Topology and its Applications 32 (1989) 267-277
North-Holland

ON TOPOLOGICAL AND LINEAR HOMEOMORPHISMS OF CERTAIN FUNCTION SPACES

Jan BAARS
Faculteit Wiskunde en Informatica, Universiteit van Amsterdam, Roetersstraat 15, 1018 WB Amsterdam, The Netherlands

Joost de GROOT
Faculteit Wiskunde en Informatica, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

Jan van MILL
Faculteit Wiskunde en Informatica, Vrije Universiteit and Universiteit van Amsterdam

Jan PELANT
Matematický Ústav, Československá Akademie Věd, Žitná 25, 115 67 Prague, Czechoslovakia

Received 20 November 1987
Revised 9 May 1988

Let X be a countable metric space which is not locally compact. We prove that the function space $C_p(X)$ is homeomorphic to σ_ω. We also give examples of countable metric spaces X and Y which are not locally compact and such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic.

AMS (MOS) Subj. Class.: 54C35, 57N17, 57N20
Hilbert cube Q-matrix   function space

Introduction

Let X be a space. Consider the spaces

$$C_p^*(X)=\{f\in\mathbb{R}^X \mid f \text{ is continuous and bounded}\}$$

and

$$C_p(X)=\{f\in\mathbb{R}^X \mid f \text{ is continuous}\}$$

as subspaces of $\mathbb{R}^X$.

In [6] van Mill showed that for a countable metric space which is not locally compact, $C_p^*(X)\approx\sigma_\omega$, where

$$\sigma_\omega=(l_f^2)^\infty \quad \text{and} \quad l_f^2=\{x\in l^2 \mid x_i=0 \text{ for all but finitely many } i\}$$

(l^2 denotes Hilbert space).

In his paper van Mill used results on Q-matrices and results of [8, 9]. The aim of this paper is to prove that $C_p(X) \approx \sigma_\omega$, by the same methods, and to give examples of countable metric spaces X and Y which are not locally compact such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic.

Observe that $C_p^*(X) \approx C_p(X)$ for X as above whereas these two spaces are not even uniformly isomorphic provided X is not compact.

Section 1 contains some definitions and theorems that we need in Section 2, where we will prove that $C_p(X) \approx \sigma_\omega$. In Section 2 we also sketch an alternative proof that $C_p(X) \approx \sigma_\omega$. In Section 3 we give examples of spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic.

1. Preliminaries

Consider *the Hilbert cube* $Q = \prod_{i=1}^{\infty} [-1, 1]_i$ with the metric

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|.$$

A space which is homeomorphic to Q is called *a Hilbert cube*. If two spaces X and Y are homeomorphic we will use the symbol $X \approx Y$.

Let X and Y be compact spaces (by a space we mean a separable metric space). Put

$$C(X, Y) = \{f: X \to Y \mid f \text{ is continuous}\}$$

and

$$\mathcal{H}(Y) = \{f: Y \to Y \mid f \text{ is a homeomorphism}\}.$$

The topology on both spaces is derived from the metric

$$\hat{d}(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\},$$

where d is an admissible metric on Y.

Let A be a closed subspace of X. A is a *Z-set* in X iff for every $f \in C(Q, X)$ and for every $\varepsilon > 0$, there is a $g \in C(Q, X)$ such that

(1) $\hat{d}(f, g) < \varepsilon$,

(2) $g(Q) \cap A = \emptyset$.

Notation: $A \in \mathcal{Z}(X)$.

1.1. Lemma. *Let* $A \subset Q$ *with* $\pi_j(A) \neq [-1, 1]$ *for infinitely many* j, *then* $A \in \mathcal{Z}(Q)$ ($\pi_j : Q \to [-1, 1]$ *is the projection on the jth coordinate*).

Let $\{A_n\}_{n\in\mathbb{N}}$ be an increasing family of Z-sets in X. Then $\{A_n\}_n \in \mathbb{N}$ is a *skeleton* in X iff for every $\varepsilon > 0$, for every $n \in \mathbb{N}$ and for every $Z \in \mathcal{Z}(X)$, there are $h \in \mathcal{H}(X)$ and $m \in \mathbb{N}$ such that

(1) $\hat{d}(h, 1) < \varepsilon$,

(2) $h|A_n = 1$,
(3) $h(Z) \subset A_m$.

The above definitions and the lemma can be found in [3]. The next three definitions are due to van Mill [6].

A *$\mathscr{Z}$-matrix* in X is a collection $\mathscr{A} = \{A_m^n \mid n, m \in \mathbb{N}\}$ of Z-sets in X such that for every $m, n \in \mathbb{N}$,
(1) $A_1^n = \emptyset$,
(2) $A_m^n \subset A_{m+1}^n$,
(3) $A_m^{n+1} \subset A_m^n$.

Define the *kernel of* $\mathscr{A}$ by $\ker \mathscr{A} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$.

