A remark on the separable extension property

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ABSTRACT

We present an example of a metrizable space having the separable extension property but which is not an Absolute Neighborhood Retract.

1. INTRODUCTION

All spaces considered in this note are metrizable; for terminology we refer the reader to [1], [2] and [3].

We say that $E$ has the separable (compact) extension property if for every space $X$ and every separable closed (compact) subspace $A$ of $X$, every continuous map $f:A \rightarrow E$ has a continuous extension $f':X \rightarrow E$.

In [5], J. van Mill constructed a separable space which has the compact extension property but is not an ANR.

In this note we present a variation of the construction from [5], which provides the following

1.1. EXAMPLE. There exists a space $E$ which has the separable extension property but which is not an ANR.

2. THE CONSTRUCTION

Let $c^+$ be the first cardinal greater than the continuum $c$. Let $B$ and $B'$ be

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the closed unit balls in the separable Hilbert space $l^2$ and in the Hilbert space $l^2(c^+)$, respectively, and let $S$ and $S'$ be the unit spheres in $B$ and $B'$, respectively. Let us notice that for every subset $A$ of $S \times S'$, the set

$$(B \times B') \setminus (S \times S') \cup A,$$

considered in the Hilbert space $l^2 \times l^2(c^+)$, is convex and hence is an AR.

2.1. THE TAYLOR MAP. By Taylor [7], there exists a compact space $T$ and a cell-like map $\tau : T \to M$, where $M$ is homeomorphic to the Hilbert cube, which is not a shape equivalence. We shall assume that $T$ is imbedded in the sphere $S$. Let $Z = BU$, $M$ and let $p : B \to Z$ be the adjunction projection; we shall identify $M$ and $p(T)$. Let us recall that $Z$ is not an ANR and that the map $p$ is cell-like, [7].

2.2. A DECOMPOSITION OF THE BAIRE SPACE OF DENSITY $c^+$. Let $C$ be a closed subspace of the sphere $S'$ which is homeomorphic to the countable infinite product of copies of the discrete space of cardinality $c^+$. There exists a decomposition $\{C_z : z \in M\}$ of $C$ into pairwise disjoint sets such that

2. For every separable set in $C$ intersects at most countably many sets $C_z$:

$$\left\{\begin{array}{c}
\text{if for every } z \in M, G_z \text{ is a } G_\delta\text{-set in } C \text{ containing } C_z, \\
\text{then } \cap \{G_z : z \in M\} \neq \emptyset.
\end{array}\right.$$  

For details related to this decomposition we refer to Elzieta Pol [6], where the decomposition was employed in a similar way as in this note.

2.3. THE SPACE $E$. The space $E$ is a non-separable analogue to the space defined in [5]. Let $Z' = (B \times B') \cup_k (Z \times C)$, where $k = p \times \text{id}_C : B \times C \to Z \times C$, where $\text{id}_C$ denotes the identity mapping on $C$, and let $q$ denote the adjunction projection; we shall identify $Z \times C$ and $q(B \times C)$. Now,

$$E = (Z' \setminus (M \times C)) \cup \bigcup \{\{z\} \times C_z : z \in M\}.  
$$

3. $E$ HAS THE SEPARABLE EXTENSION PROPERTY

To begin with, let us repeat the reasoning from the proof of lemma 2.1 in [5], to ensure that for any countable set $F$ in $M$ the space

$$E(F) = (Z' \setminus (M \times C)) \cup \bigcup \{\{z\} \times C_z : z \in F\}  
$$

is an AR. Let us note that the projection $q : B \times B' \to Z'$ is cell-like and that with

$$A = (S \times S') \setminus (T \times C) \cup \bigcup \{p^{-1}(z) \times C_z : z \in F\}  
$$

we can write $D = q^{-1}(E(F))$ in the form (1). Therefore, $E(F)$ is an image of an AR (the space $D$) under a cell-like map (the restriction of $q$ to $D$) whose set of non-degeneracy points is contained in $F \times C$ and hence is zero-dimensional. It follows that $E(F)$ is an AR, [4], [1].

