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AUTOMORPHISM GROUPS FOR MEASURABLE SPACES*

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In [1], the notion of a rigid separable measurable space was defined, and such spaces were shown to exist. We expand upon this idea and ask what are the possible automorphism groups for such spaces. We show that there is only one such non-trivial group which is Abelian (a countable product of two element groups), and that this group is realised as the automorphism group of some separable space. A particular class of such spaces is characterised in terms of rigid components.

Finally, an example of a measurable space which is rigid in the strict sense is constructed. This answers a question of K.P.S. Bhaskara Rao and B.V. Rao.

0. Preliminaries

We work in the context of separable spaces. A measurable space $(X, \mathcal{B})$ is *separable* if its Borel structure $\mathcal{B}$ is countably generated and contains all singleton subsets of X. If $(X, \mathcal{B})$ is separable, and $A \subseteq X$, then A becomes a separable space with Borel structure $\mathcal{B}(A) = \{B \cap A: B \in \mathcal{B}\}$.

0.1. Lemma. *Let $(X, \mathcal{B})$ be a separable space. There is a metric d on X such that (X, d) is homeomorphic with a subset of the line $\mathbb{R}$, and $\mathcal{B}$ is the Borel σ-algebra generated by d.*

Indication. Originally due to Marczewski. A proof may be found in [1, p. 9].

A separable space $(X, \mathcal{B})$ is *standard* if there is a complete separable metric (i.e. Polish) topology on X for which $\mathcal{B}$ is the Borel σ-algebra. In view of Lemma 0.1,

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every separable space is isomorphic with a subset of some standard space. The following result is easily proved:

0.2. Lemma. *Let $(X, \mathscr{B})$ be an uncountable separable space. Then there are pair-wise disjoint uncountable sets $B_1 B_2 \cdots$ in $\mathscr{B}$ with $X = B_1 \cup B_2 \cup \cdots$.*

If $(X_1, \mathscr{B}_1) \cdots (X_n, \mathscr{B}_n)$ are separable spaces with $X_1, \ldots, X_n$ pairwise disjoint, define

$$S = \bigcup X_i, \quad \mathscr{B} = \{\bigcup B_i \colon B_i \in \mathscr{B}_i, i = 1, \ldots, n\}.$$

Then the separable space $(S, \mathscr{B})$ is the *direct union* of the $(X_i, \mathscr{B}_i)$.

A function $f : X \to Y$ between separable spaces $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ is *measurable* if $f^{-1}(C) \in \mathscr{B}$ whenever $C \in \mathscr{C}$. If f is a one-one correspondence of X and Y, and both f and f^{-1} are measurable, then f is an *isomorphism.* An isomorphism $f : X \to X$ of $(X, \mathscr{B})$ onto itself is an *automorphism.*

0.3. Lemma. *Let $(S, \mathscr{B})$ be a standard space and $X \subseteq S$. Suppose $f : X \to X$ is an automorphism of $(X, \mathscr{B}(X))$. Then there is an automorphism $g : S \to S$ such that f is the restriction of g to X.*

Indication. This follows from the Kuratowski-Lavrentiev extension theorem [5, p. 436].

In view of Lemmas 0.1 and 0.3 and the fact that there are exactly $\mathfrak{c}$ Borel automorphisms of $\mathbb{R}$, we have:

0.4. Lemma. *Let $(X, \mathscr{B})$ be an infinite separable space. There are exactly $\mathfrak{c}$ automorphisms of X.*

Let $f : X \to X$ be an automorphism of a separable space X. We define $f^n : X \to X$ for $n \in \mathbb{Z}$ such that for $x \in X$,

$$f^0(x) = x,$$

$$f^{n+1}(x) = f(f^n(x)) \quad \text{for } n \geq 0,$$

$$f^{-n-1}(x) = f^{-1}(f^{-n}(x)) \quad \text{for } n \leq 0.$$

An *orbit of f* is a set of the form

$$\{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\}.$$

Clearly, if V is an orbit of f, then $f(V) = V$. For each $x \in X$, define

$$\mathrm{O}(f, x) = \min\{n \colon f^n(x) = x, n \geq 1\},$$

putting $\mathrm{O}(f, x) = \infty$ if $f^n(x) \neq x$ for each $n \geq 1$. We call $\mathrm{O}(f, x)$ the *order of f at x.* Note that the function $x \to \mathrm{O}(f, x)$ is measurable.

0.5. Lemma. *Let $f: X \to X$ be an automorphism of a separable space X such that $\mathrm{O}(f, x) < \infty$ for each $x \in X$. Then there is a set $B \in \mathscr{B}(X)$ that contains exactly one point from each orbit of f.*

Proof. By Lemma 0.1, we may suppose that X is a subset of $\mathbb{R}$. Set

$$B = \{x : f^n(x) \geqslant x \text{ for each } n\}.$$

Compare III. 8.3 in [3]. □

Let $(X, \mathscr{B})$ be a separable space. Let $F(X)$ be the set of all automorphisms of X. Then $F(X)$ is a group under the operation of composition. Let $C(X)$ be the normal subgroup of $F(X)$ comprising all automorphisms $f: X \to X$ such that $\{x : f(x) \neq x\}$ is countable. Define $G = F(X)/C(X)$. We call G the *reduced automorphism group* of X.

A separable space $(X, \mathscr{B})$ is *rigid* (as in [1, p. 20]) if it is uncountable and there is no automorphism $f: X \to X$ such that $\{x : f(x) \neq x\}$ is uncountable: otherwise put, the reduced automorphism group of X is trivial. In Proposition 4 of [1], the existence of rigid spaces was demonstrated. The axiom of choice was used in the proof. In this paper, we operate within *ZFC*; when the continuum hypothesis is used, it shall be indicated. We characterise rigid spaces as follows:

0.6. Lemma. *Let X be an uncountable separable space. The following are equivalent:*

(1) *X is a rigid space;*

(2) *No two disjoint uncountable sets in $\mathscr{B}(X)$ are isomorphic;*

(3) *If B_1 and B_2 are uncountable, isomorphic sets in $\mathscr{B}(X)$, then $B_1 \triangle B_2$ is countable.*

Proof. (1)⇒(2) Suppose that B_1 and B_2 are disjoint uncountable isomorphic sets in $\mathscr{B}(X)$. Let $h: B_1 \to B_2$ be an isomorphism. Then the mapping $f: X \to X$ defined by

$$f(x) = \begin{cases} h(x) & \text{if } x \in B_1, \\ h^{-1}(x) & \text{if } x \in B_2, \\ x & \text{if } x \in X \setminus (B_1 \cup B_2), \end{cases}$$

is a non-trivial automorphism of X. Thus, X is not a rigid space.

