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AN INFINITE-DIMENSIONAL PRE-HILBERT SPACE ALL BOUNDED LINEAR OPERATORS OF WHICH ARE SIMPLE

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1. Introduction. All linear spaces under discussion are separable linear spaces over K, where K denotes either $\mathbf{R}$ or $\mathbf{C}$.

The aim of this paper is to construct an infinite-dimensional pre-Hilbert space L over K with the following property: every bounded linear operator A of L has a unique eigenvalue such that the corresponding eigenspace is complemented and has finite codimension in L. Geometrically this means that for every bounded linear operator $A: L \to L$ there are two directions such that A acts as an ordinary multiplication in one direction, and A is essentially "finite-dimensional" in the other direction. Another equivalent formulation is: for every bounded linear operator A of L there is a unique scalar $\lambda \in K$ such that $A - \lambda I$ has finite rank. We conclude that L only admits "trivial" bounded linear operators. Observe that L cannot be complete, for then L is isomorphic to l_2 (see [1], 17.1), which has many bounded linear operators not of the above simple type.

I am indebted to Klaas Pieter Hart for some helpful comments.

2. Preliminaries. For all undefined notions, see [1]. We use the term pre-Hilbert space instead of inner product space.

A cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. $\mathfrak{c}$ denotes $2^{\aleph_0}$. If A is a set, then $|A|$ denotes the cardinality of A.

It is clear that the collection

$$\mathscr{F} = \{F\colon F \text{ is a bounded linear operator of } l_2\}$$

has cardinality $\mathfrak{c}$.

Let L be a linear space. If $A \subseteq L$, then $\dim A$ denotes the *algebraic dimension* of A, i.e.,

$$\dim A = \max\{\varkappa\colon \varkappa \text{ is a cardinal and there is a linearly independent } B \subseteq A \text{ such that } |B| = \varkappa\}.$$

Observe that $\dim A$ is a cardinal number and not, as usual, either one of $0, 1, 2, \ldots$ or ∞.

It is well known that there is a family $\mathscr{A}$ of $\mathfrak{c}$ infinite subsets of N such that for all distinct $A, B \in \mathscr{A}$ the set $A \cap B$ is finite [4] (for every irrational number choose a sequence of rational numbers converging to it or consider the set of paths through a Cantor tree).

The following lemma and its proof are well known:

2.1. LEMMA. $\dim l_2 = \mathfrak{c}$.

Proof. Let $\mathscr{A}$ be as above. For each $A \in \mathscr{A}$, say $A = \{n_1, n_2, \ldots\}$, where $n_1 < n_2 < \ldots$, define $x(A) \in l_2$ by

$$x(A) = (\underbrace{0, 0, \ldots, 0}_{n_1 - 1}, 1/n_1, \underbrace{0, 0, \ldots, 0}_{n_2 - n_1 - 1}, 1/n_2, 0, 0, \ldots).$$

Then $\{x(A): A \in \mathscr{A}\}$ is linearly independent and has cardinality $\mathfrak{c}$.

3. The construction. In this section we construct the example which was announced in the introduction. In the next section we will prove that the example constructed here has all the required properties.

Let $\{F_\alpha: 1 \leqslant \alpha < \mathfrak{c}, \alpha \text{ odd}\}$ be an enumeration of the family of bounded linear operators $\mathscr{F}$ which was defined in Section 2. In addition, let $\mathscr{K}$ be the family of all closed infinite-dimensional linear subspaces of l_2 and let $\{K_\alpha: \alpha < \mathfrak{c}, \alpha \text{ even}\}$ be an enumeration of $\mathscr{K}$ such that every $K \in \mathscr{K}$ is listed $\mathfrak{c}$ times (as we will see later, it is very important that $\mathscr{K}$ is enumerated in this way). It is clear that this is possible.

