General Topology and its Relations
to Modern Analysis and Algebra VI
Proc. Sixth Prague Topological Symposium 1986
Z. Frolík (ed.)

115–119

The compact extension property

D. W. Curtis, J. van Mill

Abstract: We prove that each locally compact metric space with the compact extension property is an absolute retract. In addition, assuming that there exists a cell-like dimension raising map between compact metric spaces, we construct an example of a topologically complete separable metric space with the compact extension property which is not an absolute neighborhood retract.

1980 *Mathematics Subject Classification*. Primary 55M15.

1. Introduction.

All spaces under discussion are metric.

A space X has the *compact extension property* , abbreviated **CEP**, [10], provided that for every space Y and every compact subset A of Y, every map from A into X can be extended over Y. It is known that

$$\mathbf{AR} \Rightarrow \mathbf{CEP} \Rightarrow \mathbf{C}^{\infty} \text{ and } \mathbf{LC}^{\infty}.$$

Examples in [3] and [11] show that these implications cannot be reversed.

It is clear that every compact space with the **CEP** is an **AR**. We show that this also holds for locally compact spaces.

This paper is in its final form and will not be submitted for publication elsewhere.

The counterexample in [11] is neither topologically complete nor σ-compact. We prove that if there exists a cell-like dimension raising map between compact spaces then there exists a topologically complete separable space with the **CEP** which is not an **ANR**. We have no information on σ-compact spaces.

2. The Theorem.

In this section we shall prove that every locally compact space with the **CEP** is an **AR.**

2.1. PROPOSITION: *Let* Y *be a space with the* **CEP** *and let* B *be a compact subset of* Y. *For every space* X *and every closed subset* A *of* X, *every map* $f: A \to B$ *can be extended to a map* $f': X \to Y$.

PROOF: Consider the adjunction space $Z = X \cup_f B$. Since B is compact, the identity map $B \to Y$ extends to a map $\varphi: Z \to Y$. Then define f' over $X \setminus A = Z \setminus B$ by $f'(x) = \varphi(x)$.

2.2. COROLLARY: *Let* Y *be a space with the* **CEP** *and let* U *be an open subset of* Y *with compact closure. Then* U *is an* **ANR**.

PROOF: Let X be an **AR** containing U as a closed subset, [3]. By proposition 2.1, the identity map $U \to Y$ extends to a map $f: X \to Y$. Consequently, U is a retract of $f^{-1}(U)$. Since U is open, $f^{-1}(U)$ is an **ANR**. We conclude that U is an **ANR**.

We now come to the main result in this section.

2.3. THEOREM: *Every locally compact space with the* **CEP** *is an* **AR**.

PROOF: Let X be a locally compact space with the **CEP**. By corollary 2.2, X is locally an **ANR**. By [2, II 5.1], X is an **ANR**. Since X is clearly C^∞, it follows that X is an **AR**, [5].

3. The Example.

In this section we shall prove that if there exists a cell-like dimension raising map between compact spaces then there exists a topologically complete separable space with the **CEP** which is not an **ANR**.

A space X is called *totally disconnected* if for all distinct $x,y \in X$ there is an open and closed subset $C \subseteq X$ which contains x but misses y.

The following result and its proof are implicit in Rubin, Schori and Walsh [13] and Pol [12].

3.1. PROPOSITION: *Let* X *be a compact space such that* $\dim X \geq n+1$. *Then* X *contains a totally disconnected* G_δ*-subset* S *such that* $\dim S \geq n$.

PROOF: Since $\dim X \geq n+1$ there exists a sequence $(A_0, B_0), \cdots, (A_n, B_n)$ of pairs of disjoint closed subsets of X such that if D_i is a partition between A_i and B_i for every i then $\bigcap_{0\leq i\leq n} D_i \neq \emptyset$, [6]. Let $\alpha: X \to [0,1]$ be a Urysohn map with $\alpha(A_0) = 0$ and $\alpha(B_0) = 1$. In addition, let $\Delta \subseteq (0,1)$ be a Cantor set. The collection $\mathfrak{I}$ of all subcontinua of X meeting A_0 as well as B_0, is closed in the hyperspace of all nonempty closed subsets of X. Consequently, there exists a surjective map $\beta: \Delta \to \mathfrak{I}$. Now define

$$Z = \cup\{\alpha^{-1}(t) \cap \beta(t): t \in \Delta\}.$$

It is easy to see that Z is closed in X and that $f = \alpha \mid Z: Z \to \Delta$ is surjective. By [4, chapter 9] there exists a G_δ-subset $S \subseteq Z$ which meets every fiber of f in precisely one point. Clearly, S is totally disconnected. In addition, $\dim S \geq n$ by [13].

The following lemma is well-known. For completeness sake we include an easy proof.

3.2. LEMMA: *Let* Δ *be an* n*-cell and let* X *be a subset of the boundary of* Δ. *Then* $Y = \Delta \setminus X$ *is an* **AR**.

