-matrix.

$S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$: $\exists \delta > 0$ $\forall n$-sieve $Z_1 \supseteq S_1 \supseteq \cdots \supseteq Z_n \supseteq S_n$ such that

1. *$\forall k \leqslant n$: $(A^k_{i_k}, \mathscr{A}^k_{i_k}) \leqslant (Z_k, S_k)$, $\forall T \in \mathscr{T}(\mathscr{A})$ $\forall$homeomorphism $h: T \to T$ $\forall \xi_1, \ldots, \xi_n \in \mathbb{N}$ with*
2. *$d(h, 1) < \delta$,*
3. *$\forall k \leqslant n$ $\forall l \leqslant i_k$: $h(A^k_l \cap T) = A^k_l \cap T$ and $h(\mathscr{A}^k_l \cap T) = \mathscr{A}^k_l \cap T$,*
4. *$\forall k \leqslant n$: $(h(Z_k \cap T), h(S_k \cap T)) \leqslant (A^k_{\xi_k}, \mathscr{A}^k_{\xi_k})$, there exist a homeomorphism $H: Q \to Q$ and elements $\bar{\xi}_1, \ldots, \bar{\xi}_n \in \mathbb{N}$ with*
5. *$d(H, 1) < \varepsilon$ and $H|T = h$,*
6. *$\forall k \leqslant n$ $\forall l \leqslant i_k$: $H(A^k_l) = A^k_l$ and $H(\mathscr{A}^k_l) = \mathscr{A}^k_l$,*
7. *$\forall k \leqslant n$: $(H(Z_k), H(S_k)) \leqslant (A^k_{\bar{\xi}_k}, \mathscr{A}^k_{\bar{\xi}_k})$.*

The number δ shall also be denoted by $\delta(\mathscr{A}, i_1 \ldots i_n, \varepsilon)$.

3.5. Lemma. *$\forall \mathscr{Q}$-matrix $\mathscr{A}$ $\forall \varepsilon > 0$ $\forall n \in \mathbb{N}$ $\forall$decreasing sequence $i_1 \geqslant \cdots \geqslant i_n$: $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$.*

Proof. For $n = 1$, simply apply Lemma 3.4. Suppose therefore that $S(\mathscr{A}, k_1 \geqslant \cdots \geqslant k_i, \varepsilon)$ is true for every $\mathscr{Q}$-matrix $\mathscr{A}$, for every $\varepsilon > 0$, for every $i \leqslant n - 1$ $(n \geqslant 2)$ and for every decreasing sequence $k_1 \geqslant \cdots \geqslant k_i$ of natural numbers.

Let $\mathscr{A} = \{A^n_m: n, m \in \mathbb{N}\}$ be a $\mathscr{Q}$-matrix, $\varepsilon > 0$, $i_1 \geqslant \cdots \geqslant i_n$, $Z_1 \supseteq S_1 \supseteq \cdots \supseteq Z_n \supseteq S_n$, $T \in \mathscr{T}(\mathscr{A})$, $h: T \to T$ and $\xi_1, \ldots, \xi_n \in \mathbb{N}$ be as in the definition of $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$ (we shall specify δ later of course). For every $2 \leqslant k \leqslant i_1$ let $\mathscr{B}_k = \{A^1_k \cap A^{1+n}_m: n, m \in \mathbb{N}\}$. Then $\mathscr{B}_k$ is a $\mathscr{Q}$-matrix in the Hilbert cube A^1_k, Lemma 3.2. In addition, put $\mathscr{E} = \{A^{n+1}_m: n, m \in \mathbb{N}\}$. Then $\mathscr{E}$ is a $\mathscr{Q}$-matrix, Lemma 3.2. Let $\gamma = \delta(\mathscr{E}, i_2 \geqslant \cdots \geqslant i_n, \varepsilon)$. Applying our induction hypothesis, by downward induction it is easy to find for every

$2 \leqslant k \leqslant i_1$ a $\delta(k) > 0$ such that

$$\delta(i_1) = \delta(\mathscr{B}_{i_1}, i_2 \geqslant \cdots \geqslant i_n, \gamma), \tag{1}$$

$$\text{if} \quad k < i_1 \quad \text{then} \quad \delta(k) \leqslant \delta(\mathscr{B}_k, i_2 \geqslant \cdots \geqslant i_n, \delta(k+1)), \tag{2}$$

$$\text{if} \quad k \leqslant l \leqslant i_1 \quad \text{then} \quad \delta(k) \leqslant \delta(l). \tag{3}$$

It will be convenient to write $\delta(i_1 + 1) = \gamma$. We have to find a $\delta > 0$ witnessing $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$. We put

$$\delta = \delta(2)$$

and we shall show that δ is as required.

For every $k \leqslant i_1$ we shall construct a homeomorphism $h_k \colon A_k^1 \to A_k^1$ such that

$$h_k|(A_k^1 \cap T) = h|(A_k^1 \cap T), \tag*{$(4)_k$}$$

$$d(h_k, 1) < \delta(k+1), \tag*{$(5)_k$}$$

$$\text{if} \quad l \leqslant k \quad \text{then} \quad h_k \text{ extends } h_l, \tag*{$(6)_k$}$$

$$\forall x \leqslant n \; \forall y \leqslant i_x \colon h_k(A_y^x \cap A_k^1) = A_y^x \cap A_k^1$$

and

$$h_k(\mathscr{A}_y^x \cap A_k^1) = \mathscr{A}_y^x \cap A_k^1, \tag*{$(7)_k$}$$

there are natural numbers $\eta_2, \ldots, \eta_n \in \mathbb{N}$ such that for every $2 \leqslant l \leqslant n$ we have $(h_k(Z_l \cap A_k^1), h_k(S_l \cap A_k^1)) \leqslant (A_{\eta_l}^l, \mathscr{A}_{\eta_l}^1)$. $(8)_k$

Since $A_1^1 = 0$ we let h_1 be the empty function. Then obviously $(4)_1$ through $(8)_1$ are satisfied. Suppose therefore that for some $k \leqslant i_1 (k \geqslant 1)$ the homeomorphism h_{k-1} has been defined properly.

