Totally divergent dense sets in Cantor cubes

Jan van Mill

Dedicated to Professor Miroslav Katětov on his seventieth birthday

Abstract. Let $\kappa$ be an infinite cardinal such that $2^{\log \kappa} = \kappa$. We prove that the Cantor cube $2^\kappa$ contains a dense subgroup $D$ of cardinality $\kappa$ such that for every subset $E$ of $D$ of cardinality $\kappa$ we have $|E| = 2^\kappa$.

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0. Introduction. In [3], Priestley showed that there is a countable dense $D \subseteq 2^\kappa$, where $\kappa$ denotes the cardinality of the continuum, such that no infinite set $E$ in $D$ converges uniquely to a point in $2^\kappa$. Simon [4] generalized this by showing that there is a countable dense $D \subseteq 2^\kappa$ such that for every infinite subset $E \subseteq D$ we have $|E| = 2^\kappa$; the proof is combinatorial and complicated.

He also observed that such a result cannot be obtained for every uncountable cardinal: it is consistent that every infinite subset $D$ of $2^{200}$ contains an infinite subset $E$ converging to a unique point in $2^{200}$.

Let $\kappa$ be an infinite cardinal. As usual, we put $\log \kappa = \min \{ \mu \leq \kappa : 2^\mu \geq \kappa \}$. We consider $2^\kappa$ endowed with its canonical Boolean group structure. The aim of this note is to prove the following:

0.1. THEOREM: Let $\kappa$ be an infinite cardinal such that $2^{\log \kappa} = \kappa$. Then $2^\kappa$ contains a dense subgroup $G$ of cardinality $\log \kappa$ such that for every $E \subseteq D$ of cardinality $\log \kappa$ we have $|E| = 2^\kappa$.

Since $\log \kappa = \omega$, this generalizes Simon's result in two ways.

1. The Construction. If $G$ is a group and $A \subseteq G$ then $\langle A \rangle$ denotes the subgroup of $G$ generated by $A$. A group is called Boolean if every element has order at most 2. Observe that such a group is abelian. A subset $A$ of a Boolean group $G$ is called independent if for every $a \in A$ we have $a \notin \langle \{a\} \rangle$.

Our construction depends on the following two simple results.

1.1. LEMMA: Let $G$ be a Boolean group and let $A \subseteq G$ be infinite. Then there is an independent $B \subseteq A$ such that $|B| = |A|$.
PROOF: Let $B$ be a maximal independent subset of $A$. For every $x \in A \land B$ there is a finite $F_x \subseteq B$ such that $x \in \langle F_x \rangle$. Since $G$ is Boolean, if $F \subseteq G$ is finite then so is $\langle F \rangle$. Consequently, for every finite $F \subseteq B$ there are at most finitely many $x \in A \land B$ such that $F_x = F$. Since $A$ is infinite, this implies that $|B| = |A|$. \[]

1.2. LEMMA: Let $G \subseteq B$ be independent. Then every function $f : A \rightarrow \{0,1\}$ extends to a homomorphism $\tilde{f} : G \rightarrow \{0,1\}$.

PROOF: Since $A$ is independent, it follows easily that $f$ can be extended to a homomorphism $\tilde{f} : G \rightarrow \{0,1\}$. In addition, since $G$ is Boolean, there is a subgroup $H \subseteq G$ such that $\langle A \rangle \land H = G$ (let $H$ be a maximal subgroup of $G$ with the property that $\langle A \rangle \land H = \{0\}$). Define $\tilde{f} : G \rightarrow \{0,1\}$ by

$$\tilde{f}(x + y) = \tilde{f}(x) \quad (x \in \langle A \rangle, y \in H).$$

Then $\tilde{f}$ is clearly as required. \[]

Now let $\kappa$ be an infinite cardinal such that $2^{2^\kappa} = \kappa$. For convenience, put $\mu = \log \kappa$. Let $G$ be any Boolean group of cardinality $\mu$. Observe that by lemma 1.1 there is an independent subset $A$ of $G$ of cardinality $\mu$. Since $2^\kappa = \kappa$ we can enumerate the set $\{A \subseteq G : |A| = \mu \land A$ is independent $\}$ as $\{A_\xi : \xi < \kappa\}$ (repetitions permitted). Let $\{B_\xi : \xi < \kappa\}$ be a partition of $\kappa$ into pairwise disjoint sets of cardinality $\kappa$. We identify $2^\kappa$ and the product $\prod_{\xi < \kappa} B_\xi$. Since the density of $2^\kappa$ is equal to $\mu$ (Juhász [2, 4.3]), for every $\xi < \kappa$ there is a function $f_\xi : A_\xi \rightarrow 2^{B_\xi}$ such that $f(A_\xi)$ is dense. By lemma 1.2, we can extend $f_\xi$ to a homomorphism $\tilde{f}_\xi : G \rightarrow 2^{B_\xi}$. Now define $f : G \rightarrow \prod_{\xi < \kappa} B_\xi$ by

$$f(x)_\xi = \tilde{f}_\xi(x) \quad (\xi < \kappa).$$

Observe that $f$ is a homomorphism. Put $H = \overline{f(G)}$. Observe that $H$ is a closed subgroup of $\prod_{\xi < \kappa} B_\xi$.

1.3. LEMMA: $H$ has weight $\kappa$.

PROOF: First observe that the weight of $H$ is at most $\kappa$, being a subspace of $\prod_{\xi < \kappa} B_\xi$. Conversely, pick an arbitrary $\xi < \kappa$. By construction, $H$ can be mapped onto $2^{B_\xi}$ which has weight $\kappa$. From this we conclude that the weight of $H$ is at least $\kappa$. \[]
It now follows from Hewitt and Ross [1, 25.22] that $H$ is both topologically and algebraically isomorphic to $2^\kappa$. Since, as was remarked above, the density of $2^\kappa$ is equal to $\mu$ and $f(G)$ is dense in $H$, we obtain $|f(G)| \geq \mu$. On the other hand, $|f(G)| \leq |G| = \mu$. We conclude that $|f(G)| = \mu$. Now let $B \subseteq f(G)$ be of cardinality $\mu$. Clearly, $|B| \leq 2^\kappa$. There is a set $A \subseteq G$ of cardinality $\mu$ such that $f(A) = B$. By lemma 1.1 there is a $\xi < \kappa$ with $A_\xi \subseteq A$. By construction, $f(A_\xi)$ can be mapped onto $2^\xi$. We conclude that

$$|B| \geq |f(A_\xi)| \geq 2^\xi.$$  

We are done.

References:


Faculteit Wiskunde en Informatica
Vrije Universiteit
De Boelelaan 1081
1081 HV Amsterdam
The Netherlands

and

Faculteit Wiskunde en Informatica
Universiteit van Amsterdam
Roverstraat 15
1018 WB Amsterdam
The Netherlands

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