(2)⇒(1) Suppose that X is not rigid and that $f: X \to X$ is a non-trivial automorphism of X. We may consider X as a subset of the real line (Lemma 0.1) with its usual order structure. One of the sets

$$D^- = \{x : f(x) < x\}, \qquad D^+ = \{x : f(x) > x\},$$

is an uncountable member of $\mathscr{B}(X)$. Without loss of generality, we suppose that D^- is uncountable. Then there is some $\varepsilon > 0$ such that $D(\varepsilon) = \{x : f(x) < x - \varepsilon\}$ is uncountable. Also, there is some open interval $N \subseteq \mathbb{R}$ of length ε such that $B_1 = N \cap D(\varepsilon)$ is an uncountable set in $\mathscr{B}(X)$. Whenever x and x' are elements of B_1, then $f(x) < x'$: so $B_1 \cap f(B_1) = \emptyset$. The disjoint sets B_1 and $B_2 = f(B_1)$ are uncountable and isomorphic elements of $\mathscr{B}(X)$.

(3)⇒(2) Immediate.

(2)⇒(3) Suppose that B_1 and B_2 are isomorphic uncountable sets in $\mathscr{B}(X)$. Let $f: B_1 \to B_2$ be an isomorphism. Then $B_1 \backslash B_2$ and $f(B_1 \backslash B_2)$ are disjoint, and so must be countable sets in $\mathscr{B}(X)$. The same applies to $B_2 \backslash B_1$ and $f^{-1}(B_2 \backslash B_1)$. So $B_1 \triangle B_2$ is countable. □

We conclude the introduction on an algebraic note. If G is a group, then G^ω denotes the (strong) direct product of denumerably many copies of G. Likewise, $G_1 \times G_2 \times \cdots$ denotes the direct product of the groups G_n. For each positive integer n, let S_n denote the symmetric group of all permutations of $\{1, \ldots, n\}$. Let S_∞ be the group of permutations of $\{1, 2, 3, \ldots\}$.

If $f: X \to X$ is an automorphism of X, we indicate the corresponding coset in G as $\hat{f}$.

0.7. Lemma. *Let G be the reduced automorphism group of a separable space X. The order of an element $a \in G$ is*

$$n = \inf\{m \geq 1:\ a = \hat{f} \text{ for some } f \in F(X) \text{ such that } f^m(x) = x, \text{ each } x \in X\}.$$

The result applies whether n is a positive integer or ∞.

Proof. Suppose that $a \in G$ is of order n and that $a = \hat{g}$ for some $g \in F(X)$. Then

$$n = \inf\{m \geq 1:\ \{x:\ g^m(x) \neq x\} \text{ is countable}\}.$$

Suppose that n is finite and put $U = \{x:\ g^n(x) \neq x\}$. Define the countable set

$$N = \bigcup\{g^k(U):\ k = 0, \pm 1, \pm 2, \ldots\}$$

and define $f: X \to X$ by

$$f(x) = \begin{cases} g(x), & x \in X \backslash N, \\ x, & x \in N. \end{cases}$$

Then $\hat{f} = \hat{g} = a$, and $f^n(x) = x$, each $x \in X$. The lemma follows. □

1. The structure of reduced automorphism groups

The following result discovers some of the structure of reduced automorphism groups. Note that if $k_1, \ldots, k_s$ are positive integers, then $\operatorname{lcm}(k_1, \ldots, k_s)$ is their least common multiple.

1.1. Proposition. *Let X be a separable space with reduced automorphism group G.*

(1) *If G has an element of order $n \geq 1$, then there are divisors $k_1, \ldots, k_s$ of n with $n = \operatorname{lcm}(k_1, \ldots, k_s)$ such that G contains a subgroup isomorphic with $S_{k_1}^\omega \times \cdots \times S_{k_s}^\omega$.*

(2) *If G has an element of infinite order, then there is a sequence $k_1 < k_2 < \cdots$ of positive integers such that G contains a subgroup isomorphic with $S_{k_1}^\omega \times S_{k_2}^\omega \times \cdots$.*

Demonstration. (1) Suppose that $a \in G$ is order n. Write $a = \hat{f}$ as in Lemma 0.7. For $k = 1, \ldots, n$, define $B(k) = \{x: \mathrm{O}(f, x) = k\}$. Lemma 0.5 implies the existence of a set $B \in \mathcal{B}(X)$ containing exactly one point from each orbit of f. Put $B(k, 1) = B \cap B(k)$ and for $r = 1, \ldots, k$, set $B(k, r) = f^{r-1}(B(k, 1))$. Then we have

$$X = \bigcup_{k=1}^{n} B(k)$$

as a disjoint union. Also, for each $k = 1, \ldots, n$

$$B(k) = \bigcup_{r=1}^{k} B(k, r)$$

as a disjoint union.

For some k, the set $B(k)$ is uncountable. Let $\{k_1, \ldots, k_s\}$ be the set of such k. Clearly, $k_1, \ldots, k_s$ are divisors of n. Put $m = \mathrm{lcm}(k_1, \ldots, k_s)$ and note that up to a countable set f^m is the identity map on X. This forces $m = n$.

Now for $1 \leqslant i \leqslant s$, the set $B(k_i, 1)$ is uncountable and so (Lemma 0.2) can be written as a countable disjoint union of uncountable sets in $\mathcal{B}(X)$, *viz.*

$$B(k_i, 1) = \bigcup_{t=1}^{\infty} B(k_i, 1, t).$$

For $r = 1, \ldots, k_i$, define

$$B(k_i, r, t) = f^{r-1}(B(k_i, 1, t)).$$

Notice that for each $i = 1, \ldots, s$ and $t \geqslant 1$, the sets $B(k_i, r, t)$ for $r = 1, \ldots, k_i$ are disjoint, uncountable, isomorphic elements of $\mathcal{B}(X)$.

We now construct a one-one homomorphism ϕ of $H = S^{\omega}_{k_1} \times \cdots \times S^{\omega}_{k_s}$ into G. A typical element of H is given by a matrix $(\pi(i, t))$, where for $i = 1, \ldots, s$ and $t \geqslant 1$, $\pi(i, t)$ is a permutation of $\{1, 2, \ldots, k_i\}$. We define $\phi(\pi(i, t)): X \to X$ by

$$\phi(\pi(i, t))(x) = \begin{cases} f^{\pi(i,t)(r) - r}(x) & \text{if } x \in B(k_i, r, t), \\ x & \text{if } x \notin \bigcup_{i=1}^{s} B(k_i). \end{cases}$$

Notice that

$$\phi(\pi(i, t))(B(k_i, r, t)) = B(k_i, \pi(i, t)(r), t).$$

In words, $\phi(\pi(i, t))$ permutes the isomorphic sets $B(k_i, r, t)$ according to the rule $\pi(i, t)$. It is easily checked that $\pi \to \phi(\pi)\hat{}$ is a monomorphism.