Put $E = A_{k-1}^1 \cup (T \cap A_k^1)$ and define $\varphi \colon E \to E$ by $\varphi = h \cup h_{k-1}$; by $(4)_{k-1} \varphi$ is a well-defined homeomorphism.

Claim 1. $d(\varphi, 1) < \delta(k)$ and if $l \leqslant k-1$ then φ extends h_l.

This follows directly from (3), $(5)_{k-1}$ and $(6)_{k-1}$.

Claim 2. $\forall 2 \leqslant x \leqslant n \; \forall y \leqslant i_x \colon \varphi(A_y^x \cap E) = A_y^x \cap E$ and $\varphi(\mathscr{A}_y^x \cap E) = \mathscr{A}_y^x \cap E$.

This follows from $(7)_{k-1}$ and (3) in the definition of $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$.

Claim 3. There exist $\gamma_2, \ldots, \gamma_n \in \mathbb{N}$ such that for every $2 \leqslant x \leqslant n$ we have $(\varphi(Z_x \cap E), \varphi(S_x \cap E)) \leqslant (A^x_{\gamma_x} \cap A^1_k, \mathscr{A}^x_{\gamma_x} \cap A^1_k)$.

This follows from $(8)_{k-1}$ and (4) in the definition of $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$.

Claim 4. $\varphi(\mathscr{B}_k(1) \cap E) = \mathscr{B}_k(1) \cap E$.

First observe that $\mathscr{B}_k(1) = \mathscr{A}^1_k$ and $\mathscr{B}_k(1) \cap E = \mathscr{A}^1_{k-1} \cup (\mathscr{A}^1_k \cap T)$. Consequently, the claim follows from $(7)_{k-1}$ and (3) in the definition of $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$.

Case 1. $A^1_k \subseteq T \cup A^1_{k-1}$.

Then $A^1_k = A^1_k \cap (T \cup A^1_{k-1}) = (A^1_k \cap T) \cup A^1_{k-1} = E$. Define $h_k = \varphi$. The easy proof that h_k and the numbers $\gamma_2, \ldots, \gamma_n \in \mathbb{N}$ found in Claim 3 are as required, is left to the reader.

Case 2. $A^1_k \nsubseteq T \cup A^1_{k-1}$.

Claim 5. $E \in \mathscr{T}(\mathscr{B}_k)$.

By Lemma 3.1, $E \in \mathscr{T}(\mathscr{A})$ so the claim follows from our definitions.

Claim 6. $Z_2 \cap A^1_k \supseteq S_2 \cap A^1_k \supseteq \cdots \supseteq S_n \cap A^1_k$ is an $(n-1)$ sieve in the Hilbert cube A^1_k such that
a) $\forall 2 \leqslant x \leqslant n$: $(A^x_{i_x} \cap A^1_k, \mathscr{A}^x_{i_k} \cap A^1_k) \leqslant (Z_x \cap A^1_k, S_x \cap A^1_k)$,
b) $Z_2 \cap A^1_k \subseteq \mathscr{B}_k(1) = \mathscr{A}^1_k$.

Notice that b) implies that $Z_2 \cap A^1_k \in \mathscr{Z}(A^1_k)$ since $\mathscr{A}^1_k$ is a skeletoid in A^1_k. In addition, a) follows directly from (1) in the definition of $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$. Since $Z_2 \subseteq S_1$ we also have by (1) in the definition of $S(\mathscr{A}, i \geqslant \cdots \geqslant i_n, \varepsilon)$,

$$Z_2 \cap A^1_k \subseteq S_1 \cap A^1_{i_1} \cap A^1_k = \mathscr{A}^1_{i_1} \cap A^1_k = \mathscr{A}^1_k = \mathscr{B}_k(1).$$

By Claims 1, 2, 3, 5 and 6 and by $S(\mathscr{B}_k, i_2 \geqslant \cdots \geqslant i_n, \delta(k+1))$ there exist a homeomorphism $f\colon A_k^1 \to A_k^1$ and elements $\bar\eta_2, \ldots, \bar\eta_n \in \mathbb{N}$ such that

$$d(f, 1) < \delta(k+1) \quad \text{and} \quad f|E = \varphi, \tag{9}$$

$$\forall 2 \leqslant x \leqslant n \; \forall y \leqslant i_x\colon f(A_k^1 \cap A_y^x) = A_k^1 \cap A_y^x$$

and

$$f(A_k^1 \cap \mathscr{A}_y^x) = A_k^1 \cap \mathscr{A}_y^x, \tag{10}$$

$$\forall 2 \leqslant x \leqslant n\colon (f(Z_x \cap A_k^1), f(S_x \cap A_k^1)) \leqslant (A_{\bar\eta_x}^x \cap A_k^1, \mathscr{A}_{\bar\eta_x}^x \cap A_k^1). \tag{11}$$

Let $F = f(Z_2 \cap A_k^1) \cup E$. Since both $\mathscr{A}_k^1$ and $f(\mathscr{A}_k^1)$ are skeletoids for A_k^1 there exists by Theorem 1.3 a homeomorphism $\alpha\colon A_k^1 \to A_k^1$ such that

$$d(\alpha, 1) < \delta(k+1) - d(f, 1), \tag{12}$$

$$\alpha|F = 1, \tag{13}$$

$$\alpha f(\mathscr{A}_k^1)\backslash F = \mathscr{A}_k^1\backslash F. \tag{14}$$

Put $h_k = \alpha \circ f$. We claim that h_k and the elements $\bar\eta_2, \ldots, \bar\eta_n$ are as required. That $(4)_k$, $(5)_k$ $(6)_k$ and $(8)_k$ hold is a triviality (use that $E = (T \cap A_k^1) \cup A_{k-1}^1$ and (9)–(13)). Take $2 \leqslant x \leqslant n$ ad $y \leqslant i_x$ arbitrarily. By (10), $f(A_k^1 \cap A_y^x) = A_k^1 \cap A_y^x$ and $f(A_k^1 \cap \mathscr{A}_y^x) = A_k^1 \cap \mathscr{A}_y^x$. Since

$$A_y^x \subseteq A_{i_2}^2 \subseteq Z_2 \quad \text{and} \quad \alpha|f(Z_2 \cap A_k^1) = 1$$

we conclude that

$$h_k(A_k^1 \cap A_y^x) = A_k^1 \cap A_y^x \quad \text{and} \quad h_k(A_k^1 \cap \mathscr{A}_y^x) = A_k^1 \cap \mathscr{A}_y^x.$$