Let $\mathscr{A} = \{A_m^n \mid n, m \in \mathbb{N}\}$ be a $\mathscr{Z}$-matrix in Q. Then $\mathscr{A}$ is a *Q-matrix* iff $\mathscr{A}$ has the following properties:
(1) $\forall n \in \mathbb{N}$: $\{A_m^n\}_{m>1}$ is a skeleton in Q,
and $\forall n_1 < \cdots < n_m \in \mathbb{N}$ and $\forall i_1, \ldots, i_m \in \mathbb{N} \backslash \{1\}$:
(2) $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \approx Q$,
(3) $\forall p \in \mathbb{N}$: $\{\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}\}_{i>1}$ is a skeleton in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$,
(4) $\forall n \in \mathbb{N}$ and $\forall m \in \mathbb{N}\backslash\{1\}$: $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \not\subset A_n^m \Rightarrow \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_n^m \in \mathscr{Z}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$.

In [6] van Mill proved the following theorem.

1.2. Theorem. *If $\mathscr{A}$ is a Q-matrix, then* $\ker \mathscr{A} \approx \sigma_\omega$.

Van Mill used this theorem to prove that if X is a countable metric space which is not locally compact, then $C_p^*(X) \approx \sigma_\omega$. The strategy of the proof is the following: First a nice subspace T of X is constructed and a Q-matrix $\mathscr{B}$ is found such that $\ker \mathscr{B} \approx C_p^*(T)$. So by Theorem 1.2 it follows that $C_p^*(T) \approx \sigma_\omega$. Then by applying strong results of [8, 9] he uses this result to derive that $C_p^*(X) \approx \sigma_\omega$. By the same strategy we will prove that $C_p(X) \approx \sigma_\omega$.

Let $\mathscr{A} = \{A_m^n \mid n, m \in \mathbb{N}\}$ be a $\mathscr{Z}$-matrix and let $A_{m_1}^{n_1}$ and $A_{m_2}^{n_2}$ be in $\mathscr{A}$ such that $n_1 < n_2$ and $m_1 \geqslant m_2$. Then $A_{m_2}^{n_2} \subset A_{m_1}^{n_1}$ so $A_{m_1}^{n_1} \cap A_{m_2}^{n_2} = A_{m_2}^{n_2}$. So for $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N}\backslash\{1\}$ we may assume $i_1 < \cdots < i_m$ if we are interested in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$.

The next theorem can be found in [3]. It will be used in Section 2.

1.3. Theorem. *If $\{A_i\}_{i \in \mathbb{N}}$ is an increasing family of Z-sets in Q such that*
(1) $\forall i \in \mathbb{N}$: $A_i \in \mathscr{Z}(A_{i+1})$,
(2) $\forall i \in \mathbb{N}$: *A_i is convex and infinite-dimensional,*
(3) $\bigcup_{i=1}^{\infty} A_i$ *is dense in Q,*
then $\{A_i\}_{i \in \mathbb{N}}$ is a skeleton in Q.

2. Homeomorphic function spaces

In this section we will prove that for a countable metric space X which is not locally compact the function space $C_p(X)$ is homeomorphic to σ_ω. First we define a test space T in the following way: $T=\mathbb{N}^2\cup\{\infty\}$, where each point of $\mathbb{N}^2$ is isolated and $\{(\{n,n+1,\dots\}\times\mathbb{N})\cup\{\infty\}\}_{n\in\mathbb{N}}$ is an open base at ∞.

Let $C_{p,0}(T)=\{f\in C_p(T)\,|\,f(\infty)=0\}$. We shall identify $C_p(T)$ with its subspace $\{f:T\to(-1,1)\,|\,f$ is continuous$\}$. Let $P=\prod_{i=1}^{\infty}Q_i$, where $Q_i=Q$ for every $i\in\mathbb{N}$. Notice that P is a Hilbert cube. We can embed $C_{p,0}(T)$ into P by the embedding $\phi:C_{p,0}(T)\to P$ defined by

$$\phi(f)_i=f\,|\,\{i\}\times\mathbb{N}.$$

Let $I=[-1,1]$, $I_m=[-1+1/m,1-1/m]$ for every $m\in\mathbb{N}$ and

$$B(\varepsilon)=\textstyle\prod_{i=1}^{\infty}[-\varepsilon,\varepsilon]_i\quad\text{for every } \varepsilon>0.$$

For every $n,m\in\mathbb{N}$ define $A_1^n=\emptyset$ and

$$A_m^n=\textstyle\prod_{i=1}^{m}((I_m)^n\times I\times I\times\cdots)_i\times\prod_{i=m+1}^{\infty}B_i(2^{-n})\subset\prod_{i=1}^{\infty}Q_i=P.$$

Let $\mathscr{A}=\{A_m^n\,|\,n,m\in\mathbb{N}\}$.