(2) Suppose that for some f in $F(X)$, the element $\hat{f}$ is of infinite order. There are two cases.

Case 1. The set $\{x: 0(f, x) = \infty\}$ is countable. As in the proof of part 1, we define $B(k) = \{x: 0(f, x) = k\}$. The set of k for which $B(k)$ is uncountable must be infinite: if it were finite $\{k_1, \ldots, k_s\}$, then f would be of order $n = \mathrm{lcm}(k_1, \ldots, k_s)$. Let $k_1 < k_2 < \cdots$ be the sequence of such k.

One may now proceed exactly as in the proof of part (1), defining the sets $B(k_i, r, t)$ for $i \geq 1$, $1 \leq r \leq k_i$, and $t \geq 1$. As before (*mutatis mutandis*), one may construct a monomorphism of $S_{k_1}^{\omega} \times S_{k_2}^{\omega} \times \cdots$ into G, as desired.

Case 2. The set $V = \{x: 0(f, x) = \infty\}$ is uncountable.

Case 2a. $\mathscr{B}(V)$ contains a rigid set. Let R_0 be such a rigid set. Define $R_n = f^n(R_0)$ for $n = 1, 2, \ldots$. Then no two of the isomorphic sets R_n can intersect in an uncountable set: if $R_n \cap R_m$ is uncountable for $n < m$, then put $k = m - n$ and consider $f^k(R_n \cap R_m) \subseteq R_m$. Since $R_n \cap R_m$ and $f^k(R_n \cap R_m)$ are uncountable isomorphic sets in $\mathscr{B}(R_m)$ and R_m is rigid, we must have $R_n \cap R_m = f^k(R_n \cap R_m)$ up to a countable set. Then define $h: R_n \to R_n$ by

$$h(x) = \begin{cases} f^k(x) & \text{if } x \in R_n \cap R_m, \\ x & \text{if } x \in R_n \backslash R_m. \end{cases}$$

The function h is well-defined up to a countable set and provides a non-trivial automorphism of the rigid space R_n, a contradiction.

Now define $T = \bigcup R_n \cap R_m$, where the union is over all $n \neq m$. T is countable, and so the sets $U_n = R_n \backslash T$ form a sequence of disjoint uncountable sets in $\mathscr{B}(X)$. Now suppose that π is a permutation of $\{1, 2, 3, \ldots\}$. Define an automorphism $\phi(\pi): X \to X$ by

$$\phi(\pi)(x) = \begin{cases} f^{\pi(r)-r}(x) & \text{when } x \in U_r, \\ x & \text{if } x \in X \setminus \bigcup_{r=1}^{\infty} U_r. \end{cases}$$

Note that $\phi(\pi)(U_r) = U_{\pi(r)}$, so that $\phi(\pi)$ permutes the U_r according to π. It is easy to see that $\pi \to \phi(\pi)^{\wedge}$ is a one-one homomorphism of S_∞ into G.

It remains only to note that whenever $k_1 < k_2 < \cdots$ is a sequence of positive integers, then S_∞ has a subgroup isomorphic with $S_{k_1}^{\omega} \times S_{k_2}^{\omega} \times \cdots$.

Case 2b. $\mathscr{B}(V)$ contains no rigid sets. By Lemma 0.6, there are disjoint uncountable isomorphic sets $A(0)$ and $A(1)$ in $\mathscr{B}(V)$. Applying Lemma 0.6 once more, we find disjoint uncountable isomorphic sets $A(0, 0)$, $A(0, 1)$, $A(1, 0)$, $A(1, 1)$ in $\mathscr{B}(V)$ such that

$$A(0, 0) \cup A(0, 1) \subseteq A(0), \qquad A(1, 0) \cup A(1, 1) \subseteq A(1).$$

Continue in inductive fashion to produce sets $A(\varepsilon_1, \ldots, \varepsilon_k)$ indexed by finite strings $(\varepsilon_1, \ldots, \varepsilon_k)$ of 0's and 1's. We choose these sets so that for each string $(\varepsilon_1, \ldots, \varepsilon_k)$, $A(\varepsilon_1, \ldots, \varepsilon_k, 0)$ and $A(\varepsilon_1, \ldots, \varepsilon_k, 1)$ are disjoint uncountable sets in $\mathscr{B}(V)$ such that

$$A(\varepsilon_1, \ldots, \varepsilon_k, 0) \cup A(\varepsilon_1, \ldots, \varepsilon_k, 1) \subseteq A(\varepsilon_1, \ldots, \varepsilon_k).$$

For each $k \geq 1$, define $\mathscr{A}_k$ to be the collection of all $A(\varepsilon_1, \ldots, \varepsilon_k, \eta_1, \ldots, \eta_k)$ such that $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{k-1} = 0$ and $\varepsilon_k = 1$, and $\eta_1 \cdots \eta_k$ are arbitrary. Put $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \cdots$. For each $k \geq 1$, $\mathscr{A}_k$ is a collection of 2^k isomorphic uncountable sets in $\mathscr{B}(V)$. We index the sets in $\mathscr{A}_k$ so that

$$\mathscr{A}_k = \{B(k, r): r = 1, \ldots, 2^k\}.$$

For each k, let $g_k : \bigcup \mathscr{A}_k \to \bigcup \mathscr{A}_k$ be an isomorphism such that

$$g_k(B(k, r)) = B(k, r+1), \quad r = 1, \ldots, 2^k - 1,$$
$$g_k(B(k, 2^k)) = B(k, 1).$$

Since $B(k, 1)$ is uncountable, Lemma 0.6 enables us to write

$$B(k, 1) = \bigcup_{t=1}^{\infty} B(k, 1, t)$$

as a disjoint union of uncountable sets in $\mathscr{B}(V)$. For each $k \geqslant 1$, $t \geqslant 1$, $1 \leqslant r \leqslant 2^k$, we define

$$B(k, r, t) = g_k^{r-1}(B(k, 1, t)).$$

We shall now construct a one-one homomorphism of $H = S_2^\omega \times S_4^\omega \times S_8^\omega \times \cdots$ into G.