Now take $l \leqslant i_1$ arbitrarily. If $l < k$ then $h_k(A_l^1 \cap A_k^1) = h_k(A_l^1) = A_l^1 = A_l^1 \cap A_k^1$ since h_k extends h_l. If $k \leqslant l$ then $h_k(A_l^1 \cap A_k^1) = h_k(A_k^1) = A_k^1 = A_l^1 \cap A_k^1$. Again, assume that $l < k$. Then $h_k(\mathscr{A}_l^1 \cap A_k^1) = h_k(\mathscr{A}_l^1) = h_l(\mathscr{A}_l^1) = \mathscr{A}_l^1 = \mathscr{A}_l^1 \cap A_k^1$ by $(6)_k$. If $k \leqslant l$ then $h_k(\mathscr{A}_l^1 \cap A_k^1) = h_k(\mathscr{A}_l^1) \overset{(*)}{=} \mathscr{A}_k^1 = \mathscr{A}_l^1 \cap A_k^1$ in case $(*)$ holds. This is shown in the following:

Claim 7. $h_k(\mathscr{A}_k^1) = \mathscr{A}_k^1$.

Indeed, by (14) and (13) we have

$$\begin{aligned} h_k(\mathscr{A}_k^1 \cap F) &= f(\mathscr{A}_k^1) \cap (f(Z_2 \cap A_k^1) \cup E) \\ &= f(\mathscr{A}_k^1) \cap E) \cup (f(\mathscr{A}_k^1 \cap Z_2 \cap A_k^1) \quad \text{(by (9))} \\ &= \varphi(\mathscr{A}_k^1 \cap E)) \cup (f(\mathscr{A}_k^1 \cap Z_2 \cap A_k^1) \quad \text{(by Claims 4 and 6(b))} \\ &= (\mathscr{A}_k^1 \cap E) \cup f(Z_2 \cap A_k^1). \end{aligned}$$

By (11), $f(Z_2 \cap A_k^1) \subseteq A_{\bar{\eta}_2}^2 \cap A_k^1 \subseteq \mathscr{A}_k^1$. From this we conclude that

$$h_k(\mathscr{A}_k^1 \cap F) = (\mathscr{A}_k^1 \cap E) \cup (\mathscr{A}_k^1 \cap f(Z_2 \cap A_k^1)) = \mathscr{A}_k^1 \cap F.$$

Thus the claim follows from (14) and (13).

Now let $f = h_{i_1} \cup h$ and $\bar{T} = A_{i_1}^1 \cup T$. By $(4)_{i_1}$, $f\colon \bar{T} \to \bar{T}$ is a homeomorphism. Moreover, f has clearly the following properties:

$$d(f, 1) < \gamma \quad \text{and} \quad f|T = h, \tag{15}$$

$$\forall x \leqslant n \; \forall y \leqslant i_x\colon f(A_y^x) = A_y^x \quad \text{and} \quad f(\mathscr{A}_y^x) = \mathscr{A}_y^x, \tag{16}$$

there are natural numbers $\eta_2, \ldots, \eta_n \in \mathbb{N}$ such that for every $2 \leqslant l \leqslant n$ we have $(f(Z_l \cap \bar{T}), f(S_l \cap \bar{T})) \leqslant (A_{\eta_l}^l, \mathscr{A}_{\eta_l}^l)$. (17)

(For (17) observe that if $(A, B) \leqslant (E, F)$ and $(C, D) \leqslant (E, F)$ then $(A \cap C, B \cap D) \leqslant (E, F)$). Now consider the $\mathscr{Q}$-matrix $\mathscr{E} = \{A_m^{n+1}\colon n, m \in \mathbb{N}\}$. Clearly $\bar{T} \in \mathscr{T}(\mathscr{E})$. By $S(\mathscr{E}, i_2 \leqslant \cdots \leqslant i_n, \varepsilon)$ there exist a homeomorphism $\bar{f}\colon Q \to Q$ and elements $\xi_2, \ldots, \xi_n \in \mathbb{N}$ such that

$$d(\bar{f}, 1) < \varepsilon \quad \text{and} \quad \bar{f}|\bar{T} = f, \tag{18}$$

$$\forall 2 \leqslant k \leqslant n\colon (\bar{f}(Z_k), \bar{f}(S_k)) \leqslant (A_{\xi_k}^k, \mathscr{A}_{\xi_k}^k). \tag{19}$$

Now put $T^* = \bar{T} \cup \bar{f}(Z_2)$.

Claim 8. $\exists p \in \mathbb{N}$ such that $T^* \cap A_q^1 \in \mathscr{Z}(A_q^1)$ for every $q \geqslant p$.

This follows from (19) and Lemma 3.1 (notice that $T \cup A_{\xi_2} \in \mathscr{Z}(Q)$, hence is nowhere dense, and $\bigcup_1^\infty A_m^1$ is dense in Q).

Claim 9. $\exists m \in \mathbb{N}$ such that $(\bar{f}(Z_1) \cap T^*, \bar{f}(S_1) \cap T^*) \leqslant (A^1_m, \mathscr{A}^1_m)$.