2.1. Lemma. $\ker\mathscr{A}=C_{p,0}(T)$.

Proof. Let $f\in\ker\mathscr{A}$ and $(i,j)\in\mathbb{N}^2$. By $f(i,j)$ we mean the jth coordinate of Q_i. Since $f\in\bigcup_{m=1}^{\infty}A_m^j$, there is $m\in\mathbb{N}$ with $f\in A_m^j$. If $i\leqslant m$, then $f(i,j)\in I_m\subset(-1,1)$ and if $i>m$, then $f(i,j)\in[-2^{-j},2^{-j}]\subset(-1,1)$. So f is well defined.

Now we prove that $f:T\to(-1,1)$ is continuous. Therefore we only have to prove that f is continuous at ∞. Let $\varepsilon>0$ and $n\in\mathbb{N}$ such that $2^{-n}<\varepsilon$. Let $m\in\mathbb{N}$ such that $f\in A_m^n$. Then $|f(i,j)|\leqslant 2^{-n}<\varepsilon$ for $i>m$ and $j\in\mathbb{N}$. So f is continuous at ∞. Conversely let $f\in C_{p,0}(T)$ and $n\in\mathbb{N}$. There is $m_1\in\mathbb{N}$ with $|f(i,j)|<2^{-n}$ for $i>m_1$ and $j\in\mathbb{N}$. There is $m_2\in\mathbb{N}$ such that for every $i\leqslant m_1$ and $j\leqslant n$ we have $|f(i,j)|\leqslant 1-1/m_2$. Let $m=\max(m_1,m_2)$. Then $f\in A_m^n$. $\square$

2.2. Lemma. *$\mathscr{A}$ is a $\mathscr{Z}$-matrix in P.*

Proof. By Lemma 1.1 we have for every $n,m\in\mathbb{N}$ that $A_m^n\in\mathscr{Z}(P)$. It is clear that for every $n,m\in\mathbb{N}$, $A_m^n\subset A_{m+1}^n$ and $A_m^{n+1}\subset A_m^n$. $\square$

2.3. Lemma. *$\mathscr{A}$ is a Q-matrix in P.*

Proof. By Lemma 1.1 we have for every $\varepsilon>0$ and $\delta<\varepsilon$ that $B(\delta)\in\mathscr{Z}(B(\varepsilon))$.

Claim 1. $\forall n\in\mathbb{N}$: $\{A_m^n\}_{m>1}$ is a skeleton in P.

By Lemma 1.1 we have for every $n,m\in\mathbb{N}$ that $A_m^n\in\mathscr{Z}(P)$ and $A_m^n\in\mathscr{Z}(A_{m+1}^n)$. Because each A_m^n $(m>2)$ is a product of nondegenerate intervals, it is convex and infinite-dimensional. It is easy to verify that for every $n\in\mathbb{N}$, $\bigcup_{m=1}^{\infty}A_m^n$ is dense in P. By Theorem 1.3 we have for every $n\in\mathbb{N}$ that $\{A_m^n\}_{m>1}$ is a skeleton in P.

Now let $n_1<\cdots<n_m\in\mathbb{N}$ and $i_1,\dots,i_m\in\mathbb{N}\backslash\{1\}$. We may assume $i_1<\cdots<i_m$.

Claim 2. $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \approx P$.
Because $A_{i_1}^{n_m} \subset \bigcap_{k=1}^{m} A_{i_k}^{n_k}$, $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a product of intervals and $A_{i_1}^{n_m} \approx P$ we have $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \approx P$.

Claim 3. $\forall p \in \mathbb{N}$: $\{\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}\}_{i>1}$ is a skeleton in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$.
Let $p \in \mathbb{N}$ and $i \in \mathbb{N}\backslash\{1\}$. Let k be greater than $\max(i, i_m)$. The kth factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is $B(2^{-n_m})$ and the kth factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is $B(2^{-n_m-p})$, so by Lemma 2.1 we have $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathscr{Z}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$. If $i \geqslant i_m$, then the $(i+1)$th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is $B(2^{-n_m-p})$ and the $(i+1)$th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p}$ is $B(2^{-n_m})$. If there is an $l \leqslant m$ such that $i_{l-1} < i+1 \leqslant i_l$ $(i_0 = 1)$, then the $(i+1)$th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is a Z-set in the $(i+1)$th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p}$. We conclude that for every $i \in \mathbb{N}\backslash\{1\}$, $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathscr{Z}(\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p})$. The rest of the claim can be proved as in Claim 1.

Claim 4. $\forall s \in \mathbb{N}$ and $\forall t \in \mathbb{N}\backslash\{1\}$:

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \not\subset A_t^s \Rightarrow \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathscr{Z}\left(\bigcap_{k=1}^{m} A_{i_k}^{n_k}\right).$$

If $s > n_m$, then $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathscr{Z}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$ by Claim 3. If $s \leqslant n_m$, there is $k \leqslant m$ such that $n_{k-1} < s \leqslant n_k$ (let $n_0 = 0$). This implies $t < i_k$. So there is $l \in \mathbb{N}$ such that $i_{l-1} < t+1 \leqslant i_l$ $(i_0 = 0)$. The $(t+1)$th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is $B(2^{-n_{l-1}})$ and the $(t+1)$th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s$ is $B(2^{-s})$, because $s > n_{l-1}$. So $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathscr{Z}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$.

