n-DIMENSIONAL TOTALLY DISCONNECTED TOPOLOGICAL GROUPS

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Abstract. A space X is called totally disconnected if every two distinct points $x, y \in X$ have disjoint neighborhoods which are both open and closed. We prove that if G is a LCA group with dim $G \geqslant n+1$ then G contains a totally disconnected subgroup H with dim $H \geqslant n$. If moreover G is metrizable then H can be constructed to be a Borel subset of G.

1. Introduction. A space X is called totally disconnected if every two distinct points $x, y \in X$ have disjoint neighborhoods which are both open and closed[2]. It is clear that every zero-dimensional space is totally disconnected but the converse is not true. Sierpiński [11] and Knaster and Kuratowski [6] constructed the first examples of totally disconnected 1-dimensional spaces which in addition are topologically complete and separable metrizable. Later, Mazurkiewicz [9] even constructed n-dimensional such spaces for every $n \in N$. See Rubin, Schori and Walsh [10] for elementary constructions of n-dimensional totally disconnected separable metrizable spaces.

The standard example of a totally disconnected 1-dimensional topological group is Erdös' space [4], i.e. the subspace E of Hilbert space ℓ^2 consisting of all points all of whose coordinates are rational. As far as I know, the natural question whether there exist totally disconnected n-dimensional topological groups for every $n \in N$ is unanswered. A natural candidate for a 2-dimensional totally disconnected topological group is $E \times E$. However, it is easily seen that $E \times E$ and E are

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(2) Some authors define a space to be totally disconnected if all of its components are points. We follow however the terminology in Engelking [3].

homeomorphic and consequently, dim $E^n = 1$ for every $n \in N \cup \{\infty\}$. The aim of this note is to show that every locally compact Abelian topological group G with dim $G \geqslant n+1$ contains a totally disconnected subgroup H with dim $H \geqslant n$. If G is metrizable then we can construct H in such a way that it is a Borel subset of G.

2. The construction. Throughout, T denotes the circle group and I denotes the interval $[0, 1]$. For all undefined notions see Hewitt and Ross [5]. Abelian groups are written additively.

Let G be a locally compact Abelian group with dim $G \geqslant n$. It is well-known that G contains a copy of the n-cell I^n. Since this is not stated explicitly in Hewitt and Ross [5], we will sketch a proof of this fact using only results from [5] and well-known facts from dimension theory. Let U be a symmetric neighborhood of the identity in G such that $\overline{U}$ is compact. Then $L = \bigcup_{n=1}^{\infty} n\, U$ is an open and closed subgroup of G and since $\overline{U} \subseteq 2U$ it follows that L is compactly generated. Since G is the topological sum of copies of L it follows that dim $L \geqslant n$. Since L is compactly generated there are $a, b \in N$ and a compact Abelian group F such that L is isomorphic to $R^a \times Z^b \times F$, [5, 9.8]. Since Z^b is zero-dimensional, it follows that dim $L = \dim(R^a \times F) = a + \dim F$ (the last equality is not a triviality). Let $p = \dim F$. In addition, let q denote the torsion-free rank of the character group of F. Then $p = q$, [5, 24.28], and the proof of [5, 24.28] shows that F contains a copy of I^q. Consequently, L contains a copy of $I^a \times I^p$ which is naturally homeomorphic to I^{a+p}. Since $n \leqslant a+p$, we are done.

We now come to the main result in this note. Among others, our proof depends on an interesting technique due to Rubin, Schori and Walsh [10].

2.1. Theorem. Let G be a locally compact Abelian group with dim $G \geqslant n+1$. Then G contains a totally disconnected subgroup H with dim $H \geqslant n$. If moreover G is metrizable then H can be constructed to be a Borel subset of G.