Each element of H is a matrix $(\pi(k, t))$, where for each $k \geqslant 1$ and $t \geqslant 1$, $\pi(k, t)$ is a permutation of $\{1, 2, \ldots, 2^k\}$. We define $\phi(\pi(k, t)) : X \to X$ by

$$\phi(\pi(k, t)) = \begin{cases} g_k^{\pi(k,t)(r) - r}(x) & \text{if } x \in B(k, r, t), \\ x & \text{if } x \in X \setminus \cup \mathscr{A}. \end{cases}$$

Note that

$$\phi(\pi(k, t))(B(k, r, t)) = B(k, \pi(k, t)(r), t),$$

so that, as in earlier parts of the proof, $\phi(\pi(k, t))$ permutes the isomorphic sets $B(k, r, t)$ according as $\pi(k, t)$. Again, the homomorphism $\pi \to \phi(\pi)\hat{}$ is one-one. $\square$

1.2. Corollary. *Let G be the reduced automorphism group of a separable space. Then G has cardinality either* 1 *or* $\mathfrak{c}$.

The following result shows that an Abelian automorphism group must have a very special form.

1.3. Proposition. *Let X be a separable space with reduced automorphism group G. The following are equivalent*:

(1) *G is Abelian.*

(2) *G is either trivial or isomorphic with S_2^ω.*

(3) *For each $a \in G$, one has $a^2 = \{e\}$.*

Demonstration. Suppose that G is Abelian and non-trivial. Proposition 1.1 implies that every element of G is of order 2. A well-known structure theorem [4, p. 17] says that G is a direct sum of copies of S_2. If G is non-trivial, Proposition 1.1 implies that G contains a sub-group isomorphic with S_2^ω and so has cardinality at least $\mathfrak{c}$. By Lemma 0.4, this forces G to have cardinality exactly $\mathfrak{c}$. It must be that G is a direct sum of $\mathfrak{c}$ copies of S_2.

The same structure theorem quoted above implies that S_2^ω, being of cardinality $\mathfrak{c}$, is also a weak direct sum of $\mathfrak{c}$ copies of S_2. We have proved that G and S_2^ω are isomorphic. $\square$

The next matter to be resolved is whether there is a separable space whose reduced automorphism group is non-trivially Abelian. The answer is in the affirmative, which fact will be proved in Proposition 1.4, after the following definitions.

Let Z be a separable space and let H be a group. Let $F(Z, H)$ denote the set of all functions $f: Z \to H$ such that $f^{-1}(A) \in \mathscr{B}(Z)$ whenever $A \subseteq H$. Under point-wise multiplication (in H), the set $F(Z, H)$ becomes a group.

Let $K(Z, H)$ be the set of functions $f \in F(Z, H)$ such that $\{z: f(z) \neq e\}$ is countable, where e is the identity element of H. Then $K(Z, H)$ is a normal subgroup of $F(Z, H)$, and we define $G(Z, H) = F(Z, H)/K(Z, H)$.

Note that if Z is an uncountable separable space, then $G(Z, S_n)$ contains S_n^ω as a subgroup.

1.4. Proposition. *Let X be a direct union of n copies of a rigid space Z. Let G be the reduced automorphism group of X. Then G is isomorphic with $G(Z, S_n)$.*

Demonstration. We write $X = Z_1 \cup \cdots \cup Z_n$, where the Z_i are disjoint sets in $\mathscr{B}(X)$, each isomorphic with the rigid space Z. Now fix an automorphism $f: X \to X$ such that $f(Z_i) = Z_{i+1}$ for $i = 1, \ldots, n-1$ and $f(Z_n) = Z_1$. We shall define a homomorphism from $G(Z_1, S_n)$ to G.

Given a function $h: Z_1 \to S_n$, we define an automorphism $\phi(h): X \to X$ by setting

$$\phi(h)(x) = f^{\pi(r)-r}(x)$$

whenever $x \in f^{r-1}(h^{-1}(\pi))$. Note that the sets $f^{r-1}(h^{-1}(\pi))$ form a partition of X for $r = 1, \ldots, n-1$ and $\pi \in S_n$. It is not hard to check that $h \to \phi(h)^\wedge = \phi^\wedge(h)$ is a one-one homomorphism.

We now prove that $\phi^\wedge$ maps onto G. Let $k: X \to X$ be an automorphism. For each $i = 1, \ldots, n$ and $j = 1, \ldots, n$, define

$$A(i, j) = \{x \in Z_i \colon k(x) \in Z_j\}.$$

Put

$$B(j_1, \ldots, j_n) = A(1, j_1) \cap f^{-1}(A(2, j_2)) \cap \cdots \cap f^{-n+1}(A(n, j_n)).$$

As $j_1, \ldots, j_n$ range over $\{1, \ldots, n\}$, the sets $B(j_1, \ldots, j_n)$ constitute a partition of Z_1 into sets in $\mathscr{B}(X)$.

Claim 1. If $r < s$ and $j_r = j_s$, then $B(j_1, \ldots, j_n)$ is countable.

To see the claim, note that if $B = B(j_1, \ldots, j_n)$, then $k(f^{r-1}(B))$ and $k(f^{s-1}(B))$ are isomorphic subsets of $Z_{j_r} = Z_{j_s}$. By Lemma 0.6, $k(f^{r-1}(B)) \triangle k(f^{s-1}(B))$ is countable. Then $B \triangle f^{s-r}(B) = B \cup f^{s-r}(B)$ is countable, as desired.

Now define $h : Z_1 \to S_n$ by setting

$$h(x)=\begin{cases}(j_1,\dots,j_n) & \text{if } x\in B(j_1,\dots,j_n)\\ & \text{and } B(j_1,\dots,j_n) \text{ uncountable,}\\ (1,\dots,n) & \text{if } x\in B(j_1,\dots,j_n)\\ & \text{and } B(j_1,\dots,j_n) \text{ countable.}\end{cases}$$

Claim 1 ensures that h is well-defined.

Claim 2. Except on a countable set, we have $\phi(h)=k$: Suppose that $B(j_1,\dots,j_n)$ is uncountable, so that $\pi=(j_1,\dots,j_n)$ is a permutation. Then for each fixed $r=1,\dots,n$, we note that $f^{r-1}(B(j_1,\dots,j_n))\subseteq A(r,j_r)$, so that k maps $f^{r-1}(B(j_1,\dots,j_n))$ into Z_{j_r}. So also does $\phi(h)$. The rigidity of Z_{j_r} implies that $\phi(h)=k$ on $f^{r-1}(B(j_1,\dots,j_n))$, except on a countable set. The claim follows.

We have proved that $\phi\hat{}$ is an isomorphism onto G. □

Proposition 1.4 shows that the reduced automorphism group of a separable space can indeed be Abelian. Simply take as the space the direct union of two copies of a rigid space. In the next section, we obtain a partial converse to this result.

2. Spaces with c.c.c.

Let $(X, \mathscr{B})$ be a separable space. A sub-collection $\mathscr{I}\subseteq\mathscr{B}$ is a *σ-ideal* if

(1) $\phi\in\mathscr{I}$,

(2) $N\cap B\in\mathscr{I}$ whenever $N\in\mathscr{I}$ and $B\in\mathscr{B}$,

(3) $\bigcup N_n\in\mathscr{I}$ whenever $N_1N_2\cdots\in\mathscr{I}$.