By transfinite induction, for every $\alpha < \mathfrak{c}$ we will construct linear subspaces $L_\alpha \subseteq l_2$ and subsets $V_\alpha \subseteq l_2$ such that:

(1) if $\beta < \alpha$, then $L_\beta \subseteq L_\alpha$ and $V_\beta \subseteq V_\alpha$;

(2) $L_\alpha \cap V_\alpha = \emptyset$;

(3) $\dim L_\alpha \leqslant |\alpha|$ and $|V_\alpha| \leqslant |\alpha|$;

(4) if α is even, then there is a vector $x \in (L_\alpha \cap K_\alpha) \setminus \bigcup_{\beta < \alpha} L_\beta$;

(5) if α is odd and if

$$\dim \{x \in l_2: F_\alpha(x) \notin \mathrm{sp}(\{x\} \cup \bigcup_{\beta < \alpha} L_\beta)\} = \mathfrak{c},$$

then there is a vector $x \in L_\alpha$ such that $F_\alpha(x) \in V_\alpha$.

(It is very hard to explain at this moment what the intuition is behind the inductive hypotheses (1) through (5). The reader is encouraged to follow the argumentation step by step. After reading Section 4, it will hopefully be clear why we constructed the L_α's in this peculiar way.)

The construction is a triviality. Take $x \in K_0$ arbitrarily, and define $L_0 = \mathrm{sp}\{x\}$ and $V_0 = \emptyset$. Suppose that we have completed the construction for all $\beta < \alpha$, where $\alpha < \mathfrak{c}$. For convenience, put

$$L^\alpha = \bigcup_{\beta<\alpha} L_\beta \quad \text{and} \quad V^\alpha = \bigcup_{\beta<\alpha} V_\beta .$$

Since $\dim L^\alpha \leqslant |\alpha|$ and $|V^\alpha| \leqslant |\alpha|$, the set $\mathrm{sp}(L^\alpha \cup V^\alpha)$ has algebraic dimension less than $\mathfrak{c}$. Observe that $L^\alpha \cap V^\alpha = \emptyset$.

Case 1. α is even. Observe that K_α is isomorphic to l_2 ([1], 17.1), which implies by Lemma 2.1 that $\dim K_\alpha = \mathfrak{c}$. Since $\dim \mathrm{sp}(L^\alpha \cup V^\alpha) < \mathfrak{c}$, we can therefore find a vector $x \in K_\alpha \setminus \mathrm{sp}(L^\alpha \cup V^\alpha)$. Define $L_\alpha = \mathrm{sp}(\{x\} \cup L^\alpha)$ and $V_\alpha = V^\alpha$. A routine verification shows that $L_\alpha \cap V_\alpha = \emptyset$. We conclude that L_α and V_α are properly defined.

Case 2. α is odd. Let

$$S = \{x \in l_2 \colon F_\alpha(x) \notin \mathrm{sp}(\{x\} \cup L^\alpha)\}.$$

If $\dim S < \mathfrak{c}$, then define $L_\alpha = L^\alpha$ and $V_\alpha = V^\alpha$. Suppose therefore that $\dim S = \mathfrak{c}$. As in case 1, there is a vector $x \in S \setminus \mathrm{sp}(L^\alpha \cup V^\alpha)$. Define

$$L_\alpha = \mathrm{sp}(\{x\} \cup L^\alpha) \quad \text{and} \quad V_\alpha = V^\alpha \cup \{F_\alpha(x)\}.$$

We claim that L_α and V_α are as required. All we have to prove is that $L_\alpha \cap V_\alpha = \emptyset$. Suppose therefore that $L_\alpha \cap V_\alpha \neq \emptyset$. Since $F_\alpha(x) \notin L_\alpha$, it follows that $L_\alpha \cap V^\alpha \neq \emptyset$. As in case 1, this is impossible.

This completes the transfinite construction. Now define

$$L = \bigcup_{\alpha<\mathfrak{c}} L_\alpha .$$

In Section 4 we will show that L is as required.

3.1. LEMMA. *If* $K \in \mathscr{K}$, *then* $\dim(L \cap K) = \mathfrak{c}$.

Proof. Let $A = \{\alpha < \mathfrak{c} \colon K = K_\alpha\}$. Then $|A| = \mathfrak{c}$. By (4), for every $\alpha \in A$ there is a vector

$$x_\alpha \in (L_\alpha \cap K) \setminus \bigcup_{\beta<\alpha} L_\beta .$$

We claim that the set $\{x_\alpha \colon \alpha \in A\}$ is linearly independent. Take $\alpha_1, \ldots, \alpha_n \in A$ and $\lambda_1, \ldots, \lambda_n \in K$ such that

$$\sum_{i=1}^{n} \lambda_i x_{\alpha_i} = 0.$$

Without loss of generality, $\alpha_1 < \ldots < \alpha_n$ and $\lambda_n \neq 0$. Then

$$x_{\alpha_n} = \sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} x_{\alpha_i} \in L_{\alpha_{n-1}},$$

which is a contradiction.