By Claims 1–4 we have that $\mathscr{A}$ is a Q-matrix. □

2.4. Corollary. $C_{p,0}(T) \approx \sigma_\omega$.

Proof. This follows immediately from the Lemmas 2.1, 2.3 and Theorem 1.2. □

2.5. Theorem. *Let X be a countable metric space which is not locally compact. Then $C_p(X) \approx \sigma_\omega$.*

Proof. In [6] van Mill proved that $C_{p,0}^*(T) \approx \sigma_\omega$, where

$$C_{p,0}^*(T) = \{f \in C_p^*(T) \mid f(\infty) = 0\}.$$

By using results of [8, 9], he derived from this fact that $C_p^*(X) \approx \sigma_\omega$. By using the same technique it follows from Corollary 2.4 that $C_p(X) \approx \sigma_\omega$. □

Remark. Let X be a countable metric space which is not locally compact at x_0. It is possible to find a Q-matrix $\mathscr{A}$ such that $\ker \mathscr{A} = C_{p,0}(X)$, where

$$C_{p,0}(X) = \{f \in C_p(X) \mid f(x_0) = 0\}.$$

From this it follows that $C_p(X) \approx C_{p,0}(X) \times \mathbb{R} \approx \sigma_\omega \times \mathbb{R} \approx \sigma_\omega$. The same can be done for $C_p^*(X)$. These results can be found in [2].

Remark. We are indebted to the referee for providing us with the Q-matrix for $C_{p,0}(T)$ presented in this section. This Q-matrix is much simpler than the one originally constructed in [2].

3. Function spaces that are not linearly homeomorphic

In Section 2 we proved that for countable metric spaces X and Y which are both not locally compact, the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic. In the present section we shall construct countable metric spaces X_1, X_2 and X_3 which are not locally compact, such that $C_p(X_i)$ and $C_p(X_j)$ are not *linearly* homeomorphic for $i \neq j$. First we derive the following theorem.

3.1. Theorem. *Let M be a countable metric space which is not locally compact. If M contains an infinite closed discrete set of non isolated points, then $C_p(T)$ and $C_p(M)$ are not linearly homeomorphic.*

Proof. Let $A = \{x_1, x_2, \dots\}$ be a closed discrete set of non isolated points in M. For every $n \in \mathbb{N}$, let $\{U_j^n \mid j \in \mathbb{N}\}$ be a clopen base for x_n, such that $U_{j_1}^{n_1} \cap U_{j_2}^{n_2} = \emptyset$ if $n_1 \neq n_2$ and $j_1, j_2 \in \mathbb{N}$ (this is possible since A is closed discrete and M is zero-dimensional).

Now suppose that $\phi : C_p(M) \to C_{p,0}(T)$ is a linear homeomorphism. Let g_j^n be the characteristic function of U_j^n on M. Since U_j^n is clopen, $g_j^n \in C_p(M)$. Furthermore let $h_j^n = \phi(g_j^n) \in C_{p,0}(T)$.

Claim 1. For every $n \in \mathbb{N}$ and for every $t \in T$, the set $\{h_j^n(t) \mid j \in \mathbb{N}\}$ is bounded.

Suppose the contrary. Without loss of generality we may assume that $h_j^n(t) \geq 0$ for every $j \in \mathbb{N}$. Then for every $k \in \mathbb{N}$, there is $j_k \in \mathbb{N}$ such that $h_{j_k}^n(t) \geq 2^k$. Notice that $f = \sum_{k=1}^{\infty} 2^{-k} g_{j_k}^n \in C_p(M)$. But then $\phi(f) = \sum_{k=1}^{\infty} 2^{-k} h_{j_k}^n \in C_{p,0}(T)$. Since $\phi(f)(t) = \sum_{k=1}^{\infty} 2^{-k} h_{j_k}^n(t)$ is divergent, we have a contradiction.

Claim 2. For every $t \in T$ there are only finitely many $n \in \mathbb{N}$ such that there is $j_n \in \mathbb{N}$ with $h_{j_n}^n(t) \neq 0$.

Suppose there is $t \in T$ such that there are infinitely many $n \in \mathbb{N}$, say $n_1, n_2, \dots$ with the property that for every $i \in \mathbb{N}$, there is $j_i \in \mathbb{N}$ such that $h_{j_i}^{n_i}(t) \neq 0$. Without loss of generality we may assume that $h_{j_i}^{n_i}(t) > 0$ for every $i \in \mathbb{N}$. Let $\lambda_i = (h_{j_i}^{n_i}(t))^{-1}$. Notice that $f = \sum_{i=1}^{\infty} \lambda_i g_{j_i}^{n_i} \in C_p(M)$, so $\phi(f) = \sum_{i=1}^{\infty} \lambda_i h_{j_i}^{n_i} \in C_{p,0}(T)$. Since $\phi(f)(t) = \sum_{i=1}^{\infty} \lambda_i h_{j_i}^{n_i}(t) = \sum_{i=1}^{\infty} 1$ is divergent, we have a contradiction.