A σ-ideal $\mathscr{I}$ is *continuous* if it contains all singleton subsets of X. Say that $\mathscr{I}$ satisfies the *countable chain condition* (*c.c.c.*) if every sub-collection of $\mathscr{B}\backslash\mathscr{I}$ comprising pair-wise disjoint sets is necessarily countable (one also says that $\mathscr{I}$ is "ω_1-saturated"). A separable space $(X, \mathscr{B})$ satisfies *c.c.c.* if every sub-collection of $\mathscr{B}$ comprising pair-wise disjoint uncountable sets is countable.

Let $(S, \mathscr{B})$ be a separable space, and let $\mathscr{I}\subseteq\mathscr{B}$ be a σ-ideal with c.c.c. Suppose $X\subseteq S$ has the property that $X\cap N$ is countable whenever $N\in\mathscr{I}$. If X is also uncountable, we say that X is *$\mathscr{I}$-Lusin*. Such a space X must have c.c.c. Two examples of this phenomenon are rather well known (see [2] for a survey):

(1) Let $S=[0,1]$, let $\mathscr{B}$ be the Borel σ-algebra on S, and let $\mathscr{I}$ be the σ-ideal of Borel sets of Lebesgue measure zero. The $\mathscr{I}$-Lusin sets are called *Sierpiński sets.*

(2) Let S and $\mathscr{B}$ be as in 1. Let $\mathscr{I}$ be the σ-ideal of first category Borel subsets of S. The $\mathscr{I}$-Lusin sets are called *Lusin sets.*

The existence of $\mathscr{I}$-Lusin sets can be demonstrated using the continuum hypothesis (CH). The reason for introducing them at this juncture is that the structure of such c.c.c. sets is accessible through the reduced automorphism group. First, we show that (under CH) rigid c.c.c. sets exist. (Compare Proposition 4 in [1].)

2.1. Proposition (CH). *Let S be an uncountable standard space and let $\mathcal{J}$ be a continuous σ-ideal in $\mathcal{B}(S)$ with* c.c.c. *There is a rigid set $X \subseteq S$ which is $\mathcal{J}$-Lusin.*

Construction. Let $f_0 f_1 \cdots f_\alpha \cdots \alpha < \omega_1$ be a transfinite listing of all automorphisms of S such that $A_\alpha = \{x : f(x) \neq x\}$ is a set in $\mathcal{B}(S) \backslash \mathcal{J}$. Let $N_0 N_1 \cdots N_\alpha \cdots \alpha < \omega_1$ be a listing of all sets in $\mathcal{J}$.

For each $A \subseteq S$ and $\alpha < \omega_1$, define $\mathrm{O}_\alpha(A)$ to be the smallest set containing A and closed under f_β and f_β^{-1} for all $\beta \leq \alpha$. It is easily checked that if $\mathrm{card}(A) < \omega_1$, then $\mathrm{card}(\mathrm{O}_\alpha(A)) < \omega_1$. We choose points $x_0 x_1 \cdots x_\alpha \cdots \alpha < \omega_1$ inductively so that

$$x_\alpha \in S - \mathrm{O}_\alpha\{x_\beta : \beta < \alpha\} - \bigcup\{N_\beta : \beta \leq \alpha\}.$$

Put $X = \{x_\alpha : \alpha < \omega_1\}$. Suppose that $g : X \to X$ is an automorphism. Then there is an automorphism $f : S \to S$ whose restriction to X is g.

Case 1. If $\{s : f(s) \neq s\} \in \mathcal{J}$, then $\{x : g(x) \neq x\}$ is countable.

Case 2. If $\{s : f(s) \neq s\} \notin \mathcal{J}$, then $f = f_\beta$ for some $\beta < \omega_1$. Then we claim the following:

Claim. Whenever $\beta < \alpha < \omega_1$, then $f_\beta(x_\alpha) = x_\alpha$. We know that $f_\beta(x_\alpha) = g(x_\alpha) \in X$. If $f_\beta(x_\alpha) = x_\gamma$ for $\gamma > \alpha$, then the choice of x_γ is contradicted. If $f_\beta(x_\alpha) = x_\gamma$ for $\gamma < \alpha$, then $x_\alpha = f_\beta^{-1}(x_\gamma)$ contradicts the choice of x_α.

Therefore $g : X \to X$ moves only countably many points of X. $\square$

What follows is a structure theorem for c.c.c. sets with Abelian automorphism group.

2.2. Proposition. *Let X be an uncountable separable space with reduced automorphism group G. Consider the following conditions:*

(1) *X is the direct union of sets Z_0, Z_1, Z_2, such that*

  (a) *$Z_0 \cup Z_1$ and $Z_0 \cup Z_2$ are rigid;*

  (b) *Z_1 and Z_2 are isomorphic.*

(2) *G is Abelian.*

Then (1)$\Rightarrow$(2); *if X has c.c.c., then* (2)$\Rightarrow$(1). *Additionally,* (1) *implies that G is trivial if and only if* $\mathrm{card}(Z_1) = \mathrm{card}(Z_2)$ *is countable.*

Demonstration. (1)$\Rightarrow$(2)

Case 1. Assume Z_0 is countable. Then X is isomorphic to the direct union of the rigid spaces Z_1 and Z_2. Propositions 1.4 and 1.3 imply that G is isomorphic with S_2^ω.

Case 2. Assume Z_0 is uncountable. Then Z_0 is a rigid space. Let $f : X \to X$ be an automorphism. Since $Z_0 \cup Z_1$ and $Z_0 \cup Z_2$ are rigid, $f(Z_0) \cap Z_1$ and $f(Z_0) \cap Z_2$ are countable. So, up to a countable set, $f(Z_0) = Z_0$, forcing f to be the identity map when restricted to Z_0. It is then clear that G is isomorphic with the reduced automorphism group of $Z_1 \cup Z_2$. Again, if Z_1 and Z_2 are uncountable, Propositions 1.3 and 1.4 apply.

(2)$\Rightarrow$(1) (Assuming X c.c.c.): Consider the collection $\mathcal{P}$ of all pairs (A, B), where A and B are disjoint uncountable isomorphic sets in $\mathcal{B}(X)$. We introduce a partial order on $\mathcal{P}$ by setting $(A, B) < (A', B')$ in case

$$A' = A \cup A'' \quad \text{and} \quad B' = B \cup B'',$$

where

$$A \cap A'' = \emptyset \quad \text{and} \quad B \cap B'' = \emptyset$$

and

$$(A'', B'') \in \mathcal{P}.$$

If G is non-trivial and Abelian, then by Proposition 1.3 it is isomorphic with S_2^ω. Since X is not rigid, $\mathcal{P}$ is non-empty.