Now let $f : A \to E$ be a continuous map defined on a separable closed subspace
A of a space $X$. Since $f(A)$ is separable, so is the projection of $f(A) \cap (M \times C)$ onto the $C$-axis, and by property (2) there exists a countable set $F \subset M$ such that $f(A)$ is contained in $E(F)$, see (4) and (5). Since $E(F)$ is an AR, the map $f: A \rightarrow E(F)$ can be extended to a continuous map $f': X \rightarrow E(F) \subset E$.

4. E IS NOT AN ANR

This part of the proof corresponds to the proof of lemma 2.2 in [5]. Let us consider $Z$ as a closed subspace of a normed linear space $L$ and let

$$N = ((L \times C) \setminus (Z \times C)) \cup H,$$

where

$$H = ((Z \setminus M) \times C) \cup \bigcup \{\{z\} \times C_z : z \in M\}.$$

Striving for a contradiction, assume that $E$ is an ANR. Since $E$ has the separable extension property, it is $C^\omega$ and therefore an AR by [3]. Consequently, the identity embedding $e: H \rightarrow E$ of the closed subset $H$ of $N$ into $E$ can be extended to a continuous map $f: N \rightarrow E$, i.e. $f(z, y) = (z, y)$ for each $(z, y) \in H$. Since $Z'$ is a completely metrizable space containing $E$, by the Lavrentieff Theorem, there exists a $G_\delta$-set $G$ in $L \times C$ containing $N$ such that $f$ extends to a continuous map $g: G \rightarrow Z'$. For each $z \in M$, $G_z = \{y \in C : (z, y) \in G\}$ is a $G_\delta$-set in $C$ containing $C_z$, and by property (3), there exists an $a \in \bigcap \{G_z : z \in M\}$. Let us notice that $G \supseteq L \times \{a\}$ and therefore one can define a continuous map $s: L \rightarrow Z'$ by the formula $s(x) = g(x, a)$; observe that if $x \in Z \setminus M$ then $(x, a) \in H$, so $g(x, a) = (x, a)$ and hence $s(x) = (x, a)$ for every $x \in Z$, the set $Z \setminus M$ being dense in $Z$. To finish the proof let us consider the following commutative diagram:

$$\begin{CD}
B \times B' @>{p \times id_B}>> Z \times B' \\
@A{q}AA @VV{l}V \\
Z' @>>> Z \times B',
\end{CD}$$

where $l$ is the uniquely defined continuous map whose restriction to $Z \times C$ is the identity. Let us define $r: L \rightarrow Z$ by $r(x) = \text{proj}(l(s(x)))$, $\text{proj}$ being the projection onto the $Z$-axis. For each $z \in Z$ we have $s(z) = (z, a)$ and $l(s(z)) = (z, a)$ and consequently, $r(z) = z$. We conclude that $r$ is a retraction of the normed linear space $L$ onto $Z$, which contradicts the fact that $Z$ is not an AR.

5. REMARK

Let the “density $\lambda$ extension property” be defined by replacing the separability condition in the definition of the separable extension property in § 1 by the condition “density $\leq \lambda$”.

Let $E$ be the space defined in § 2. For any convex set $W \subset B'$ the space $E_W = q(B \times W) \cap E$ has the separable extension property; this can be verified by similar arguments as the ones in § 3. Let $K$ be a convex subset of $B'$ of minimal possible density $\lambda$ such that $E_K$ is not an AR (notice that $\lambda > \aleph_0$). Let
A be a closed subset of a space $X$, let $f: A \to E_K$ be a continuous map and let us assume that the density of $A$ is less than $\lambda$. Since $f(A)$ is contained in a set of type $q(B \times W)$, where $W$ is a convex set of $K$ of density less than $\lambda$, by the minimality of $\lambda$, the map $f: A \to E_W$ has a continuous extension $f': X \to E_W \subseteq E_K$. The space $E' = E_K$ has therefore the following properties:

$E'$ is a space of density $\lambda > \aleph_0$ which has the density $\kappa$ extension property for each $\kappa < \lambda$, but $E'$ is not an AR.

Under the Continuum Hypothesis, the cardinal number $\lambda$ is either $\aleph_1$ or $\aleph_2$, but our reasoning does not decide which one of these possibilities occurs.

REFERENCES