4. *L* is as required. In this section we will show that L is as required. To this end, let $A \colon L \to L$ be a bounded linear operator. By [1], Exercise X.5, there is a bounded linear operator $\bar{A}: l_2 \to l_2$ extending A. Consequently, $\bar{A} \in \mathscr{F}$. Choose an index $\alpha < \mathfrak{c}$ such that $\bar{A} = F_\alpha$. For convenience, define

$$L^\alpha = \bigcup_{\beta<\alpha} L_\beta \quad \text{and} \quad V^\alpha = \bigcup_{\beta<\alpha} V_\beta.$$

4.1. LEMMA. *If $\lambda, \mu \in K$ are distinct eigenvalues of $\bar{A}$, then at most one of the corresponding eigenspaces is infinite-dimensional.*

Proof. Let W_λ (resp. W_μ) be the eigenspace of $\bar{A}$ corresponding to the eigenvalue λ (resp. μ). Striving for a contradiction, assume that both W_λ and W_μ are infinite-dimensional. Observe that both W_λ and W_μ are isomorphic to l_2, whence, by Lemma 2.1, $\dim W_\lambda = \dim W_\mu = \mathfrak{c}$. Let $E = W_\lambda \oplus W_\mu$ and, for every vector $x \in E$, write x uniquely in the form $x = x_\lambda + x_\mu$, where $x_\lambda \in W_\lambda$ and $x_\mu \in W_\mu$. Define

$$L^\alpha_\lambda = \{x_\lambda : x \in L^\alpha \cap E\} \quad \text{and} \quad L^\alpha_\mu = \{x_\mu : x \in L^\alpha \cap E\}.$$

Since $\dim L^\alpha < \mathfrak{c}$, it follows easily that both L^α_λ and L^α_μ have algebraic dimension less than $\mathfrak{c}$.

By transfinite induction, for every $\xi < \mathfrak{c}$ take $x_\xi \in W_\lambda$ and $y_\xi \in W_\mu$ such that

$$x_\xi \notin \operatorname{sp}(L^\alpha_\lambda \cup \{x_\eta : \eta < \xi\}) \quad \text{and} \quad y_\xi \notin \operatorname{sp}(L^\alpha_\mu \cup \{y_\eta : \eta < \xi\}).$$

It is possible to do this, since at each stage of the induction we have

$$\dim(L^\alpha_\lambda \cup L^\alpha_\mu \cup \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\}) < \mathfrak{c} \quad \text{and} \quad \dim W_\lambda = \dim W_\mu = \mathfrak{c}.$$

As in the proof of Lemma 3.1 it follows that $X = \{x_\xi : \xi < \mathfrak{c}\}$ and $Y = \{y_\xi : \xi < \mathfrak{c}\}$ both are linearly independent. For every $\xi < \mathfrak{c}$ define $p_\xi = x_\xi + y_\xi$. Since $W_\lambda \cap W_\mu = \{0\}$, it follows easily that $\{p_\xi : \xi < \mathfrak{c}\}$ is linearly independent. Suppose that there exist a $\xi < \mathfrak{c}$ and a scalar $\gamma \in K$ such that $\bar{A}p_\xi - \gamma p_\xi \in L^\alpha$. Since

$$z = \bar{A}p_\xi - \gamma p_\xi = \lambda x_\xi + \mu y_\xi - \gamma x_\xi - \gamma y_\xi = (\lambda - \gamma) x_\xi + (\mu - \gamma) y_\xi \in L^\alpha \cap E,$$

we conclude that

$$z_\lambda = (\lambda - \gamma) x_\xi \quad \text{and} \quad z_\mu = (\mu - \gamma) y_\xi.$$

Since $x_\xi \notin L^\alpha_\lambda$, we obtain $\lambda - \gamma = 0$, and similarly $\mu - \gamma = 0$, whence $\lambda = \mu$, which is a contradiction.

