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Totally divergent dense sets in Cantor cubes

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Dedicated to Professor Miroslav Katětov on his seventieth birthday

Abstract. Let κ be an infinite cardinal such that $2^{\log\kappa} = \kappa$. We prove that the Cantor cube 2^{κ} contains a dense subgroup D of cardinality κ such that for every subset E of D of cardinality κ we have $|\overline{E}| = 2^{\kappa}$.

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0. Introduction. In [3], Priestley showed that there is a countable dense $D \subseteq 2^{\mathfrak{c}}$, where $\mathfrak{c}$ denotes the cardinality of the continuum, such that no infinite set E in D converges uniquely to a point in $2^{\mathfrak{c}}$. Simon [4] generalized this by showing that there is a countable dense $D \subseteq 2^{\mathfrak{c}}$ such that for every infinite subset $E \subseteq D$ we have $|\overline{E}| = 2^{\mathfrak{c}}$: the proof is combinatorial and complicated. He also observed that such a result cannot be obtained for every uncountable cardinal: it is consistent that every infinite subset D of 2^{ω_1} contains an infinite subset E converging to a unique point in 2^{ω_1}.

Let κ be an infinite cardinal. As usual, we put $\log\kappa = \min\{\mu \le \kappa : 2^{\mu} \ge \kappa\}$. We consider 2^{κ} endowed with its canonical Boolean group structure. The aim of this note is to prove the following:

0.1. THEOREM: *Let κ be an infinite cardinal such that $2^{\log\kappa} = \kappa$. Then 2^{κ} contains a dense subgroup* G *of cardinality* $\log\kappa$ *such that for every* $E \subseteq D$ *of cardinality* $\log\kappa$ *we have* $|\overline{E}| = 2^{\kappa}$.

Since $\log\mathfrak{c} = \omega$, this generalizes Simon's result in two ways.

1. The Construction. If G is a group and $A \subseteq G$ then «A» denotes the subgroup of G generated by A. A group is called *Boolean* if every element has order at most 2. Observe that such a group is abelian. A subset A of a Boolean group G is called *independent* if for every $a \in A$ we have $a \notin$ «$A\setminus\{a\}$».

Our construction depends on the following two simple results.

1.1. LEMMA: *Let* G *be a Boolean group and let* $A \subseteq G$ *be infinite. Then there is an independent* $B \subseteq A$ *such that* $|B| = |A|$.

PROOF: Let B be a maximal independent subset of A. For every $x \in A \backslash B$ there is a finite $F_x \subseteq B$ such that $x \in «F_x»$. Since G is Boolean, if $F \subseteq G$ is finite then so is «F». Consequently, for every finite $F \subseteq B$ there are at most finitely many $x \in A \backslash B$ such that $F_x = F$. Since A is infinite, this implies that $|B| = |A|$. □

1.2. LEMMA: *Let* G *be a Boolean group and let* $A \subseteq G$ *be independent. Then every function* f: $A \to \{0,1\}$ *extends to a homomorphism* $\bar{f}: G \to \{0,1\}$.

PROOF: Since A is independent, it follows easily that f can be extended to a homomorphism $\bar{\bar{f}}$: $«A» \to \{0,1\}$. In addition, since G is Boolean, there is a subgroup $H \subseteq G$ such that $«A» \oplus H = G$ (let H be a maximal subgroup of G with the property that $«A» \cap H = \{0\}$). Now define $\bar{f}$: G $\to \{0,1\}$ by

$$\bar{f}(x+y) = \bar{\bar{f}}(x) \qquad (x \in «A», y \in H).$$

Then $\bar{f}$ is clearly as required. □

Now let κ be an infinite cardinal such that $2^{\log \kappa} = \kappa$. For convenience, put $\mu = \log \kappa$. Let G be any Boolean group of cardinality μ. Observe that by lemma 1.1 there is an independent subset A of G of cardinality μ. Since $2^\mu = \kappa$ we can enumerate the set $\{A \subseteq G: |A| = \mu$ and A is independent$\}$ as $\{A_\xi: \xi < \kappa\}$ (repetitions permitted). Let $\{E_\xi: \xi < \kappa\}$ be a partition of κ into pairwise disjoint sets of cardinality κ. We identify 2^κ and the product $\Pi_{\xi<\kappa}2^{E_\xi}$. Since the density of 2^κ is equal to μ (Juhász [2, 4.5]), for every $\xi < \kappa$ there is a function f_ξ: $A_\xi \to 2^{E_\xi}$ such that $f(A_\xi)$ is dense. By lemma 1.2, we can extend f_ξ to a homomorphism $\bar{f}_\xi$: $G \to 2^{E_\xi}$. Now define f: G $\to$ $\Pi_{\xi<\kappa}2^{E_\xi}$ by

$$f(x)_\xi = \bar{f}_\xi(x) \qquad (\xi < \kappa).$$

Observe that f is a homomorphism. Put $H = \overline{f(G)}$. Observe that H is a closed subgroup of $\Pi_{\xi<\kappa}2^{E_\xi}$.

1.3. LEMMA: H *has weight* κ.

PROOF: First observe that the weight of H is at most κ, being a subspace of $\Pi_{\xi<\kappa}2^{E_\xi}$. Conversely, pick an arbitrary $\xi < \kappa$. By construction, H can be mapped onto 2^{E_ξ} which has weight κ. From this we conclude that the weight of H is at least κ. □

It now follows from Hewitt and Ross [1, 25.22] that H is both topologically and algebraically isomorphic to 2^κ. Since, as was remarked above, the density of 2^κ is equal to μ and f(G) is dense in H, we obtain $|f(G)| \geq \mu$. On the other hand, $|f(G)| \leq |G| = \mu$. We conclude that $|f(G)| = \mu$. Now let $B \subseteq f(G)$ be of cardinality μ. Clearly, $|\overline{B}| \leq 2^\kappa$.There is a set $A \subseteq G$ of cardinality μ such that $f(A) = B$. By lemma 1.1 there is a $\xi < \kappa$ with $A_\xi \subseteq A$. By construction, $\overline{f(A_\xi)}$ can be mapped onto 2^{E_ξ}. We conclude that

$$|\overline{B}| \geq |\overline{f(A_\xi)}| \geq 2^\kappa.$$

We are done.

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