A remark on the separable extension property

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ABSTRACT

We present an example of a metrizable space having the separable extension property but which is not an Absolute Neighborhood Retract.

1. INTRODUCTION

All spaces considered in this note are metrizable; for terminology we refer the reader to [1], [2] *and* [3].

We say that E has the *separable* (*compact*) extension property if for every space X and every separable closed (compact) subspace A of X, every continuous map $f: A \to E$ has a continuous extension $f': X \to E$.

In [5], J. van Mill constructed a separable space which has the compact extension property but is not an **ANR**.

In this note we present a variation of the construction from [5], which provides the following

1.1. EXAMPLE. *There exists a space E which has the separable extension property but which is not an* **ANR**.

2. THE CONSTRUCTION

Let $\mathfrak{c}^+$ be the first cardinal greater than the continuum $\mathfrak{c}$. Let B and B' be

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the closed unit balls in the separable Hilbert space l^2 and in the Hilbert space $l^2(\mathfrak{c}^+)$, respectively, and let S and S' be the unit spheres in B and B', respectively. Let us notice that for every subset A of $S\times S'$, the set

(1) $((B\times B')\setminus(S\times S'))\cup A,$

considered in the Hilbert space $l^2\times l^2(\mathfrak{c}^+)$, is convex and hence is an **AR**.

2.1. THE TAYLOR MAP. By Taylor [7], there exists a compact space T and a cell-like map $\tau: T\to M$, where M is homeomorphic to the Hilbert cube, which is not a shape equivalence. We shall assume that T is imbedded in the sphere S. Let $Z=B\cup_\tau M$ and let $p: B\to Z$ be the adjunction projection; we shall identify M and $p(T)$. Let us recall that Z is not an **ANR** and that the map p is cell-like, [7].

2.2. A DECOMPOSITION OF THE BAIRE SPACE OF DENSITY $\mathfrak{c}^+$. Let C be a closed subspace of the sphere S' which is homeomorphic to the countable infinite product of copies of the discrete space of cardinality $\mathfrak{c}^+$. There exists a decomposition $\{C_z: z\in M\}$ of C into pairwise disjoint sets such that

(2) every separable set in C intersects at most countably many sets C_z;

(3) $\begin{cases} \text{if for every } z\in M,\ G_z \text{ is a } G_\delta\text{-set in } C \text{ containing } C_z, \\ \text{then } \bigcap\{G_z: z\in M\}\neq\emptyset. \end{cases}$

For details related to this decomposition we refer to Elzbieta Pol [6], where the decomposition was employed in a similar way as in this note.

2.3. THE SPACE E. The space E is a non-separable analogue to the space defined in [5]. Let $Z'=(B\times B')\cup_k(Z\times C)$, where $k=p\times \mathrm{id}_C: B\times C\to Z\times C$, where id_C denotes the identity mapping on C, and let q denote the adjunction projection; we shall identify $Z\times C$ and $q(B\times C)$. Now,

(4) $E=(Z'\setminus(M\times C))\cup\bigcup\{\{z\}\times C_z: z\in M\}.$

3. E HAS THE SEPARABLE EXTENSION PROPERTY

To begin with, let us repeat the reasoning from the proof of lemma 2.1 in [5], to ensure that for any countable set F in M the space

(5) $E(F)=(Z'\setminus(M\times C))\cup\bigcup\{\{z\}\times C_z: z\in F\}$

is an **AR**. Let us note that the projection $q: B\times B'\to Z'$ is cell-like and that with

$$A=((S\times S')\setminus(T\times C))\cup\bigcup\{p^{-1}(z)\times C_z: z\in F\}$$

we can write $D=q^{-1}(E(F))$ in the form (1). Therefore, $E(F)$ is an image of an **AR** (the space D) under a cell-like map (the restriction of q to D) whose set of non-degeneracy points is contained in $F\times C$ and hence is zero-dimensional. It follows that $E(F)$ is an **AR**, [4], [1].