We therefore conclude that $\bar{A}p_\xi \notin \operatorname{sp}(\{p_\xi\} \cup L^\alpha)$ for every $\xi < \mathfrak{c}$. Since $\dim\{p_\xi : \xi < \mathfrak{c}\} = \mathfrak{c}$, we have

$$\dim\{x \in l_2 : \bar{A}x \notin \operatorname{sp}(\{x\} \cup L^\alpha)\} = \mathfrak{c},$$

and, consequently, by (5) of Section 3, there is a vector $x \in L_\alpha \subseteq L$ such that

$$\bar{A}x = F_\alpha(x) \in V_\alpha \subseteq l_2 \setminus L.$$

However, $\bar{A}$ extends A, whence $\bar{A}x = Ax \in L$. This is a contradiction.

4.2. LEMMA. *Every orthogonal set of eigenvectors of $\bar{A}$ corresponding to distinct eigenvalues is finite.*

Proof. Striving for a contradiction, suppose that λ_n, $n \in N$, are infinitely many distinct eigenvalues of $\bar{A}$ and that x_n is an eigenvector of $\bar{A}$ corresponding to the eigenvalue λ_n, while moreover the set $\{x_n: n \in N\}$ is orthonormal. Let $\mathscr{B}$ be a family of $\mathfrak{c}$ infinite subsets of N such that for all distinct $B, B' \in \mathscr{B}$ the set $B \cap B'$ is finite (see Section 2). For every $B \in \mathscr{B}$ define $x(B) \in l_2$ by

$$x(B) = \sum_{n \in B} 2^{-n} x_n.$$

It is clear that $x(B)$ is well defined (because of the factor 2^{-n}).

CLAIM 1. *$\{x(B): B \in \mathscr{B}\}$ is linearly independent.*

Take $B_1, \ldots, B_n \in \mathscr{B}$ and $\gamma_1, \ldots, \gamma_n \in K$ such that

$$\sum_{i=1}^{n} \gamma_i x(B_i) = 0.$$

Without loss of generality, $\gamma_1 \neq 0$. Take $m \in B_1 \setminus (B_2 \cup \ldots \cup B_n)$. Then

$$0 = \langle 0, x_m \rangle = \left\langle \sum_{i=1}^{n} \gamma_i x(B_i), x_m \right\rangle = \sum_{i=1}^{n} \gamma_i \langle x(B_i), x_m \rangle$$
$$= \sum_{i=1}^{n} \gamma_i \left\langle \sum_{j \in B_i} 2^{-j} x_j, x_m \right\rangle = \sum_{i=1}^{n} \gamma_i \sum_{j \in B_i} 2^{-j} \langle x_j, x_m \rangle = \gamma_1 \cdot 2^{-m} \langle x_m, x_m \rangle.$$

We conclude that $\gamma_1 = 0$, which is a contradiction.

CLAIM 2. *If $\mathscr{M} = \{B \in \mathscr{B}: \bar{A}x(B) \in \operatorname{sp}(\{x(B)\} \cup L^\alpha)\}$, then $|\mathscr{M}| < \mathfrak{c}$.*

Suppose not. For each $B \in \mathscr{M}$ take $\gamma(B) \in K$ such that

$$\bar{A}x(B) - \gamma(B)\, x(B) \in L^\alpha.$$

Observe that L^α is a union of fewer than $\mathfrak{c}$ finite-dimensional linear subspaces. Since by assumption $|\mathscr{M}| = \mathfrak{c}$, there are infinitely many distinct $B_n \in \mathscr{M}$, $n \in N$, and a finite-dimensional linear subspace $E \subseteq L^\alpha$ such that

$$\bar{A}x(B_n) - \gamma(B_n)\, x(B_n) \in E$$

for all $n \in N$. Let $m = \dim E$ and find $\delta_i \in K$ for $i \leqslant m+1$ such that

$$\sum_{i=1}^{m+1} \delta_i \left(\bar{A}x(B_i) - \gamma(B_i)\, x(B_i)\right) = 0,$$

while moreover $\delta_i \neq 0$ for certain $i \leqslant m+1$. Without loss of generality, $\delta_1 \neq 0$. Consequently,

$$\sum_{i=1}^{m+1} \delta_i \Big(\sum_{j \in B_i} \lambda_j \cdot 2^{-j} x_j - \gamma(B_i) \sum_{j \in B_i} 2^{-j} x_j \Big) = \sum_{i=1}^{m+1} \delta_i \Big(\sum_{j \in B_i} 2^{-j} (\lambda_j - \gamma(B_i))\, x_j \Big) = 0.$$