Now let $n \in \mathbb{N}$. Then $g_j^n \to \chi_{\{x_n\}}$ (the characteristic function of $\{x_n\}$) pointwise $(j \to \infty)$. Observe that $\chi_{\{x_n\}}$ is not continuous, since x_n is a non isolated point. For every $n \in \mathbb{N}$ we define a sequence $(f_k^n)_{k \in \mathbb{N}}$ in $C_{p,0}(T)$ as follows: Since T is countable, we can enumerate the elements of $T \backslash \{\infty\}$ as $\{t_1, t_2, \dots\}$. Inductively for every $l \in \mathbb{N}$, we find converging sequences $(h_{j_k^l}^n(t_l))_{k \in \mathbb{N}}$ as follows: Since $\{h_j^n(t_1) \mid j \in \mathbb{N}\}$ is bounded (Claim 1), there is a converging subsequence $(h_{j_k^1}^n(t_1))_{k \in \mathbb{N}}$. Suppose the sequence is found for $i = 1, \dots, l$. Since $\{h_{j_k^l}^n(t_{l+1}) \mid k \in \mathbb{N}\}$ is bounded (Claim 1), there is a converging subsequence $(h_{j_k^{l+1}}^n(t_{l+1}))_{k \in \mathbb{N}}$.

Now let $f_k^n = h_{j_k}^{n_k}$ ($k \in \mathbb{N}$). By construction, for every $t \in T$, $\sigma_n(t) = \lim_{k\to\infty} f_k^n(t)$ exists. So $\sigma_n : T \to \mathbb{R}$ is well defined. Observe that $\sigma_n(\infty) = 0$. Suppose σ_n is continuous. Then $f_k^n \to \sigma_n$ in $C_{p,0}(T)$. Since ϕ^{-1} is continuous, $\phi^{-1}(f_k^n) = g_{j_k}^{n_k} \to \phi^{-1}(\sigma_n)$ in $C_p(M)$, so $\chi_{\{x_n\}} = \lim_{k\to\infty} g_{j_k}^{n_k}$ is continuous; a contradiction.

Since σ_n is well defined and $T\backslash\{\infty\}$ is discrete, σ_n is discontinuous at ∞. It follows that there is a sequence $(y_l^n)_{l\in\mathbb{N}}$ in T, converging to ∞ such that $|\sigma_n(y_l^n)| > \varepsilon_n$ for some $\varepsilon_n > 0$ and for every $l \in \mathbb{N}$. Since ϕ is linear we may assume that $\varepsilon_n = 1$ for every $n \in \mathbb{N}$.

We now inductively construct sequences $(n_i)_{i\in\mathbb{N}}$, $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ and $(t_i)_{i\in\mathbb{N}}$ in T such that

(i) $n_1 < n_2 < \cdots$,
(ii) $t_i \to \infty$,
(iii) $|f_{k_i}^{n_i}(t_i)| > 1$ for every $i \in \mathbb{N}$,
(iv) $f_{k_i}^{n_i}(t_j) = 0$ for every $i \in \mathbb{N}$ and $j < i$,
(v) $|f_{k_j}^{n_j}(t_i)| < 1/(2(i-1))$ for every $i \in \mathbb{N}$ and $j < i$,

as follows:

Let $n_1 = 1$. Since $\lim_{k\to\infty} |f_k^{n_1}(y_1^{n_1})| = |\sigma_{n_1}(y_1^{n_1})| > 1$, there is $k_1 \in \mathbb{N}$ such that $|f_{k_1}^{n_1}(y_1^{n_1})| > 1$. Let $t_1 = y_1^{n_1}$.

Suppose $n_1, \ldots, n_i$, $k_1, \ldots, k_i$ and $t_1, \ldots, t_i$ are found. By Claim 2, for every $j \leqslant i$ there are only finitely many $n \in \mathbb{N}$ such that $f_m^n(t_j) \neq 0$ for some $m \in \mathbb{N}$. It follows that there is $n_{i+1} > n_i$ such that for every $j \leqslant i$ and $m \in \mathbb{N}$, $f_m^{n_{i+1}}(t_j) = 0$, so (i) and (iv) are satisfied. Since $y_l^{n_{i+1}} \to \infty$ ($l \to \infty$) and $f_{k_j}^{n_j} \in C_{p,0}(T)$ for $j \leqslant i$, there is $l_0 \in \mathbb{N}$ such that for $t_{i+1} = y_{l_0}^{n_{i+1}}$ we have

$$|f_{k_j}^{n_j}(t_{i+1})| < 1/(2i) \quad (j \leqslant i),$$

and the first coordinate of t_{i+1} is greater than the first coordinates of $t_1, \ldots, t_i$. With this t_{i+1} (v) is satisfied.