Proof. By the above remarks, G contains a copy A of the cube I^{n+1}. We may assume that $0 \in A$. Take a point $x \in A \setminus \{0\}$ and let $X: G \to T$ be a continuous homomorphism with $X(x) \neq 0$, [5, 23.26]. Then $X(A)$ is a nondegenerate subcontinuum of T and therefore contains a nonempty connected open set U. We choose U to be a proper subset of T which implies that U is homeomorphic to an interval (a, b) for certain $a, b \in R$ with $a < b$. By abuse of notation we assume that $U = (a, b)$. Let $V = X^{-1}(U) \cap A$. Then V is a nonempty open subset of A and therefore has only countably many components. If the image under X of each of these components is degenerate then U is countable, which is not the case. Consequently, there is a connected (relatively) open set $E \subset A$ such that $X(E) \subset U$ and $X(E)$ is nondegenerate. Take two points $p, q \in E$ with $a < X(p) < X(q) < b$. Choose points $\bar{a}$ and $\bar{b}$ in U such that

$$a < X(p) < \bar{a} < \bar{b} < X(q) < b.$$

Since $X^{-1}((a, \bar{a}))$ and $X^{-1}((\bar{b}, b))$ are open neighborhoods of p and q, respectively, it is clear that there is a copy B of I^{n+1} in E such that for some pair of opposite faces (F_1, F_2) of B we have

$$X(F_1) \subset (a, \bar{a}) \text{ and } X(F_2) \subset (\bar{b}, b).$$

Observe that $X(B) \subset (a, b)$. So again by abuse of notation we assume that $I^{n+1} \subset G$ and that

(1) $X(I^{n+1}) \subset (a, b)$,

(2) $X(\{0\} \times I^n) \subset (a, \bar{a})$ and $X(\{1\} \times I^n) \subset (\bar{b}, b)$.

By Mauldin [8] there is an (algebraically) independent Cantor set in T. An inspection of Mauldin's proof yields that independent Cantor sets exist in every nonempty open subset of T. Consequently, there is an independent Cantor set $K \subset (\bar{a}, \bar{b})$ Let $r \in K$ be a point of infinite order and let $\Delta \subset K \setminus \{r\}$ be a Cantor set. By independence, the subgroups generated by Δ and by $\{r\}$ only meet in the identity element of T. Consequently, since the subgroup generated by $\{n\}$ is dense in T, the subgroup generated by Δ is zero-dimensional.

Now we closely follow a construction in Rubin, Schori and Walsh [10]. Let $\mathscr{C}$ denote the collection of all continua in I^{n+1} meeting both $\{\alpha\} \times I^n$ and $\{1\} \times I^n$. Then $\mathscr{C}$, topologized by the Hausdorff metric, is a compact metrizable space and consequently there exists a continuous surjection $\alpha : \Delta \to \mathscr{C}$. If $C \in \mathscr{C}$ then both $X(C) \cap (a, \bar{a})$ and $X(C) \cap (\bar{b}, b)$ are nonempty and consequently, by connectivity of C and by the fact that $X(C) \subset (a, b)$, we have $\Delta \subset X(C)$. Put

$$Z = \cup \{ X^{-1}(t) \cap \alpha(t) : t \in \Delta \} .$$

It is easy to see that Z is compact (use that Δ is compact and that α is continuous). Since $\Delta \subset X(C)$ for every $C \in \mathscr{C}$ it follows that $X^{-1}(t) \cap \alpha(t) \neq \phi$ for every $t \in \Delta$. Consequently, the function $\pi = X|Z : Z \to \Delta$ is a continuous surjection. By Rubin, Schori and Walsh [10, Theorem 4. 2], every subset of Z that intersects every fiber of π is at least n-dimensional. By Bourkaki [1, page 262, exercise 9 a], there is a G_δ-subset P of Z such that P intersects every fiber of π in precisely one point (observe that Z is metrizable). Now let $<P>$ and $<\Delta>$ denote the subgroups of G and T generated by P and Δ, respectively. Since $X(P) = \Delta$, clearly $X(<P>) = <\Delta>$. Take a point $z \in <P>$ such that $X(z) = 0$. There exist distinct $p_1, \ldots, p_n \in P$ and elements $m_1, \ldots, m_n \in Z$ such that $z = m_1 p_1 + \ldots + m_n p_n$. Then $0 = X(z) = m_1 X(p_1) + \ldots + m_n X(p_n)$. Since X restricted to P is one to one, the $X(p_i)$'s are pairwise distinct. The fact that Δ is independent now implies that $m_1 = \ldots = m_n = 0$. Consequently, X restricted to $<P>$ is one to one from which follows that $<P>$ is totally disconnected since we already observed that $<\Delta>$ is zero-dimensional.