Choose $k \in B_1 \setminus (B_2 \cup \ldots \cup B_{m+1})$ such that $\lambda_k \neq \gamma(B_1)$ (there is at most one $n \in N$ with $\lambda_n = \gamma(B_1)$ and there are infinitely many indices in $B_1 \setminus \bigcup_{i=2}^{m+1} B_i$). Then

$$\begin{aligned} 0 = \langle 0, x_k \rangle &= \Big\langle \sum_{i=1}^{m+1} \delta_i \Big(\sum_{j \in B_i} 2^{-j} (\lambda_j - \gamma(B_i)) x_j \Big), x_k \Big\rangle \\ &= \sum_{i=1}^{m+1} \delta_i \sum_{j \in B_i} 2^{-j} (\lambda_j - \gamma(B_i)) \langle x_k, x_j \rangle \\ &= \delta_1 \cdot 2^{-k} (\lambda_k - \gamma(B_1)) \langle x_k, x_k \rangle. \end{aligned}$$

We conclude that $\delta_1 (\lambda_k - \gamma(B_1)) = 0$, which is impossible since $\delta_1 \neq 0$ and $\lambda_k \neq \gamma(B_1)$.

Claims 1 and 2 imply that

$$\dim \{x \in l_2 \colon \bar{A}x \notin \mathrm{sp}(\{x\} \cup L^\alpha)\} = \mathfrak{c}.$$

By (5) of Section 3, there is a vector $x \in L_\alpha \subseteq L$ such that

$$\bar{A}x = F_\alpha(x) \in V_\alpha \subseteq l_2 \setminus L.$$

However, $\bar{A}$ extends A, whence $\bar{A}x = Ax \in L$. This is a contradiction.

4.3. LEMMA. *Let $E \subseteq L$ be a linear subspace such that* $\dim E = \mathfrak{c}$. *Then E contains an eigenvector of A.*

Proof. Let $S = \{x \in l_2 \colon \bar{A}x \notin \mathrm{sp}(\{x\} \cup L^\alpha)\}$. If $\dim S = \mathfrak{c}$, then we can derive a contradiction in precisely the same way as in the proofs of the previous lemmas. Therefore, $\dim S < \mathfrak{c}$. Let $H = \mathrm{sp}(L^\alpha \cup S)$. Then

$$\dim H \leqslant \max \{\dim L^\alpha, \dim S\} < \mathfrak{c}.$$

By transfinite induction, for every $\xi < \mathfrak{c}$ define a vector $t_\xi \in E$ such that

$$t_\xi \notin \mathrm{sp}(H \cup \{t_\eta \colon \eta < \xi\}).$$

This is possible, since at step ξ of the transfinite construction we have

$$\dim \mathrm{sp}(H \cup \{t_\eta \colon \eta < \xi\}) < \mathfrak{c} \quad \text{and} \quad \dim E = \mathfrak{c}.$$

Let $G = \mathrm{sp}\{t_\xi \colon \xi < \mathfrak{c}\}$. We claim that $G \cap H = \{0\}$. Suppose not. Then there is a vector $x \in G \cap H$ such that $x \neq 0$. We can write x in the form $\sum_{i=1}^{n} \lambda_i t_{\xi_i}$, where $\xi_1 < \xi_2 < \ldots < \xi_n$ and $\lambda_i \neq 0$ for every $i \leqslant n$. Then

$$t_{\xi_n} = x - \sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} t_{\xi_i} \in \mathrm{sp}(H \cup \{t_\eta \colon \eta < \xi_{n-1}\}),$$

which contradicts the definition of t_{ξ_n}. We conclude that $G \cap H = \{0\}$.