Indeed, let $m = \max(\bar{\xi}_2, \xi_1, i_1)$. Then

$$\begin{aligned}\bar{f}(Z_1) \cap T^* &= (\bar{f}(Z_1) \cap \bar{T}) \cup \bar{f}(Z_2)\\ &= (\bar{f}(Z_1) \cap A^1_{i_1}) \cup (\bar{f}(Z_1) \cap T) \cup \bar{f}(Z_2)\\ &\subseteq A^1_{i_1} \cup A^1_{\xi_1} \cup A^2_{\bar{\xi}_2} \subseteq A^1_m.\end{aligned}$$

In addition,

$$\begin{aligned}\mathscr{A}^1_m \cap \bar{f}(Z_1) \cap T^* &= (\mathscr{A}^1_m \cap \bar{f}(Z_1) \cap A^1_{i_1}) \cup (\mathscr{A}^1_m \cap \bar{f}(Z_1) \cap T)\\ &\quad \cup (\mathscr{A}^1_m \cap \bar{f}(Z_2)) \qquad \text{(by (16) and (15))}\\ &= (\mathscr{A}^1_m \cap \bar{f}(Z_1 \cap A^1_{i_1})) \cup (\mathscr{A}^1_m \cap h(Z_1 \cap T)) \cup \bar{f}(Z_2)\\ &= (\mathscr{A}^1_m \cap \bar{f}(A^1_{i_1})) \cup h(S_1 \cap T) \cup \bar{f}(Z_2) \qquad \text{(by (16))}\\ &= \mathscr{A}^1_{i_1} \cup h(S_1 \cap T) \cup \bar{f}(Z_2).\end{aligned}$$

Also,

$$\begin{aligned}\bar{f}(S_1) \cap T^* &= (\bar{f}(S_1) \cap A^1_{i_1}) \cup (\bar{f}(S_1) \cap T) \cup (\bar{f}(S_1)\\ &\quad \cap \bar{f}(Z_2)) \qquad \text{(by (16))}\\ &= (\bar{f}(S_1 \cap A^1_{i_1}) \cup (f(S_1) \cap T) \cup (\bar{f}(S_1 \cap Z_2))\\ &= \bar{f}(\mathscr{A}^1_{i_1}) \cup h(S_1 \cap T) \cup \bar{f}(Z_2) \qquad \text{(by (16))}\\ &= \mathscr{A}^1_{i_1} \cup h(S_1 \cap T) \cup \bar{f}(Z_2),\end{aligned}$$

which proves the claim.

By Lemma 3.4 there exist a homeomorphism $\alpha: Q \to Q$ and an element $\bar{\xi}_1 \in \mathbb{N}$ such that

$$d(\alpha, 1) < \varepsilon - d(\bar{f}, 1) \quad \text{and} \quad \alpha | T^* = 1, \tag{20}$$

$$(\alpha\bar{f}(Z_1), \alpha\bar{f}(S_1)) \leqslant (A^1_{\bar{\xi}_1}, \mathscr{A}^1_{\bar{\xi}_1}). \tag{21}$$

Now put $H = \alpha \circ \bar{f}$. We claim that H is as required. Clearly, $d(H, 1) < \varepsilon$ and since $\alpha|T^* = 1$ we have $H|T = h$ by (18) and (15). That condition (6) of the definition holds and that

$$\forall 2 \leqslant k \leqslant n\colon (H(Z_k), H(S_k)) \leqslant (A^k_{\xi_1}, \mathscr{A}^k_{\xi_k})$$

follows from (16), (19) and (20) since $A^x_y \cup \bar{f}(Z_k) \subseteq A^1_{i_1} \cup \bar{f}(Z_2) \subseteq T^*$ and $\alpha|T^* = 1$.

Finally, that $(H(Z_1), H(S_1)) \leqslant (A^1_{\xi_1}, \mathscr{A}^1_{\xi_1})$ follows directly from (21). □

We now come to the main result in this section.

3.6. THEOREM. *Let $\mathscr{A} = \{A^n_m\colon n, m \in \mathbb{N}\}$ be a $\mathscr{Q}$-matrix in Q. Then $\mathscr{A}$ is an $\mathbb{N}$-skeleton.*

Proof. Take $T = \varnothing$ in $S(\mathscr{A}, i_1 \geqslant \cdots \geqslant i_n, \varepsilon)$ and apply Lemma 3.5. □

4. Some $\mathbb{N}$-skeletons

In this section we shall prove that $\mathscr{Q}$-matrices exist. Compared to all the work we did so far this turns out to be surprisingly simple.

Let us fix some notation. For every $n \in \mathbb{N}$, let A_n be as in Theorem 1.2, i.e.

$$A_n = \prod_{i=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]_i.$$

Let $\hat{Q} = \Pi_1^\infty Q_i$, where $Q_i = Q$ for every i. Clearly, $\hat{Q} \approx Q$. For every $n, m \in \mathbb{N}$ define $A^n_m \subseteq \hat{Q}$ as follows:

$$\begin{cases} A^n_1 = \varnothing & \text{for every } n \in \mathbb{N}, \\ A^n_m = \underbrace{A_m \times A_m \times \cdots \times A_m}_{n} \times Q \times Q \times Q \times \ldots & \text{for } n \in \mathbb{N} \text{ and } m \geqslant 2. \end{cases}$$

4.1. THEOREM. *$\mathscr{A} = \{A^n_m\colon n, m \in \mathbb{N}\}$ is a $\mathscr{Q}$-matrix in $\hat{Q}$.*

Proof. To see that

$$\forall n \in \mathbb{N}\colon \{A^n_m\}_{m>1} \tag{1}$$

is a skeleton, use Theorem 1.2 and Chapman [8].

That

$$\forall n_1 < \cdots < n_m \in \mathbb{N}\ \forall i_1, \ldots i_m \in \mathbb{N}\backslash\{1\}: \bigcap_{k=1}^{m} A_{i_k}^{n_k} \approx Q \tag{2}$$

is easy since choose indices $n_1 < \cdots < n_m \in \mathbb{N}$, $i_1, \ldots, i_m \in \mathbb{N}\backslash\{1\}$ and define natural numbers $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_m$ by

$$p_k = \min\{i_k, \ldots, i_m\} \quad (1 \leqslant k \leqslant m\}).$$

Then

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} = \underbrace{A_{p_m} \times \cdots \times A_{p_m}}_{n_1} \times \underbrace{A_{p_{m-1}} \times \cdots \times A_{p_{m-1}}}_{n_2 - n_1} \times \ldots$$

$$\times \underbrace{A_{p_1} \times \cdots \times A_{p_1}}_{n_m - n_{m-1}} \times Q \times Q \times \cdots \times Q \times \ldots \tag{$*$}$$

is a product of Hilbert cubes and hence is a Hilbert cube itself. That

$$\forall n_1 < \cdots < n_m \in \mathbb{N}\ \forall i_1, \ldots, i_m \in \mathbb{N}\backslash\{1\}\ \forall p \in \mathbb{N}: \left\{\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}: i \in \mathbb{N}\right\} \tag{3}$$

is a skeleton in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ follows easily from Chapman [8] and the formula ($*$).