Finally, since $\lim_{k\to\infty} |f_k^{n_{i+1}}(t_{i+1})| = |\sigma_{n_{i+1}}(t_{i+1})| > 1$, there is $k_{i+1} \in \mathbb{N}$ such that

$$|f_{k_{i+1}}^{n_{i+1}}(t_{i+1})| > 1,$$

so (iii) is satisfied and the induction is completed. By construction (ii) is also satisfied.

Notice that by (i), $f = \sum_{j=1}^{\infty} \phi^{-1}(f_{k_j}^{n_j}) \in C_p(M)$. So $\phi(f) = \sum_{j=1}^{\infty} f_{k_j}^{n_j} \in C_{p,0}(T)$. Since by (ii) $t_i \to \infty$, $\phi(f)(t_i) \to 0$. But

$$\begin{aligned}
|\phi(f)(t_i)| &= \left| \sum_{j=1}^{\infty} f_{k_j}^{n_j}(t_i) \right| \\
&= \left| \sum_{j=1}^{i-1} f_{k_j}^{n_j}(t_i) + f_{k_i}^{n_i}(t_i) \right| \quad \text{(by (iv))} \\
&\geqslant \left| |f_{k_i}^{n_i}(t_i)| - \left| \sum_{j=1}^{i-1} f_{k_j}^{n_j}(t_i) \right| \right| \\
&> 1 - (i-1)/(2(i-1)) \quad \text{(by (iii) and (v))} \\
&= 1 - \tfrac{1}{2} = \tfrac{1}{2}.
\end{aligned}$$

A contradiction.

Since $C_{p,0}(T)$ and $C_p(T)$ are linearly homeomorphic, we proved the theorem. $\square$

Now let $\mathbb{Q}$ be the set of rationals and $\sum T = \sum_{i=1}^{\infty} T_i$ the topological sum of infinitely many copies of T. For convenience let ∞_i be the non isolated point in T_i. Notice that $\mathbb{Q}$ and $\sum T$ are countable metric spaces which are not locally compact. By Theorem 3.1 we have the following corollary.

3.2. Corollary. (a) *$C_p(\mathbb{Q})$ and $C_p(T)$ are not linearly homeomorphic.*
(b) *$C_p(\sum T)$ and $C_p(T)$ are not linearly homeomorphic.*

However Theorem 3.1 does not decide whether $C_p(\mathbb{Q})$ and $C_p(\sum T)$ are linearly homeomorphic. In the sequel we will show that this is not the case. First we need some results from [1].

For every space X let $C(X)=\{f: X\to\mathbb{R} \mid f \text{ is continuous}\}$. If we endow $C(X)$ with the compact-open topology we write $C_0(X)$. A subbase for $C_0(X)$ is

$$\{A_X(K, U) \mid K\subset X \text{ compact and } U\subset\mathbb{R} \text{ open}\},$$

where $A_X(K, U)=\{f\in C(X) \mid f(K)\subset U\}$.

3.3. Theorem (Arhangelskiĭ [1]). *If $\theta: C_p(X)\to C_p(Y)$ is a linear homeomorphism, then θ considered as a function from $C_0(X)\to C_0(Y)$ is also a linear homeomorphism.*

Let X and Y be spaces and $\theta: C(X)\to C(Y)$ a linear mapping.

3.4. Definition. For every $y\in Y$, the *support* of y in X is defined to be the set $\operatorname{supp}(y)$ of all $x\in X$ satisfying the condition that for every neighborhood U of $x\in X$ there is an $f\in C(X)$ such that $f(X\setminus U)\subset\{0\}$ and $\theta(f)(y)\neq 0$.

Notice that for every $y\in Y$, $\operatorname{supp}(y)$ is closed.

3.5. Definition. θ is said to be *effective* if for every $f, g\in C(X)$ and $y\in Y$, such that f and g coincide on a neighborhood of $\operatorname{supp}(y)$, $\theta(f)(y)=\theta(g)(y)$.

3.6. Proposition (Arhangelskiĭ [1]). *Let $\theta: C_0(X)\to C_0(Y)$ be a linear homeomorphism. Then*
(a) *θ is effective,*
(b) *for every compact $K\subset Y$ we have that $\overline{\bigcup_{y\in K}\operatorname{supp}(y)}$ is compact.*

Remark. In fact Arhangelskiĭ proved that for every bounded set $K\subset Y$ (that means for every continuous real-valued f on Y, $f(K)$ is bounded in $\mathbb{R}$), $\bigcup_{y\in K}\operatorname{supp}(y)$ is bounded in X. For metric spaces we then have the formulation in Proposition 3.6(b).

We shall prove that $C_p(\sum T)$ and $C_p(\mathbb{Q})$ are not linearly homeomorphic. To derive a contradiction we assume a linear homeomorphism $\theta: C_0(\sum T)\to C_0(\mathbb{Q})$. For the proof we need a property which can be found in [4].

3.7. Definition. Let E be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence in E. We say that $x_n \to x$ weakly iff $f(x_n) \to f(x)$ for every (norm) continuous linear functional f on E.