Claim. $\mathcal{P}$ has a maximal element. To see this, construct a transfinite series (A_α, B_α) of elements of $\mathcal{P}$ indexed by $\alpha < \omega_1$ such that

$$(A_\alpha, B_\alpha) < (A_{\alpha+1}, B_{\alpha+1}) \quad \text{all } \alpha < \omega_1,$$

$$\left.\begin{aligned} A_\alpha &= \bigcup\{A_\beta : \beta < \alpha\} \\ B_\alpha &= \bigcup\{B_\beta : \beta < \alpha\} \end{aligned}\right\} \quad \alpha \text{ a limit ordinal.}$$

Because X has c.c.c., this induction cannot be continued through all $\alpha < \omega_1$. It will terminate in a maximal element (Z_1, Z_2).

Define $Z_0 = X \setminus (Z_1 \cup Z_2)$. We must prove that $Z_0 \cup Z_1$ is rigid. Symmetry will then imply that $Z_0 \cup Z_1$ is rigid. So suppose that U and V are disjoint uncountable isomorphic sets in $\mathcal{B}(Z_0 \cup Z_1)$. Let $f: U \to V$ supply the isomorphism. Also, let $g: Z \to Z_2$ be an isomorphism.

Claim. The set $U \cap Z_1$ is countable. If not, then $U \cap Z_1$, $f(U \cap Z_1)$, and $g(U \cap Z_1)$ are three disjoint uncountable sets in $\mathcal{B}(X)$. By cyclically permuting these, one obtains an element of G of order 3. Proposition 1.1 implies that G is not Abelian. This is a contradiction.

A similar argument shows that $V \cap Z_1$ is countable. So $U \cap Z_0$ and $V \cap Z_0$ are disjoint uncountable isomorphic elements of $\mathcal{B}(Z_0)$. This, however, contradicts the maximality of (Z_1, Z_2) in $\mathcal{P}$. So $Z_0 \cup Z_1$ is rigid. $\square$

We now proceed to show that the requirement that X be c.c.c. cannot simply be removed from the implication (2)$\Rightarrow$(1) in Proposition 2.2. A little machinery is needed.

Let X be a separable space. Say that X satisfies condition $\triangle$ if

($\triangle$) X is the direct union of sets $Z_0 Z_1 Z_2$ such that

(1) $Z_0 \cup Z_1$ and $Z_0 \cup Z_2$ are rigid;

(2) Z_1 and Z_2 are isomorphic and uncountable.

If G is the reduced automorphism group of a separable space X, then there is a partial order $<$ on G defined by declaring $\hat{f} < \hat{g}$ when

$$\{x : f(x) \neq x \text{ and } g(x) = x\} \text{ is countable}$$

and

$$\{x\colon g(x)\neq x \text{ and } f(x)=x\} \text{ is uncountable.}$$

2.3. Lemma. *Let X be a separable space satisfying condition $\triangle$. Then there is an element of the reduced automorphism group G which is largest for the partial order $<$.*

Proof. We return to the proof of Proposition 1.4. Let $f\colon Z_1\to Z_2$ be an isomorphism and consider once more the group isomorphism $\phi\hat{}\colon G(Z_1, S_2)\to G$. We write $S_2=\{0, 1\}$.

Claim. For $h_1, h_2\in G(Z_1, S_2)$, one has $\phi(h_1)\hat{}<\phi(h_2)\hat{}$ if and only if

$$\{x\colon h_1(x)=1 \text{ and } h_2(x)=0\} \text{ is countable}$$

and

$$\{x\colon h_1(x)=0 \text{ and } h_2(x)=1\} \text{ is uncountable.}$$

Verification of the claim is routine. It then follows that $\phi(h)\hat{}$ is the largest element in G, where $h(x)=1$ for all $x\in Z_1$. □

2.4. Lemma. *Let S be an uncountable standard space. There is a function $f\colon S\to S$ such that*

(1) *f is a one-one correspondence of S onto itself such that $f=f^{-1}$;*

(2) *If $E\in\mathscr{B}(S)$ and $g\colon E\to S$ is a one-one measurable function, then $\{s\in S\colon f(s)=g(s)\}$ is of cardinality less than $\mathfrak{c}$.*

Proof. We may take $S=[0, 1)\cup[1, 2)$ under its usual Borel structure. List the points of $[0, 1)$ without repetitions as $x_0x_1\cdots x_\alpha\cdots\alpha<\mathfrak{c}$, where $x_0=0$. List all one-one measurable functions $g\colon E\to(0, 1)$, where E is an uncountable set in $\mathscr{B}[0, 1)$, as $g_0g_1\cdots g_\alpha\cdots\alpha<\mathfrak{c}$ and define, for each $\alpha<\mathfrak{c}$, $N_\alpha=\bigcup\{\text{graph}(g_\beta)\colon\beta\leq\alpha\}$. We define points y_α and z_α for $\alpha<\mathfrak{c}$. Put $y_0=z_0=0$. The process is inductive: suppose that $0<\alpha<\mathfrak{c}$ and that

$$Y_\alpha=\{y_\beta\colon\beta<\alpha\}\cup\{x_\beta\colon\beta\leq\alpha\},\qquad Z_\alpha=\{z_\alpha\colon\beta<\alpha\}\cup\{x_\beta\colon\beta<\alpha\}.$$

If possible, choose y_α so that

$$(x_\alpha, y_\alpha)\notin N_\alpha\cup(Z_\alpha\times[0, 1))\cup([0, 1)\times Y_\alpha).$$

If the choice is not possible, it must be that $x_\alpha=z_\beta$ for some $\beta<\alpha$. In this case, put $y_\alpha=0$. Likewise, if possible, choose z_α so that

$$(z_\alpha, x_\alpha)\notin N_\alpha\cup((Z_\alpha\cup\{x_\alpha\})\times[0, 1))\cup([0, 1)\times(Y_\alpha\cup\{y_\alpha\})).$$

If the choice is not possible, it must be that $x_\alpha=y_\beta$ for some $\beta\leq\alpha$. In this case, put $z_\alpha=0$. Then define

$$G=\{(x_\alpha, y_\alpha), (z_\alpha, x_\alpha)\colon y_\alpha\neq 0, z_\alpha\neq 0, \alpha<c\}\cup\{(0, 0)\}.$$

Then it is easily verified that G is the graph of a one-one correspondence $f_0:[0,1)\to[0,1)$. Define $f:S\to S$ by

$$f(s)=\begin{cases} f_0(s)+1, & 0\leq s<1,\\ f_0^{-1}(s-1), & 1\leq s<2.\end{cases}$$

The function f has the desired properties. □

2.5. Lemma. *Let S be an uncountable standard space and suppose that $A\in\mathcal{B}(S\times S)$. Let $p:S\times S\to S$ be projection to the first factor. If $p(A)$ is uncountable, then there is an uncountable $E\in\mathcal{B}(S)$ with $E\subseteq p(A)$ and a measurable function $k:E\to S$ such that graph $(k)\subseteq A$.*

Indication. This follows from the Jankov-von Neumann selection theorem. See [6, p. 871].