We shall prove that

$$\forall n, m \in \mathbb{N}: A_m^n \in \mathscr{T}(\mathscr{A}). \tag{4}$$

So take $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N}$ arbitrarily and suppose that

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \not\subseteq A_m^n.$$

By the above formulas we can write

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} = \prod_1^\infty E_k \quad \text{and} \quad \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_m^n = \prod_1^\infty F_k,$$

where all but finitely many E_k's (resp. F_k's) are equal to Q and the remaining finitely many E_k's) are elements of $\{A_n\}_{n>1}$. Since $\Pi_1^\infty F_k$ is a proper subset of $\Pi_1^\infty E_k$ we have

$$F_k \subseteq E_k \text{ for every } k \text{ and there exists a } k \in \mathbb{N} \text{ such that } F_k \text{ is a proper subset of } E_k. \tag{5}$$

Then clearly $F_k \in \mathscr{X}(E_k)$ from which easily follows that $\Pi_1^\infty F_k \in \mathscr{X}(\Pi_1^\infty E_k)$. □

4.2. Corollary. *Let $\mathscr{A}$ be an $\mathbb{N}$-skeleton in Q. Then* ker $(\mathscr{A})$ *is homeomorphic to σ_ω.*

Proof. Let $\Sigma = \bigcup_1^\infty A_n$, where A_n is defined as above. The $\mathscr{Q}$-matrix $\mathscr{A}$ constructed in Theorem 4.1 is an $\mathbb{N}$-skeleton by Theorem 3.6 and ker $(\mathscr{A})$ obviously equals Σ^∞. It is known that Σ is homeomorphic to

$$l_f^2 \times Q.$$

For details see [6]. Consequently,

$$\sigma_\omega = (l_f^2)^\infty \approx (l_f^2 \times I)^\infty \approx (l_f^2)^\infty \times Q \approx (l_f^2)^\infty \times Q^\infty \approx (l_f^2 \times Q)^\infty \approx \Sigma^\infty.$$

So the desired result now follows from Corollary 2.3. □

We shall now give a different example of an $\mathbb{N}$-skeleton in Q. This example will be of importance in the announced "applications" to function spaces.

It will be convenient to fix some notation. For $x \in Q$ write $\|x\| = \sup_i |x_i|$. Denote points of $\hat{Q}$ by $x = (x(i))$, where $x(i) \in Q$ for all i. In addition, write $B_m^n = \{x \in \hat{Q}\colon \|x(i)\| \leqslant 1 - 1/m \text{ if } i \leqslant m \text{ and } \|x(i)\| \leqslant 2^{-n} \text{ otherwise}\}$.

It is clear that $\mathscr{B} = \{B_m^n\colon n, m \in \mathbb{N}\}$ is a $\mathscr{X}$-matrix in Q.

4.3. Lemma. $\forall n \in \mathbb{N}$: $\{B_m^n\}_{m>1}$ *is a skeleton*

Proof. The standard proof that $\{A_n\}_1^\infty$ is a skeleton, where each A_n is as in Theorem 1.2, can easily be adapted to show that $\{B_m^n\}_{m>1}$ is a skeleton. For details, see [6], Proposition 3.1 on page 156. □

4.4. Lemma. $\forall n_1 < \cdots < n_m \in \mathbb{N}\ \forall i_1, \ldots, i_m \in \mathbb{N}\setminus\{1\}$: $\bigcap_{k=1}^m B_{i_k}^{n_k} \approx Q$. □

4.5. LEMMA. $\forall p, q \in \mathbb{N}$: $B_q^p \in \mathscr{T}(\mathscr{B})$.

Proof. Take $n_1 < \cdots < n_m \in \mathbb{N}$ and $p, q, i_1, \ldots, i_n \in \mathbb{N}$ and write $B = \bigcap_{k=1}^m B_{i_k}^{n_k}$. With $A(\varepsilon) = \{x \in Q: \|x\| \leqslant \varepsilon\}$ there are sequences (ε_i) and (δ_i) of positive numbers such that $B = \Pi_i A(\varepsilon_i)$, $B \cap B_q^p = \Pi_i A(\delta_i)$. Observe that $\delta < \varepsilon$ implies that $A(\delta) \in \mathscr{Z}(A(\varepsilon))$. Therefore, if $B \cap B_q^p \notin \mathscr{Z}(B)$ then $\delta_i \geqslant \varepsilon_i$ for each i, i.e., $B \subseteq B \cap B_q^p$. □

4.6. LEMMA. $\forall n_1 < \cdots < n_m \in \mathbb{N}\ \forall i_1, \ldots, i_m \in \mathbb{N}\backslash\{1\}\ \forall p \in \mathbb{N}$: $\{\bigcap_{k=1}^m B_{i_k}^{n_k} \cap B_i^{n_m+p}\}_{i>1}$ *is a skeleton in* $\bigcap_{k=1}^m B_{i_k}^{n_k}$.

Proof. Write $B = \bigcap_{k=1}^m B_{i_k}^{n_k}$. By lemma 4.5, $B \cap B_i^{n_m+p} \in \mathscr{Z}(B)$ for each i.

A straightforward adaptation of the standard proof that $\{A_n\}_1^\infty$ is a skeleton, where each A_n is as in Theorem 1.2, now also shows that $\{\bigcap_{k=1}^m B_{i_k}^{n_k} \cap B_i^{n_m+p}\}_{i>1}$ is a skeleton in $\bigcap_{k=1}^m B_{i_k}^{n_k}$. □

4.7. COROLLARY. $\mathscr{B} = \{B_m^n: n, m \in \mathbb{N}\}$ *is an* $\mathbb{N}$*-skeleton in* $\hat{Q}$.

proof. Apply Lemmas 4.3–4.6 and Theorem 3.6. □

We shall now try to find a description of ker $(\mathscr{B})$ in terms of function spaces. To this end we shall first introduce a test space T. The underlying set of T is $(\mathbb{N} \times \mathbb{N}) \cup \{\infty\}$.