3.8. Definition. A Banach space E has the *weak Banach–Saks property* iff for every sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $x_n \to 0$ weakly, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $\|\sum_{k=1}^{j} x_{n_k}/j\| \to 0$.

It is easy to see that the weak Banach–Saks property is a property which is preserved by linear homeomorphisms and which is hereditary.

For a space X we denote $X^{(1)}$ the set of accumulation points of X. Inductively we can define $X^{(n)} = (X^{(n-1)})^{(1)}$ for every $n \in \mathbb{N}$ and $X^{(\omega)} = \bigcap_{n=1}^{\infty} X^{(n)}$.

3.9. Theorem [4, p. 85]. *Let X be a compact space. Then $C_0(X)$ has the weak Banach–Saks property iff $X^{(\omega)} = \emptyset$.*

Let K be a copy of $\omega^\omega + 1$ in $\mathbb{Q}$ and $L = \overline{\bigcup_{y\in K} \operatorname{supp}(y)}$. Notice that K is compact and therefore by Proposition 3.6(b), L is compact. This means that there is $p \in \mathbb{N}$ such that $L \subset \sum_{i=1}^{p} T_i$. Furthermore by Theorem 3.9, $C_0(L)$ has and $C_0(K)$ does *not* have the weak Banach–Saks property.

3.10. Lemma. *For every $f, g \in C_0(\mathbb{Q})$ with $f|K \neq g|K$ it follows that*

$$\theta^{-1}(f)\,|\,L \neq \theta^{-1}(g)\,|\,L.$$

Proof. Let $y \in K$ be such that $f(y) \neq g(y)$ and let R_0 and R_1 be disjoint open neighborhoods of $f(y)$ and $g(y)$, respectively. Then $A_{\mathbb{Q}}(\{y\}, R_0)$ and $A_{\mathbb{Q}}(\{y\}, R_1)$ are disjoint open neighborhoods of f and g in $C_0(\mathbb{Q})$. So $\theta^{-1}(A_{\mathbb{Q}}(\{y\}, R_0))$ and $\theta^{-1}(A_{\mathbb{Q}}(\{y\}, R_1))$ are disjoint open neighborhoods of $\theta^{-1}(f)$ and $\theta^{-1}(g)$ in $C_0(\sum T)$. There consequently exist compact $K_1, \ldots, K_n, L_1, \ldots, L_m \subset \sum T$ and open $U_1, \ldots, U_n, V_1, \ldots, V_m \subset \mathbb{R}$ such that

$$\theta^{-1}(f) \in \bigcap_{i=1}^{n} A_{\sum T}(K_i, U_i) \subset \theta^{-1}(A_{\mathbb{Q}}(\{y\}, R_0)),$$

and

$$\theta^{-1}(g) \in \bigcap_{j=1}^{m} A_{\sum T}(L_j, V_j) \subset \theta^{-1}(A_{\mathbb{Q}}Q(\{y\}, R_1)).$$

We claim there is a $z \in \operatorname{supp}(y) \subset L$ such that $\theta^{-1}(f)(z) \neq \theta^{-1}(g)(z)$ (and then we are done). Striving for a contradiction, assume the contrary.

For every $s \leq p$ let $I_s = \{i \leq n \,|\, \infty_s \notin K_i\}$, $J_s = \{j \leq m \,|\, \infty_s \notin L_j\}$ and

$$P_s = \bigcup_{i\in I_s} (K_i \cap T_s) \cup \bigcup_{j\in J_s} (L_j \cap T_s).$$

Then P_s is compact in T_s and $\infty_s \notin P_s$, so P_s is finite. Let $P = \bigcup_{s \leq p} P_s$. Then P is finite and $P \cap \{\infty_1, \ldots, \infty_p\} = \emptyset$.

Define $f' : \sum T \to \mathbb{R}$ by

$$f'(x) = \begin{cases} \theta^{-1}(f)(x) & \text{if } x \in P \cup \sum_{i=p+1}^{\infty} T_i, \\ \theta^{-1}(f)(\infty_s) & \text{if } x \in T_s \backslash P_s \text{ for } s \leq p. \end{cases}$$

Define $g' : \sum T \to \mathbb{R}$ by

$$g'(x) = \begin{cases} \theta^{-1}(g)(x) & \text{if } x \in P \cup \sum_{i=p+1}^{\infty} T_i, \\ \theta^{-1}(g)(\infty_s) & \text{if } x \in T_s \backslash P_s \text{ for } s \leq p. \end{cases}$$

Then f' and g' are continuous because $T_s \backslash P_s$ is a neighborhood of ∞_s.

Claim 1. f' and g' coincide on a neighborhood of $\operatorname{supp}(y)$.

Let $M = \{s \leq p \mid \infty_s \in \operatorname{supp}(y)\}$. Then by assumption we have that for every $s \in M$ $\theta^{-1}(f)(\infty_s) = \theta^{-1}(g)(\infty_s)$. Let $U = \sum_{s \in M} T_s \backslash P_s \cup \operatorname{supp}(y)$. Then U is a neighborhood of $\operatorname{supp}(y)$ on which f' and g' coincide.