Let $T = \{t_\xi : \xi < \mathfrak{c}\}$. Take distinct $x, y \in T$. Since $G \cap H = \{0\}$, we have

$$\bar{A}x \in \operatorname{sp}(\{x\} \cup L^\alpha) \quad \text{and} \quad \bar{A}y \in \operatorname{sp}(\{y\} \cup L^\alpha).$$

Take $\lambda, \mu \in K$ and $a, b \in L^\alpha$ such that

$$\bar{A}x = \lambda x + a \quad \text{and} \quad \bar{A}y = \mu y + b.$$

Since $x - y \in G$, there are $\gamma \in K$ and $c \in L^\alpha$ (both possibly 0) such that

$$\bar{A}(x-y) = \gamma(x-y) + c.$$

Consequently, $(\lambda x + a) - (\mu y + b) = \gamma(x-y) + c$, and therefore

$$(\lambda - \gamma)x + (\gamma - \mu)y = c - a + b \in L^\alpha \subseteq H.$$

Since $(\lambda-\gamma)x + (\gamma-\mu)y \in G$ and $G \cap H = \{0\}$, this implies that

$$(\lambda-\gamma)x + (\gamma-\mu)y = 0.$$

As in the proof of Lemma 3.1 it follows that T is linearly independent. We therefore can conclude that $\lambda - \gamma = 0$ and $\gamma - \mu = 0$. Consequently, $\lambda = \mu$.

We infer that there is a fixed scalar $\lambda \in K$ such that $\bar{A}t - \lambda t \in H$ for every $t \in T$. Observe that H is the union of fewer than $\mathfrak{c}$ finite-dimensional linear subspaces. Since $|T| = \mathfrak{c}$, as in the proof of Lemma 4.2, we can find distinct vectors t_n, $n \in N$, and a finite-dimensional linear subspace $B \subseteq H$ such that $\bar{A}t_n - \lambda t_n \in B$ for all $n \in N$. Let $m = \dim B$ and find $\delta_i \in K$ for $i \leqslant m+1$, such that

$$\sum_{i=1}^{m+1} \delta_i (\bar{A}t_i - \lambda t_i) = 0,$$

while moreover $\delta_i \neq 0$ for certain $i \leqslant m+1$. For each $i \leqslant m+1$, let $b_i = \bar{A}t_i - \lambda t_i$. Then

$$\bar{A}\Big(\sum_{i=1}^{m+1} \delta_i t_i\Big) = \sum_{i=1}^{m+1} \delta_i \bar{A}t_i = \sum_{i=1}^{m+1} \delta_i \lambda t_i + \sum_{i=1}^{m+1} \delta_i b_i$$
$$= \lambda\Big(\sum_{i=1}^{m+1} \delta_i t_i\Big).$$

Since $\delta_i \neq 0$ for certain $i \leqslant m+1$ and since the t_i's are linearly independent, $\sum_{i=1}^{m+1} \delta_i t_i$ is an eigenvector of A belonging to the linear subspace E.

4.4. LEMMA. *There is a unique eigenvalue of A such that the corresponding eigenspace is complemented and has finite codimension (in L).*

Proof. Define $S_0 = \operatorname{Ker} \bar{A}$ and $\lambda_0 = 0$. By induction, for every $n \in N$ we will construct a sequence of closed linear subspaces $S_n \subseteq l_2$ and a sequence of scalars $\lambda_n \in K$ such that, for all $n \in N$,

(1) if $\dim(S_0 \cup S_1 \cup \ldots \cup S_{n-1})^\perp$ is finite, then $S_n = S_{n-1}$ and $\lambda_n = \lambda_{n-1}$;
(2) if $\dim(S_0 \cup S_1 \cup \ldots \cup S_{n-1})^\perp$ is infinite, then
(a) λ_n is an eigenvalue of $\bar{A}$,
(b) if W_{λ_n} is the eigenspace corresponding to the eigenvalue λ_n, then

$$S_n = W_{\lambda_n} \cap (S_0 \cup \ldots \cup S_{n-1})^\perp,$$

(c) $S_n \neq \{0\}$.