Now let $f: A\to E$ be a continuous map defined on a separable closed subspace

A of a space X. Since $f(A)$ is separable, so is the projection of $f(A) \cap (M \times C)$ onto the C-axis, and by property (2) there exists a countable set $F \subseteq M$ such that $f(A)$ is contained in $E(F)$, see (4) and (5). Since $E(F)$ is an **AR**, the map $f: A \to E(F)$ can be extended to a continuous map $f': X \to E(F) \subseteq E$.

4. E IS NOT AN **ANR**

This part of the proof corresponds to the proof of lemma 2.2 in [5]. Let us consider Z as a closed subspace of a normed linear space L and let

$$N = ((L \times C) \setminus (Z \times C)) \cup H,$$

where

$$H = ((Z \setminus M) \times C) \cup \bigcup \{\{z\} \times C_z : z \in M\}.$$

Striving for a contradiction, assume that E is an **ANR**. Since E has the separable extension property, it is C^∞ and therefore an **AR** by [3]. Consequently, the identity embedding $e: H \to E$ of the closed subset H of N into E can be extended to a continuous map $f: N \to E$, i.e. $f(z, y) = (z, y)$ for each $(z, y) \in H$. Since Z' is a completely metrizable space containing E, by the Lavrentieff Theorem, there exists a G_δ-set G in $L \times C$ containing N such that f extends to a continuous map $g: G \to Z'$. For each $z \in M$, $G_z = \{y \in C : (z, y) \in G\}$ is a G_δ-set in C containing C_z and by property (3), there exists an $a \in \bigcap \{G_z : z \in M\}$. Let us notice that $G \supseteq L \times \{a\}$ and therefore one can define a continuous map $s: L \to Z'$ by the formula $s(x) = g(x, a)$; observe that if $x \in Z \setminus M$ then $(x, a) \in H$, so $g(x, a) = (x, a)$ and hence $s(x) = (x, a)$ for every $x \in Z$, the set $Z \setminus M$ being dense in Z. To finish the proof let us consider the following commutative diagram:

$$\begin{array}{ccc} & B \times B' & \\ {\scriptstyle q} \swarrow & & \searrow {\scriptstyle p \times \mathrm{id}_{B'}} \\ Z' & \xrightarrow{\quad l \quad} & Z \times B', \end{array}$$

where l is the uniquely defined continuous map whose restriction to $Z \times C$ is the identity. Let us define $r: L \to Z$ by $r(x) = \mathrm{proj}(l(s(x)))$, proj being the projection onto the Z-axis. For each $z \in Z$ we have $s(z) = (z, a)$ and $l(s(z)) = (z, a)$ and consequently, $r(z) = z$. We conclude that r is a retraction of the normed linear space L onto Z, which contradicts the fact that Z is not an **AR**.

5. REMARK

Let the "density λ extension property" be defined by replacing the separability condition in the definition of the separable extension property in § 1 by the condition "density $\leq \lambda$".

Let E be the space defined in § 2. For any convex set $W \subseteq B'$ the space $E_W = q(B \times W) \cap E$ has the separable extension property; this can be verified by similar arguments as the ones in § 3. Let K be a convex subset of B' of minimal possible density λ such that E_K is not an **AR** (notice that $\lambda > \aleph_0$). Let

A be a closed subset of a space X, let $f: A \to E_K$ be a continuous map and let us assume that the density of A is less than λ. Since $f(A)$ is contained in a set of type $q(B \times W)$, where W is a convex set of K of density less than λ, by the minimality of λ, the map $f: A \to E_W$ has a continuous extension $f': X \to E_W \subseteq \subseteq E_K$. The space $E' = E_K$ has therefore the following properties:

> *E' is a space of density $\lambda > \aleph_0$ which has the density κ extension property for each $\kappa < \lambda$, but E' is not an* **AR**.

Under the Continuum Hypothesis, the cardinal number λ is either $\aleph_1$ or $\aleph_2$, but our reasoning does not decide which one of these possibilities occurs.

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