We shall now verify that $\dim <P> \geqslant n$. Observe that $<P>$ is a continuous image of the topological sum of countably many finite powers of P. Since P is separable and metrizable, it therefore follows that $<P>$ is Lindelöf, hence normal. Also X restricted to $<P>$ is one to one, from which follows that

$$<P> \cap Z = P,$$

i.e. P is closed in $<P>$. Since $\dim P \geqslant n$, the normality of $<P>$ now implies that $\dim <P> \geqslant n$, [3, Theorem 7. 18].

It remains to verify that $<P>$ is Borel if G is metrizable. To this end, let G be metrizable. Since P is a G_δ-subset of Z and Z is closed in G it folows that P is a Borel subset of G. Let $\mathscr{B}$ be a countable open basis for P. Fix a pairwise disjoint collection $B_1, \ldots, B_n \in \mathscr{B}$ and a point $(m_1, \ldots, m_n) \in Z^n \setminus \{(0, \ldots, 0)\}$. Define a function $\Phi : \prod_{i=1}^{n} B_i \to <P>$ by

$$\Phi(b_1, \ldots, b_n) = m_1 b_1 + \ldots + m_n b_n .$$

We claim that Φ is one to one. To this end, take distinct points $(b_1, \ldots, b_n)$,

$(e_1, \ldots, e_n) \in \prod_{i=1}^{n} B_i$. Without loss of generality assume that for a certain $k \leqslant n$ we have $b_i = e_i$ iff $i < k$. Suppose that $\Phi(b_1, \ldots, b_n) = \Phi(e_1, \ldots, e_n)$. Then clearly

$$\sum_{\ell=k}^{n} m_\ell b_\ell - \sum_{\ell=k}^{n} m_\ell e_\ell = 0$$

from which follows that

$$\sum_{\ell=k}^{n} m_\ell X(b_\ell) - \sum_{\ell=k}^{n} m_\ell X(e_\ell) = 0 .$$

Since X is one to one on P and since the sets $B_k, \ldots, B_n$ are pairwise disjoint, it follows that the points $X(b_k), \ldots, X(b_n)$, $X(e_k), \ldots, X(e_n)$ are pairwise distinct. Since Δ is independent it therefore follows that $m_k = \ldots = m_n = 0$, which is a contradiction.

Consequently Φ is one to one and clearly continuous. Since $\prod_{i=1}^{n} B_i$ is topologically complete and separable metrizable it now follows that $\Phi(\prod_{i=1}^{n} B_i)$ is a Borel subset of G, Kuratowski [7, R 39 V]. Since $< P > \setminus \{0\}$ is clearly a countable union of sets of the form $\Phi(\prod_{i=1}^{n} B_i)$, we find that $< P >$ is a Borel subset of G.

2. 2. Corollary. R^{n+1} contains an n-dimensional subgroup H which is dense, Borel and totally disconnected.

Proof. By Theorem 2. 1. R^{n+1} contains a subgroup H with $\dim H \geqslant n$ and which is Borel and totally disconnected. We claim that H is dense in R^{n+1}. Suppose that H is not dense and let $\overline{H}$ denote the closure of H. Then $\overline{H}$ is nowhere dense, being a subgroup of R^{n+1}. By Engelking [3, Exercise 7 . 4 . 18] it follows that $\dim \overline{H} \leqslant n$ and since $\dim H \geqslant n$ we get $\dim \overline{H} = n$. By Hewitt and Ross [5, Theorem 9 . 11], $\overline{H}$ is isomorphic to $R^a \times Z^b$ for certain $a, b \in N$. Since

Z^b is zero-dimensional, it clearly follows that $\dim(R^a \times Z^b) = a$, [3, 7.13.19]. Consequently, $\bar{H}$ is isomorphic to $R^n \times Z^b$ for certain $b \in N$ Let C be the component of the identity in $\bar{H}$. If $\dim(C \cap H) \leqslant n-1$ then since Z^b is countable, by the Countable Sum Theorem, [3, 7.2.1], it follows that $\dim H \leqslant n-1$, which is a contradiction. Consequently, $\dim(C \cap H) = n$. Since C is open and closed in $\bar{H}$ it follows that $C \cap H$ is dense in C. In addition, since C is isomorphic to R^n, it follows from [3, Exercise 7.4.18] that every n-dimensional subset of C has nonempty interior in C. Consequently, H contains a nonempty open set which has compact closure in H. Therefore, being a topological group, H is locally compact. However, H is totally disconnected and every totally disconnected locally compact metrizable space is clearly zero-dimensional. This is a contradiction.