Suppose that we have constructed S_i and λ_i for all i $(0 \leqslant i \leqslant n-1)$. Define T by

$$T = (S_0 \cup S_1 \cup \ldots \cup S_{n-1})^\perp.$$

If $\dim T$ is finite, then (1) tells us how to define S_n and λ_n. Suppose therefore that $\dim T$ is infinite. By Lemma 3.1, $\dim(L \cap T) = \mathfrak{c}$. From Lemma 4.3 it therefore follows that $L \cap T$ contains an eigenvector of A, say corresponding to the eigenvalue λ. This eigenvector is of course also an eigenvector of $\bar{A}$ corresponding to the eigenvalue λ. Define $\lambda_n = \lambda$ and let S_n be defined by $S_n = W_\lambda \cap T$, where W_λ denotes the eigenspace of $\bar{A}$ corresponding to the eigenvector λ. This completes the induction.

Let k be the smallest integer for which $\dim(S_0 \cup \ldots \cup S_k)^\perp$ is finite, if such an integer exists, or ∞ otherwise.

CLAIM. *If* $n < m < k$, *then* $\lambda_n \neq \lambda_m$.

Striving for a contradiction, assume that $\lambda_n = \lambda_m = \lambda$. Let

$$M = S_0 \cup \ldots \cup S_{n-1}.$$

Since

$$S_n = W_\lambda \cap M^\perp \supseteq W_\lambda \cap (M \cup S_n \cup S_{n+1} \cup \ldots \cup S_{m-1})^\perp = S_m,$$

we have $S_m \subseteq S_n$ and $S_m \perp S_n$. This implies that $S_m = \{0\}$, which contradicts (c).

Suppose first that $k = \infty$. By (c), we can take $x_n \in S_n \setminus \{0\}$ for all $n \in N$. Then $\{x_n \colon n \in N\}$ is an orthogonal set of eigenvectors which, by the Claim, correspond to distinct eigenvalues. This contradicts Lemma 4.2.

We conclude that $k \in N$. Observe that

$$l_2 = S_0 \oplus \ldots \oplus S_k \oplus (S_0 \cup \ldots \cup S_k)^\perp.$$

Since l_2 is infinite-dimensional, there is an i with $0 \leqslant i \leqslant k$ such that S_i is infinite-dimensional. By Lemma 4.1 this i is unique (use the Claim). Define

$$E = S_0 \oplus \ldots \oplus S_{i-1} \oplus S_{i+1} \oplus \ldots \oplus S_k \oplus (S_0 \cup \ldots \cup S_k)^\perp.$$

Then E is a finite-dimensional (closed) subspace of l_2 such that $l_2 = E \oplus S_i$.

Let $\tilde{S}_i = S_i \cap L$. Then $\tilde{S}_i$ is the eigenspace of A corresponding to the eigenvalue λ_i. We will show that $\tilde{S}_i$ has a finite-dimensional complement in L. The argument we give is routine; it is only included for completeness sake. Let F be a linear subspace of L such that $L = F \oplus \tilde{S}_i$. Let $P \colon l_2 \to E$

be the orthogonal projection. Since $\operatorname{Ker} P = S_i$ and $F \cap S_i = \{0\}$, P must be one-to-one on F. We conclude that $\dim F \leqslant \dim E < \infty$.

All there remains to prove now is that λ_i is unique. But this is a direct consequence of Lemma 4.1 (use that L is infinite-dimensional).

We have completed the proof of the following

4.5. THEOREM. *There is an infinite-dimensional pre-Hilbert space L over K with the following property: every bounded linear operator A of L has a unique eigenvalue such that the corresponding eigenspace is complemented and has finite codimension in L.*

4.6. Remark. As was observed by M. A. Kaashoek, the above property of L is equivalent to the following one: for every bounded linear operator A of L there is a unique scalar $\lambda \in K$ such that $A - \lambda I$ has finite rank.

4.7. Remark. For related results, see [3] and [2].

REFERENCES

[1] I. Gohberg and S. Goldberg, *Basic Operator Theory*, Boston 1981.
[2] J. van Mill, *A topological group having no homeomorphisms other than translations*, Trans. Amer. Math. Soc. 280 (1983), pp. 491–498.
[3] T. Ohkuma, *Sur quelques ensembles ordonnés linéairement*, Fund. Math. 43 (1955), pp. 326–337.
[4] W. Sierpiński, *Sur une décomposition d'ensembles*, Monatsh. Math. Phys. 35 (1928), pp. 239–242.

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