The points of $\mathbb{N} \times \mathbb{N}$ are isolated and a basic neighborhood of ∞ has the form

$$(\{n, n+1, \ldots\} \times \mathbb{N}) \cup \{\infty\} \quad (n \in \mathbb{N}).$$

Clearly, T is a countable metric space. Define

$$C_{p,0}^*(T) = \{f \in C_p^*(T): f(\infty) = 0\}.$$

4.8. LEMMA.

$$C_p^*(T) \approx C_{p,0}^*(T) \times \mathbb{R}, \tag{1}$$

$$C_{p,0}^*(T) \approx \ker(\hat{\mathscr{B}}) \approx \sigma_\omega. \tag{2}$$

Proof. For (1), simply observe that the function $\varphi\colon C_p^*(T) \to C_{p,0}^*(T) \times \mathbb{R}$ defined by

$$\varphi(f) = (f - f(\infty), f(\infty))$$

is a homeomorphism.

Define

$$B = \{x \in \hat{Q}\colon \sup_k \|x(k)\| < 1\},$$

and

$$Z = \{x \in B\colon \lim_{k\to\infty} \|x(k)\| = 0\},$$

respectively. Define a function $\Psi\colon C_{p,0}^*(T) \to Z$ by

$$\Psi(f)_n = \frac{2}{\pi}(\arctan f(1, n), \arctan f(2, n), \ldots, \arctan f(m, n), \ldots).$$

it is easy to see that Ψ is a homeomorphism. So in order to prove the first part of (2), it suffices to show that $Z = \ker(\mathscr{B})$. This follows from the following observation:

$$\begin{aligned} Z &= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left\{x \in \hat{Q}\colon \|x(k)\| \leqslant 1 - \frac{1}{m} \text{ and } \|x(k)\| \leqslant 2^{-m} \text{ for } k \geqslant m\right\} \\ &= \ker(\mathscr{B}). \end{aligned}$$

□

5. The space σ_ω

In this section we shall derive some topological properties of the space σ_ω which are probably known already since they follow easily and rather mechanically from known results.

As noticed in the proof of Corollary 4.2, σ_ω is homeomorphic to $\Sigma^\infty = \Pi_{i=1}^\infty \Sigma_i$, where each Σ_i is a copy of the space

$$\Sigma = \left\{x \in Q\colon \exists n \in \mathbb{N}\ \forall i \in \mathbb{N}\colon |x_i| \leqslant 1 - \frac{1}{n}\right\}.$$

5.1. LEMMA. *Let X be a σ-compact space. Then X admits a closed imbedding in the space Σ.*

Proof. Let γX be a compactification of X. Without loss of generality, $\gamma X \subseteq Q$ and $\gamma X \in \mathscr{Z}(Q)$ (apply for example Theorem 1.7; easier proofs are known of course). By Theorems 1.2 and 1.3 it follows that Σ is an absorber in Q. By Lemma 3.3 there exists a homeomorphism α: $Q \to Q$ such that $\alpha(\gamma X) \cap \Sigma = \alpha(X)$. Consequently, $\alpha(X)$ is a closed copy of X in Σ. □

5.2. COROLLARY. *Let X be an absolute $F_{\sigma\delta}$. Then X admits a closed imbedding in $\Sigma^\infty \approx \sigma_\omega$.*

Proof. Assume that $X \subseteq Q$ and let $F_i \subseteq Q$ be σ-compact for every $i \in \mathbb{N}$ such that $X = \bigcap_{i=1}^\infty F_i$. By Lemma 5.1 there exists for every $i \in \mathbb{N}$ a closed imbedding f_i: $F_i \to \Sigma$. Now define f: $X \to \Sigma^\infty$ by

$$f(x) = (f_1(x), f_2(x), \ldots).$$

It is well-known, and easy to prove, that f is a closed imbedding. □

By a *linear space* we mean a topological vector space over the reals. Let E be a linear space and put

$$\Sigma E = \{x \in E^\infty : x_i = 0 \text{ for all but finitely many } i\}.$$

In [23], Toruńczyk proved that for every locally convex (metric) linear space E we have

$$\Sigma E^\infty \approx \Sigma\mathbb{R} \times E^\infty \approx l_f^2 \times E^\infty.$$

Consequently, if we take $E = \sigma_\omega$ we obtain

$$\Sigma\sigma_\omega \approx \Sigma(\sigma_\omega)^\infty \approx l_f^2 \times (\sigma_\omega)^\infty \approx \sigma_\omega.$$

This consequence of Toruńczyk's result can also be derived by the apparatus developed in this paper since it is easy to represent $\Sigma\sigma_\omega$ as the kernel of some $\mathscr{Q}$-matrix, see §4.

For information on absolute (neighborhood) retracts (abbreviated: A(N)R's), see Borsuk [7].

5.3. THEOREM. *Let X be a space. The following statements are equivalent:*
1. $X \times \sigma_\omega \approx \sigma_\omega$,
2. *X is an absolute $F_{\sigma\delta}$ and an* AR.

Proof. 1 $\Rightarrow$ 2 is a triviality since σ_ω is an AR by Dugundji's Theorem [7, 7.1], and σ_ω is clearly an absolute $F_{\sigma\delta}$. For 2 $\Rightarrow$ 1, let X be an AR and an absolute $F_{\sigma\delta}$. By Corollary 5.2, X admits a closed imbedding in σ_ω. By Toruńczyk [20] we may conclude that $X \times \Sigma\sigma_\omega \approx \Sigma\sigma_\omega$. Since, as was observed above, $\Sigma\sigma_\omega \approx \sigma_\omega$ the desired conclusion follows. □

Let X be a space Y is called an *X-manifold* if Y admits an open covering by sets homeomorphic to open subsets of X. The above theorem generalizes without much difficulty to the following result: For any space X the following statements are equivalent: 1) $X \times \sigma_\omega$ is a σ_ω-manifold, and 2) X is an absolute $F_{\sigma\delta}$ and an ANR. Let us also mention that by results of Henderson, σ_ω-manifolds are characterized by homotopy type, i.e. if X and Y are σ_ω-manifolds then X is homeomorphic to Y if and only if X and Y have the same homotopy type. For details see [6], chapter 9.