Now by Claim 1 and the fact that θ is effective we have $\theta(f')(y) = \theta(g')(y)$.

Claim 2. $f' \in \bigcap_{i=1}^{n} A_{\sum T}(K_i, U_i)$ and $g' \in \bigcap_{j=1}^{m} A_{\sum T}(L_j, V_j)$.

Let $i \leq n$ and $x \in K_i$. If $x \in P \cup \sum_{i=p+1}^{\infty} T_i$, then $f'(x) = \theta^{-1}(f)(x) \in U_i$ because $\theta^{-1}(f) \in \bigcap_{i=1}^{n} A_{\sum T}(K_i, U_i)$, and if $x \notin P \cup \sum_{i=p+1}^{\infty} T_i$ we have $x \in T_s \backslash P_s$ for some $s \leq p$. Because $x \in K_i \cap T_s$ and $x \notin P_s$ we have $\infty_s \in K_i$, so $f'(x) = \theta^{-1}(f)(\infty_s) \in U_i$. The remaining part of the claim can be proved similarly.

Now we have $\theta(f') \in A_{\mathbb{Q}}(\{y\}, R_0)$ and $\theta(g') \in A_{\mathbb{Q}}(\{y\}, R_1)$. But this means $\theta(f')(y) \neq \theta(g')(y)$; a contradiction. $\square$

Because $K \subset \mathbb{Q}$ is compact and $L \subset \sum T$ is compact we can find retractions $r : \mathbb{Q} \to K$ and $s : \sum T \to L$ (see [5]). Define

$$\psi : C_0(K) \to C_0(L) \quad \text{by } \psi(f) = \theta^{-1}(f \circ r) \mid L,$$

and

$$\phi : C_0(L) \to C_0(K) \quad \text{by } \phi(g) = \theta(g \circ s) \mid K.$$

3.11. Lemma. *ψ is a linear embedding.*

Proof. It is easy to see that ψ and ϕ are well-defined continuous functions. We claim that for every $h \in C_0(K)$ we have $\phi(\psi(h)) = h$. To the contrary suppose $\theta(\psi(h) \circ s) \mid K \neq h = (h \circ r) \mid K$. By Lemma 3.10 we have $(\psi(h) \circ s) \mid L \neq \theta^{-1}(h \circ r) \mid L$. But this implies $\psi(h) \neq \psi(h)$; a contradiction. It easily follows that ψ is a linear embedding. $\square$

3.12. Proposition. *$C_p(\sum T)$ and $C_p(\mathbb{Q})$ are not linearly homeomorphic.*

Proof. Suppose $C_p(\sum T)$ and $C_p(\mathbb{Q})$ are linearly homeomorphic. By Theorem 3.3 $C_0(\sum T)$ and $C_0(\mathbb{Q})$ are linearly homeomorphic. Then by Lemma 3.11 we have a linear embedding $\psi: C_0(K) \to C_0(L)$. $C_0(L)$ has the weak Banach-Saks property which $C_0(K)$ does not have; a contradiction. □

Remark. In [7] Pelant proved that the function spaces $C_p^*(T)$ and $C_p^*(\mathbb{Q})$ are not linearly homeomorphic. His proof does not seem to generalize to get our result that $C_p(T)$ and $C_p(\mathbb{Q})$ are not linearly homeomorphic.

Note added in proof

Dobrowolski, Gulko and Mogilski recently proved that $C_p(X) \approx \sigma_\omega$ provided X is a countable nondiscrete metric space. This result generalizes Theorem 2.5 of the present paper.

References

[1] A.V. Arhangelskiĭ, On linear homeomorphisms of function spaces, Soviet Math. Dokl. 25 (1982) 852–855.

[2] J. Baars, J. de Groot and J. van Mill, Topological equivalence of certain function spaces II, VU (Amsterdam) Report 321, 1986.

[3] C. Bessaga and A. Pełczyński, Selected Topics in Infinite-dimensional Topology (PWN, Warsaw, 1975).

[4] J. Diestel, Geometry of Banach-spaces—Selected Topics (Springer, Berlin, 1975).

[5] R. Engelking, On closed images of the space of irrationals, Proc. Amer. Math. Soc. 21 (1969) 583–586.

[6] J. van Mill, Topological equivalence of certain function spaces, Compositio Math. 63 (1987) 159–188.

[7] J. Pelant, A remark on spaces of bounded continuous functions, Indag. Math. 91 (1988) 335–338.

[8] H. Toruńczyk, On cartesian factors and the topological classification of linear metric spaces, Fund. Math. 88 (1975) 71–86.

[9] H. Toruńczyk, (G, K)-absorbing and skeletonized sets in metric spaces, Dissertationes Math., to appear.

[10] T. Dobrowolski, S.P. Gulko and J. Mogilski, Function spaces homeomorphic to the countable product of l_2^f, Topology Appl., to appear.