2.6. Proposition. *There is a separable space X with reduced automorphism group G such that*

(1) *G is Abelian, but non-trivial*;

(2) *X does not satisfy condition $\triangle$.*

Demonstration. Let S be an uncountable standard space and let $f:S\to S$ be as in Lemma 2.4. Define $F:S\times S\to S\times S$ by $F(s,t)=(f(s),t)$. Let $g_0g_1\cdots g_\alpha\cdots\alpha<\mathfrak{c}$ be a listing in transfinite series of all automorphisms of $S\times S$. Given $A\subseteq S\times S$ and $\alpha<\mathfrak{c}$, define $\mathcal{O}_\alpha(A)$ to be the smallest set containing A and closed under the functions g_β, g_β^{-1} and $F=F^{-1}$ for all $\beta\leq\alpha$. If card$(A)<\mathfrak{c}$, then card $\mathcal{O}_\alpha(A)<\mathfrak{c}$. Let $B_0B_1\cdots B_\alpha\cdots\alpha<\mathfrak{c}$ be a listing of all uncountable sets in $\mathcal{B}(S\times S)$. We choose points $x_0x_1\cdots x_\alpha\cdots\alpha<\mathfrak{c}$ inductively so that

$$x_\alpha\in B_\alpha-\mathcal{O}_\alpha\{x_\beta:\beta<\alpha\}.$$

Put $X=\{x_\alpha:\alpha<\mathfrak{c}\}\cup\{F(x_\alpha):\alpha<\mathfrak{c}\}$.

Claim 1. If $\beta\leq\alpha<\mathfrak{c}$, then $g_\beta(\{x_\alpha,F(x_\alpha)\})\subseteq\{x_\alpha,F(x_\alpha)\}\cup((S\times S)\backslash X)$. Verification of the claim is routine, given the selection of x_α and the definition of $\mathcal{O}_\alpha$.

The reduced automorphism group G of X is non-trivial. To see this, note that for any countable set $C\subseteq S$ the function $h_C:X\to X$ defined by

$$h_C(s,t)=\begin{cases} F(s,t), & s\in C\cup f(C),\\ (s,t), & s\notin C\cup f(C),\end{cases}$$

is an automorphism of X. Since X intersects every uncountable set in $\mathcal{B}(S\times S)$, the map h_C moves uncountably many points of X.

The group G is, however, Abelian. For every automorphism of X extends to an automorphism of $S\times S$ (Lemma 0.3). This automorphism has been listed as some g_β. From Claim 1, it follows that $\{x\in X: \mathrm{O}(g_\beta,x)>2\}$ is of cardinality less than $\mathfrak{c}$.

Since X meets every uncountable set in $\mathscr{B}(S\times S)$ in $\mathfrak{c}$ many points, it follows that $\{s\in S\colon O(g_\beta, S)>2\}$ is countable. Therefore, $g_\beta^2(s)=s$ for all but countably many s. So G is Abelian.

We now show that X does not satisfy condition $\triangle$. Suppose it did. By Lemma 2.3, there is some element $\hat{g}$ of G which is largest for the partial order $<$. This automorphism g extends to some automorphism g_β of $S\times S$. Certainly $T=\{s\in S\colon O(g_\beta, s)=2\}$ is an uncountable set in $\mathscr{B}(S\times S)$.

Claim 2. Except for a set of points of cardinality less than $\mathfrak{c}$, $g_\beta=F$ on $T\cap X$.

Claim 2 follows from Claim 1. Note that $\operatorname{card}(T\cap X)=\mathfrak{c}$.

Now apply Lemma 0.5 to find a set $B\in\mathscr{B}(S\times S)$ which contains exactly one point from each orbit of g_β. Let $p\colon S\times S\to S$ be projection to the first factor.

Claim 3. The set $p(B\cap T)$ is uncountable.

This follows from the maximality of g_β. Otherwise, g_β will move points in only countably many vertical sections of $S\times S$. This contradicts the existence of the automorphisms h_C mentioned above.

Apply Lemma 2.5 to find an uncountable set $E\in\mathscr{B}(S)$ with $E\subseteq p(B\cap T)$ and a measurable function $k\colon E\to S$ such that graph $(k)\subseteq B\cap T$. Consider the mapping $l\colon E\to S$ defined by $l(s)=p(g_\beta(s, k(s)))$.

Claim 4. The Borel set $g_\beta(\operatorname{graph}(k))$ has countable vertical sections.

Suppose not, and let $s_0\in S$ be such that $D=\{(s,t)\colon s=s_0\}\cap g_\beta(\operatorname{graph}(k))$ is uncountable. The $X\cap g_\beta^{-1}(D)\subseteq X\cap B\cap T$ has cardinality $\mathfrak{c}$. Now $p(g_\beta(a,b))=s_0$ for all $(a,b)\in g_\beta^{-1}(D)$. But by Claim 2, $g_\beta=F$ on $\mathfrak{c}$ many points of $X\cap g_\beta^{-1}(D)$. Noting that $p\circ F$ is one-one on graph(k) provides a contradiction.

Apply the selection theorem III.9.4 (p. 137) of [3] to find an uncountable $H\in\mathscr{B}(S\times S)$ with $H\subseteq g_\beta(\operatorname{graph}(k))$ such that every vertical section of H is either empty or singleton. Then $p\circ g_\beta$ is one-one on $g_\beta^{-1}(H)$. Define $E_0=p(g_\beta^{-1}(H))$, an uncountable set in $\mathscr{B}(S)$. Define l_0 to be the restriction of l to E_0. Then l_0 is a one-one measurable function.

Using claim 2, we find a set $Z\subseteq X\cap g_\beta^{-1}(H)\subseteq X\cap T$ of cardinality $\mathfrak{c}$ such that $g_\beta=F$ on Z. Then card $(p(Z))=\mathfrak{c}$ and $l_0=f$ on $p(Z)$. This contradicts the construction of f given in Lemma 2.4. $\square$

3. A strictly rigid measurable space

In this section, we consider measurable spaces $(X,\mathscr{B})$ which are not separable. As before, an *automorphism* of $(X,\mathscr{B})$ is a one-one correspondence $f\colon X\to X$ such that $f(B)\in\mathscr{B}$ if and only if $B\in\mathscr{B}$. Say that $(X,\mathscr{B})$ is *strictly rigid* if the only automorphism $f\colon X\to X$ is the identity map. We prove the existence of a strictly rigid measurable space, thus solving Problem $P3$ (p. 21) of [1].