6. Function spaces

If X is a space then, as noted in the introduction, $C^*(X)$ denotes the set of continuous bounded real-valued functions on X endowed with the topology of uniform convergence. It is known that if X and Y are uncountable compact (metric) spaces then $C^*(C)$ and $C^*(Y)$ are linearly homeomorphic ([18], 21.5.10). If X and Y are countably infinite compact (metric) spaces then $C^*(X)$ and $C^*(Y)$ need not be linearly homeomorphic, ([18], 21.5.13(A)). In fact, there are uncountably many types among the spaces $C^*(X)$, where X is countable and compact (metric) [5].

The Anderson–Kadec Theorem that all separable infinite-dimensional Fréchet spaces are homeomorphic (in the topological sense), see [6], chapter 6, for background information, shows that all function spaces of the form $C^*(X)$, where X is any infinite compact space, are homeomorphic. In fact, a more general result can be formulated by applying results from [22].

There is another topology on function spaces that is of interest in abstract functional analysis, namely the topology of point-wise convergence. To avoid confusion, for any space X the set of continuous real-valued bounded functions on X endowed with the topology of point-wise convergence shall be denoted by $C_p^*(X)$. By [3], if X and Y are compact Hausdorff and $C_p^*(X)$ is linearly homeomorphic to $C_p^*(Y)$ then X and Y have the same dimension (for X and Y compact metric this was first proved in [16]). From this we see that for example the function spaces $C_p^*([0, 1])$ and $C_p^*([0, 1]^2)$ are not linearly homeomorphic. There even exist countable infinite compact metric spaces X and Y for which $C_p^*(X)$ and $C_p^*(Y)$ are not linearly homeomorphic.

This can be seen by combining [18], 21.5.13(A), and the result due to Pavlowskii (see [4] for details) that for compact Hausdorff spaces X and Y each linear homeomorphism f: $C_p^*(X) \to C_p^*(Y)$ is also a linear homeomorphism from $C^*(X)$ onto $C^*(Y)$. These results indicate that a classification result for the spaces $C_p^*(X)$ in the linear world promises to be extremely complicated.

Trying to get a topological classification result for the spaces $C_p^*(X)$ seems to be more promising since no "obstacles" such as in [3] and [16] for topological homeomorphism are known. On the other hand, recent results in [11] show that $C_p^*(X)$ for non-trivial X is always a rather complex subspace of $\mathbb{R}^X$ which indicates that topological homeomorphisms may be hard to get.

In this section we shall consider spaces $C_p^*(X)$ for spaces X of the simplest possible non-trivial form, namely countable infinite (metric) spaces. It is easy to show that if X is discrete then $C_p^*(X) \approx \Sigma$, where Σ is the space defined in the proof of Corollary 4.2. We conjecture the following:

6.1. Conjecture. *If X is countable infinite and not discrete then $C_p^*(X)$ is homeomorphic to σ_ω.*

The aim of this section is to verify this conjecture under the extra assumption that X is not locally compact. Our main result is:

6.2. Theorem. *If X is countable infinite (metric) and fails to be locally compact then $C_p^*(X) \approx \sigma_\omega$.*

6.3. Corollary. *Let $\mathbb{D}$ denote the space of rational numbers. Then $C_p^*(\mathbb{Q}) \approx \sigma_\omega$.*

That X is required to be metric in Theorem 6.2 is essential. In [14] it was shown that $C_p^*(X)$ need not be a Borel subset of $\mathbb{R}^X$, even for countable non-metric X. Let T be the test space introduced in §4.

6.4. Lemma. *Let X be a (metric) space. The following statements are equivalent:*
1. *X is not locally compact,*
2. *X contains a closed homeomorph of T.*

Proof. Observe that $2 \Rightarrow 1$ is a triviality. For $1 \Rightarrow 2$, let $x \in X$ be a point at which X fails to be locally compact.

We can find a family $\{U_n\colon n \in \mathbb{N}\}$ of open subsets of X such that

1. $\forall n \in \mathbb{N}\colon x \in U_n$,
2. $\forall n \in \mathbb{N}\colon \operatorname{diam} U_n < 2^{-n}$.

Fix $n \in \mathbb{N}$ arbitrarily. Since $\bar{U}_n$ is not compact, it contains an infinite closed discrete subset, say D_n. Without loss of generality, $x \notin D_n$. It is clear that we can moreover choose the D_n's to be disjoint. Put

$$S = \{x\} \cup \bigcup_1^\infty D_n.$$

Obviously, S is closed in X and homeomorphic to T. $\square$

6.5. LEMMA. *Let X be a countable space. Then $C_p^*(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$.*

Proof. Let

$$C_p(X) = \{f \in \mathbb{R}^X\colon f \text{ is continuous}\}$$

and

$$B = \{f \in \mathbb{R}^X\colon f \text{ is bounded}\}.$$

Clearly, $B = \bigcup_1^\infty [-n, n]^X$ is an F_σ-subset of $\mathbb{R}^X$. In [11] it was shown that $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$. Since clearly $C_p^*(X) = C_p(X) \cap B$, it follows that $C_p^*(X)$ is also an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$. $\square$

6.6. Proof of Theorem 6.2. We may assume that $T \subseteq X$ and that T is closed, Lemma 6.4. Define $\varrho\colon C_p^*(X) \to C_p^*(T)$ by $\varrho(f) = f|T$. Then ϱ is clearly a continuous linear operator. By [13], there exists a retraction $r\colon X \to T$. Define $\xi\colon C_p^*(T) \to C_p^*(X)$ by $\xi(f) = f \circ r$. Then ξ is clearly well-defined and continuous. Obviously,

$$\varrho \circ \xi = 1_{C_p^*(T)}.$$

As in the proof of the Bartle and Graves Theorem, define a function h: $C_p^*(X) \to \ker \varrho \times C_p^*(T)$ by

$$h(f) = (f - \xi\varrho(f), \varrho(f)).$$

A straightforward check shows that h is a homeomorphism.