PROOF: Let Z be any space, $A \subseteq Z$ be closed and $f: A \to Y$ be a map. Since Δ is an **AR**, there is a map $f': Z \to \Delta$ which extends f. There clearly exists a homotopy $H: \Delta \times I \to \Delta$ such that H_0 is the identity while moreover $H(\Delta \times (0,1])$ is contained in the interior of Δ. Let d be an admissible metric for Z such that the diameter of Z is at most 1. Now define $g: Z \to Y$ by

$$g(x) = H_{d(x,A)}(f'(x)).$$

An easy check shows that g extends f.

We now come to the main result in this section.

3.3. THEOREM: *If there exists a cell-like dimension raising map between compact spaces then there exists a topologically complete separable space* Z *with the* **CEP** *such that* Z *is not an* **ANR.**

PROOF: Let X and Y be compact, $f: X \to Y$ be cell-like, and assume that dim Y > dim X. Then dim Y = ∞ by [9] (see also [1]).

By [6] we may assume that X is a subset of the boundary of some n-cell Δ. By proposition 3.1 there exists a totally disconnected G_δ-subset S of Y such that dim S > n. Consider the adjunction space $\Delta \cup_f Y$ and put

$$Z = \text{Int}\,\Delta \cup S \subseteq \Delta \cup_f Y$$

(here Int Δ refers to the geometrical interior of Δ). We claim that Z is the required example.

We first claim that if $A \subseteq S$ is finite-dimensional then Int $\Delta \cup A$ is an **AR**. Simply observe that by lemma 3.2 it follows that Int $\Delta \cup A$ is a cell-like image of an **AR** such that the nondegeneracy set of the map is finite-dimensional. This implies that Int $\Delta \cup A$ is an AR by [9] (see also [1]).

We next claim that Z has the **CEP**. To this end, let F be any space, let $E \subseteq F$ be compact and let $g: E \to Z$ be a map. Then g(E) is compact and since S is closed in Z, it follows that $g(E) \cap S = A$ is compact as well. Since S is totally disconnected, dim $A \leq 0$ and it follows from above that Int $\Delta \cup A$ is an **AR**. Consequenly, g can be extended to a map $g': F \to \text{Int}\,\Delta \cup A \subseteq Z$.

Since Z is clearly topologically complete, there remains to prove that Z is not an **ANR**. But this is a triviality. Simply observe that by lemma 3.2, Z is the cell-like image of an **AR** of smaller dimension than Z. Consequently, Z is not an **ANR** by [9] (see also [1]).

3.4. *Remarks:* We do not know whether every σ-compact space with the **CEP** is an **AR**. This seems to be a difficult problem.

A linear space E is *admissible* if every compact subset of E can be pushed by arbitrarily small maps into finite-dimensional linear subspaces of E. Every locally convex space is admissible, but there exist nonlocally convex spaces which are also admissible, e.g. l^p for $0 < p < 1$, [8]. It is known that every admissible topologically complete linear space has the **CEP**. Apparently, it is still unknown whether *every* linear space is admissible.

References:

[1] F.D. Ancel, *The role of countable dimensionality in the theory of cell-like relations,* Trans. Am. Math. Soc. 287(1985)1-40.

[2] C. Bessaga and A. Pelczynski, *Selected topics in infinite-dimensional topology,* PWN, Warszawa, 1975.

[3] K. Borsuk, *Theory of retracts,* PWN, Warszawa, 1967.

[4] N. Bourbaki, *Topologie Générale,* 2nd ed., Chap. 9, Hermann, Paris, 1958.

[5] S.T. Hu, *Theory of retracts,* Wayne State University Press, Detroit, 1965.

[6] W. Hurewicz and H. Wallman, *Dimension theory,* Van Nostrand, Princeton, N.J., 1948.

[7] V.L. Klee, *Shrinkable neighborhoods in Hausdorff linear spaces,* Math. Ann. 141(1960)281-285.

[8] V.L. Klee, *Leray - Schauder theory without local convexity,* Math. Ann. 141(1960)286-296.
[9] G. Kozlowski, *Images of A(N)R's,* unpublished manuscript.
[10] K. Kuratowski, *Sur quelques problèmes topologiques concernant le prolongement des functions continues,* Coll. Math. 2(1951)186 - 191.
[11] J. van Mill, *Another counterexample in ANR theory,* Proc. Am. Math. Soc. 97(1986)136-138.
[12] R. Pol, *A weakly infinite-dimensional compactum which is not countable -dimensional,* Proc. Am. Math. Soc. 82(1981)634-636.
[13] L. Rubin, R.M. Schori and J.J. Walsh, *New dimension-theory techniques for constructing infinite-dimensional examples,* Gen. Top. Appl. 10(1979)93 - 102.

Department of Mathematics
Louisiana State University
Baton Rouge
LA, 70803
USA

Subfaculteit Wiskunde
Vrije Universiteit
Amsterdam
The Netherlands

and

Mathematisch Instituut
Universiteit van Amsterdam
Amsterdam
The Netherlands