A completely regular topological space X is a *P-space* if every G_δ subset of X is open. If X is a P-space, and $\mathscr{B}$ is the collection of clopen subsets of X, then $\mathscr{B}$ is a σ-algebra.

3.1. Lemma. *Let X be a P-space and let $\mathscr{B}$ be the σ-algebra of clopen subsets of X. Every automorphism of $(X, \mathscr{B})$ is a (topological) homeomorphism of X onto itself.*

Proof. Immediate, noting that $\mathscr{B}$ is a base for X. □

The rest of this section is devoted to a proof of the following:

3.2. Proposition. *There is a non-trivial strictly rigid measurable space $(X, \mathscr{B})$.*

Demonstration. We construct an infinite 0-dimensional P-space X which is topologically rigid, i.e. the only homeomorphism of X onto itself is the identity map. If $\mathscr{B}$ is the clopen σ-algebra of X, then Lemma 3.1 implies that $(X, \mathscr{B})$ is strictly rigid.

Let $X_0 = \{0\}$. Suppose that X_n has been constructed for some $n < \omega$. For each $x \in X_n$, there clearly exists a set $B(x, n)$ such that

(1) $B(x, n) \cap X_n = \emptyset$;

(2) if $x, y \in X_n$ are distinct, then $B(x, n) \cap B(y, n) = \emptyset$;

(3) $\operatorname{card}(B(x, n))$ is regular and has uncountable cofinality;

(4) $\operatorname{card}(B(x, n)) > \operatorname{card}(X_n)$;

(5) if $x, y \in X_n$ are distinct, then $\operatorname{card}(B(x, n)) \neq \operatorname{card}(B(y, n))$.

Define $X_{n+1} = \bigcup\{B(x, n): x \in X_n\}$. Also put $X = \bigcup\{X_n: n < \omega\}$. For every $x \in X$, there is a unique $n(x) < \omega$ with $x \in X_{n(x)}$. For every $x \in X$ and $n < \omega$, define $T(x, n) \subseteq X$ by

$$T(x, 0) = \{x\}, \qquad T(x, n+1) = \bigcup\{B(y, n(x)+n): y \in T(x, n)\}.$$

Also, put $T(x) = \bigcup\{T(x, n): n < \omega\}$.

3.3. Lemma. *If $x, y \in X$ and $T(x) \cap T(y) \neq \emptyset$, then either $T(x) \subseteq T(y)$ or $T(y) \subseteq T(x)$.*

Proof. If $n(x) = n(y)$ and $x \neq y$, then clearly $T(x) \cap T(y) = \emptyset$. So suppose $n(x) < n(y)$. If $y \notin T(x, n(y) - n(x))$, then clearly $T(x) \cap T(y) = \emptyset$. So $y \in T(x, n(y) - n(x))$, from which it follows that $T(y) \subseteq T(x)$. □

For every $x \in X_n$ and $F \subseteq B(x, n)$ such that $\operatorname{card}(F) < \operatorname{card}(B(x, n))$, put

$$U(x, F) = \{x\} \cup \bigcup\{T(y): y \in B(x, n) \setminus F\}.$$

Let $\mathscr{U}$ be the set of all such $U(x, F)$.

3.4. Lemma. *For all $U_0, U_1 \in \mathscr{U}$ and $x \in U_0 \cap U_1$, there exists $F \subseteq B(x, n(x))$ such that $\operatorname{card}(F) < \operatorname{card}(B(x, n(x)))$ and $U(x, F) \subseteq U_0 \cap U_1$.*

Proof. Let $U_i = U(p_i, F_i)$, with appropriate p_i and F_i, $i = 0, 1$. Since $T(p_0) \cap T(p_1) \neq \emptyset$, we may assume $T(p_0) \subseteq T(p_1)$ by Lemma 3.3.

Case 1. $p_0 = p_1$. If $x = p_0 = p_1$, then put $F = F_0 \cup F_1$. If $x \neq p_0$, put $F = \emptyset$.

Case 2. $p_0 \neq p_1$. Then $U_0 \subseteq T(p_0) \subseteq U(p_1, F_1)$. Consequently $x \in U_0$. If $x = p_0$, then put $F = F_0$. If $x \neq p_0$, then put $F = \emptyset$. □

Lemma 3.4 implies that $\mathcal{U}$ is the base for a topology on X. We supply X with this topology.

3.5. Lemma. *For every $U \in \mathcal{U}$ and $p \notin U$, there is some $V \in \mathcal{U}$ with $p \in V$ and $U \cap V = \emptyset$.*

Proof. Let $U = U(\bar{p}, F)$ with appropriate $\bar{p}$ and F.

Case 1. $p \in T(\bar{p})$. Then $T(p) \cap U = \emptyset$. So without loss of generality $p \notin T(\bar{p})$.

Case 2. $\bar{p} \in T(p)$. There is some $q \in B(p, n(p))$ such that $\bar{p} \in T(q)$. Put $F = \{q\}$ and let $V = U(p, F)$. So without loss of generality $\bar{p} \notin T(p)$.

By Lemma 3.3, we now have $T(p) \cap T(\bar{p}) = \emptyset$. So put $V = T(p)$. $\square$

3.6. Lemma. *For all distinct $x, y \in X$, there exists some $U \in \mathcal{U}$ with $x \in U$ and $y \notin U$.*

Proof. If $y \notin T(x)$, then we are done. If $y \in T(x)$, then there exists $q \in B(x, n(x))$ with $y \in T(q)$. Put $F = B(x, n(x)) \setminus \{q\}$ and let $U = U(x, F)$. $\square$

From Lemmas 3.5 and 3.6 we conclude that the topology on X is Hausdorff and that elements of $\mathcal{U}$ are clopen; hence, the topology is 0-dimensional and completely regular. By construction and Lemma 3.4, X is a P-space. Since the least cardinality of a local base at $x \in X$ (the character at x) equals card $(B(x, n))$, and the cardinalities of the sets $B(x, n)$ are pairwise distinct for distinct $x, y \in X$, it follows that the character at x and at y are distinct. Hence X is topologically rigid. $\square$

The referee has suggested an alternate proof: Construct X as a tree of height ω in which every element has a different uncountable, regular cardinal number of successors. Take $\mathcal{B}$ as the σ-algebra generated by the sets $\{y: y \geqslant x\}$. Then show that for any x, the number of successors of x equals its pseudo-character (the smallest number of measurable sets with intersection $\{x\}$).

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