Since $\ker \varrho$ is a linear subspace of the locally convex space $C_p^*(X)$ we conclude that ker ϱ is locally convex. From Dugundji's Theorem, [7], 7.1, we infer that ker ϱ is an AR. Since ker ϱ is closed in $C_p^*(X)$ and $C_p^*(X)$ is an absolute $F_{\sigma\delta}$ by Lemma 6.5, ker ϱ is an absolute $F_{\sigma\delta}$ as well. Now observe that by Lemma 4.9 we have

$$C_p^*(T) \approx C_{p,0}^* \times \mathbb{R} \approx \sigma_\omega \times \mathbb{R} \approx \sigma_\omega$$

from which, by the above and Theorem 5.3, follows that

$$C_p^*(X) \approx \ker \varrho \times C_p^*(T) \approx \ker \varrho \times \sigma_\omega \approx \sigma_\omega. \qquad \square$$

7. Remarks

There are beautiful topological characterizations of Hilbert space and Hilbert cube manifolds due to Toruńczyk [21], [22]. In the proofs of these results the topological completeness of the spaces involved is used in an essential way. Due to this, finding topological characterizations of certain incomplete interesting spaces, such as σ_ω, seems complicated. Mogilski [15] has recently found a characterization of all l_f^2-manifolds. Of course, l_f^2 is not complete. However, being σ-compact, it is the union of countably many sets of "smaller complexity" and this was used essentially in Mogilski's proof. Since σ_ω is not a $G_{\delta\sigma}$, it is not a countable union of "nice" sets which makes convergence procedures delicate. We were able to pass by these complications by approximating σ_ω from the outside. In this way we did not characterize σ_ω topologically but we characterized the way canonical copies of σ_ω are placed in Q. This is interesting in its own right but will probably not be of help in solving the problem of finding good usable topological characterizations of σ_ω. Recently, Bestvina and Mogilski in their paper "*Characterizing certain incomplete infinite-dimensional Absolute Retracts*" found an interesting topological characterization of σ_ω (and many other spaces). This characterization, however, is in terms of a difficult to verify mapping replacement condition. It seems not clear how to get our results from theirs in a direct way. I can only verify their condition in the function spaces $C_p^*(X)$ by going through similar constructions as in this paper. Results from Steel [19] can be used to show that $\mathbb{Q}^\infty$, the product of countably many copies of the space of rational numbers, is (topologically) the unique zero-dimensional $F_{\sigma\delta}$ which is nowhere a $G_{\delta\sigma}$. Recently, van Engelen [12] independently found a more elementary proof of this particularly consequence of Steel's result. A step in van Engelen's proof more or less motivated us to consider $\mathscr{X}$-matrices and to formulate the concept of an $\mathbb{N}$-skeleton.

Remarks added in December 1986

J. Pelant has shown that the spaces $C_p^*(T)$ and $C_p^*(\mathbb{Q})$ are not linearly homeomorphic. J. Baars, J. de Groot and J. van Mill have shown that if X is any countable, metric space which is not locally compact then the function space $C_p(X)$ is homeomorphic to σ_ω.

References

1. R.D. Anderson: On topological infinite deficiency. *Mich. Math. J.* 14 (1967) 365–383.
2. R.D. Anderson: *On sigma-compact subsets of infinite-dimensional spaces.* Unpublished manuscript.
3. A.V. Arhangelskii: The principle of τ-approximation and a test for equality of dimension of compact Hausdorff spaces. *Soviet Math. Dokl.* 21 (1980) 805–809.
4. A.V. Arhangelskii: On linear homeomorphisms of function spaces. *Soviet Math. Dokl.* 25 (1982) 852–855.
5. C. Bessaga and A. Pelczyński: Spaces of continuous functions (IV) (On isomorphical classification of function spaces C(S)). *Studia Math.* 19 (1960) 53–62.
6. C. Bessaga and A. Pelczyński: *Selected topics in infinite-dimensional topology.* PWN, Warsaw (1975).
7. K. Borsuk: *Theory of retracts.* PWN, Warsaw (1967).
8. T.A. Chapman: Dense sigma-compact subsets of infinite-dimensional manifolds. *Trans. Amer. Math. Soc.* 154 (1971) 399–426.
9. T.A. Chapman: *Lectures on Hilbert cube manifolds. CMBS Regional Conf. Series in Math.* no. 28. Amer. Math. Soc., Providence, R.I. (1976).
10. J.J. Dijkstra: *Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility. CWI Tract* 2. Amsterdam, (1984).
11. J.J. Dijkstra, T. Grilliot, D.J. Lutzer and J. van Mill: Function spaces of low Borel complexity. *Proc. Amer. Math. Soc.* 94 (1985) 703–710.
12. F. van Engelen: *Characterization of the countable infinite product of rationals and some related problems.* To appear.
13. R. Engelking: On closed images of the space of irrationals. *Proc. Amer. Math. Soc.* 21 (1969) 583–586.
14. D.J. Lutzer, J. van Mill and R. Pol: Descriptive complexity of function spaces. *Trans. Amer. Math. Soc.* 291 (1985) 121–128.
15. J. Mogilski: Characterizing the topology of infinite-dimensional σ-compact manifolds. *Proc. Amer. Math. Soc.* 92 (1984) 111–118.
16. S. Negrepontis: *Banach spaces and topology*, Handbook of Set theoretic Topology, (K. Kunen and J.E. Vaughan, eds.). North-Holland (1984) 1045–1142.
17. D. Pavlovskii: On spaces of continuous functions. *Soviet Math. Dokl.* 22 (1980) 34–37.
18. Z. Semadeni: *Banach spaces of continuous functions.* PWN, Warsaw (1971).
19. J.R. Steel: Analytic sets and Borel isomorphisms. *Fund. Math.* 108 (1980) 83–88.
20. H. Toruńczyk On cartesian factors and the topological classification of linear metric spaces. *Fund. Math.* 88 (1975) 71–86.
21. H. Toruńczyk: On CE-images of the Hilbert cube and characterization of Q-manifolds. *Fund. Math.* 108 (1980) 31–40.
22. H. Toruńczyk: Characterizing Hilbert space topology. *Fund. Math.* 111 (1981) 247–262.
23. H. Toruńczyk: (G,K)-absorbing and skeletonized sets in metric spaces. *Diss. Math.* (to appear).
24. J.E. West: The ambient homeomorphy of an incomplete subspace of infinite-dimensional Hilbert spaces. *Pacific J. Math.* 34 (1970) 257–267.