By similar arguments as above it also follows that $\dim H \neq n+1$, i.e. $\dim H = n$.

2.3. Remark. We adopt the notation in the proof of Theorem 2.1 and we assume that G is metrizable. There exists a Cantor set $\Delta_0 \subseteq (\bar{a}, \bar{b}) \setminus \Delta$ such that $\Delta_0 \cup \Delta$ is independent while moreover the group generated by $\Delta_0 \cup \Delta$ is zero-dimensional. There exists a compact set Z_0 in I^{n+1} such that $X(Z_0) = \Delta_0$. Let $\pi_0 = X \mid Z_0 : Z_0 \to \Delta_0$ and let P_0 be a G_δ subset of Z_0 intersecting each fiber of π_0 in precisely one point. Let N_0 be a G_δ-subset of P_0 homeomorphic to the space of irrational numbers, Kuratowski [7, § 36 V Corollary 2]. The space N_0 contains Borel sets of arbitrarily large complexity, [7, § 30 XIV]. These sets are also Borel subsets of G of arbitrarily large complexity. If B is such a set then the subgroup of G generated by $B \cup P$ is totally disconnected, its dimension is at least n and it clearly contains B as a closed set. This proves that G contains totally disconnected Borel subgroups which are at least n-dimensional and which are in addition of arbitrarily large Borel complexity. Consequently, R^{n+1} contains uncountably may pairwise nonhomeomorphic, n-dimensional, dense, totally disconnected Borel subgroups.

2.4. Remark. By the method of Theorem 2.2 it is impossible to construct topologically complete, separable metrizable, n-dimensional topological groups which in addition are totally disconnected ($n \geqslant 1$). Simply observe that every topologically complete subgroup of a locally compact group is locally compact and that every locally compact totally disconnected metrizable space is zero-dimensional. This leaves open the following question :

> Do there exist for every $n \geqslant 1$ topologically complete, separable metrizable, n-dimensional, totally disconnected topological groups ?

For $n = 1$, the answer to this question is in the affirmative. Brechner [2] constructed a compact metrizable space X the autohomeomorphism group of which is totally disconnected, topologically complete and 1-dimensional.

References

[1] N. Bourbaki : General Topology (Part 2) (English version), Hermann and Addison-Wesley, 1966.

[2] B. Brechner : On the dimensions of certain spaces of homeomorphisms, Trans. Am. Math. Soc. **121** (1966), 516–548.

[3] R. Engelking : General Topology, PWN, 1977.

[4] P. Erdös : The dimension of the rational points in Hilbert space, Ann. of Math. **41** (1940), 734–736.

[5] E. Hewitt and K. A. Ross : Abstract Harmonic Analysis (Part 1), Springer, 1963.

[6] B. Knaster and K. Kuratowski : Sur les ensembles connexes, Fund. Math. **2** (1921), 206–255.

[7] K. Kuratowski : Topology (Part 1) (English version), Academic Press and PWN, 1966.

[8] R. D. Mauldin : On the Borel subspaces of algebraic structures, Indiana Univ. Math. J. **29** (1980), 261–265.

[9] S. Mazurkiewicz : Sur les problèmes κ et λ de Urysohn, Fund. Math. **10** (1927), 311–319.

[10] L. R. Rubin, R. M. Schori and J. J. Walsh : New dimension-theory techniques for constructing infinite-dimensional spaces, Gen. Top. Appl. **10** (1979), 93–102.

[11] W. Sierpiński : Sur les ensembles connexes et non connexes, Fund. Math. **2** (1921), 81–95.