Topology and its Applications 33 (1989) 21-45
North-Holland

CONCERNING CONNECTED, PSEUDOCOMPACT ABELIAN GROUPS

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Received 5 January 1988
Revised 23 June 1988

It is known that if **P** is either the property ω-bounded or countably compact, then for every cardinal $\alpha \geqslant \omega$ there is a **P**-group G such that $wG = \alpha$ and no proper, dense subgroup of G is a **P**-group. What happens when **P** is the property pseudocompact? The first-listed author and Robertson have shown that every zero-dimensional Abelian **P**-group G with $wG > \omega$ has a proper, dense, **P**-group. Turning to the case of connected **P**-groups, the present authors show the following results: Let G be a connected, pseudocompact, Abelian group with $wG = \alpha > \omega$. If any one of the following conditions holds, then G has a proper, dense (necessarily connected) pseudocompact subgroup: (a) $wG \leqslant c$; (b) $|G| \geqslant \alpha^{\omega}$; (c) α is a strong limit cardinal and $\text{cf}(\alpha) > \omega$; (d) $|\text{tor}\ G| > c$; (e) G is not divisible.

AMS (MOS) Subj. Class.: Primary 54A05; secondary 20K45, 22C05
pseudocompact group   connected group   dense subgroup

1. Introduction

All topological spaces considered here (in particular, all topological groups) are assumed to be completely regular, Hausdorff spaces, i.e., Tychonoff spaces. Following Engelking [12], we use w, χ, ψ and d to denote weight, local weight (=character), pseudocharacter and density character, respectively. As usual, a space X is said to be:

ω-bounded if $\text{cl}_X A$ is compact whenever $A \subseteq X$ and $|A| \leqslant \omega$;

countably compact if each infinite subset of X has an accumulation point in X; and

pseudocompact if every locally finite family of nonempty open subsets of X is finite—equivalently, if every real-valued continuous function on X is bounded.

For extensive information and references to the literature concerning these properties, the reader may consult [12].

* The second-listed author acknowledges partial support from the National Science Foundation (USA) under grant NSF-DMS-8606113. He is pleased to thank also the Department of Mathematics at Wesleyan University for generous hospitality and support during the summers of 1986 and 1987.

1.1. History of the problem

Of course, every ω-bounded space is countably compact. It has been known for some time [8, 3.3] that for every infinite cardinal α there is an ω-bounded group G with $wG = \alpha$ such that no proper, dense, subgroup of G is countably compact. For an example to this effect let G be the Σ-product in T^α (first defined in a related context in [15, 10, 21]) by

$$G = \{x \in T^\alpha : |\{\xi < \alpha : x_\xi \neq 1\}| \leq \omega\}.$$

For every countable $A \subseteq G$ there is countable $C \subseteq \alpha$ such that

$$A \subseteq T^C \times \{1\}_{\alpha \setminus C} \subseteq G,$$

so G is ω-bounded. Now let D be a dense, countably compact subspace of G. As with every dense, pseudocompact space, D is then G_δ-dense in G. To prove that $D = G$ it is enough to show that for every $p \in G$ there is a sequence $x(n)$ in D such that $x(n) \to p$. To see this, for each $x \in G$ write

$$E(x) = \{\xi < \alpha : x_\xi \neq 1\},$$

choose $x(0) \in D$ so that

$$x(0)_\xi = p_\xi \quad \text{for } \xi \in E(p)$$

and recursively, $x(k)$ having been defined in D for all $k < n$, choose $x(n) \in D$ so that

$$x(n)_\xi = \begin{cases} p_\xi & \text{for } \xi \in E(p), \\ 1 & \text{for } \xi \in \left(\bigcup_{k<n} E(x(k)) \right) \setminus E(p). \end{cases}$$

(The availability for $0 \leq n < \omega$ of such points $x(n) \in D$ is immediate from the fact that D is G_δ-dense in G.)

The group G just defined is connected. If a zero-dimensional example is desired, replace T by $\{0, 1\}$. For a torsion-free group which is zero-dimensional or connected (and of weight α), replace T with Δ_a or Σ_a as in [17, 25.8]. Obviously non-Abelian modifications are also readily constructed.

The foregoing remarks and the close relationship between pseudocompactness and countable compactness [12] make it attractive to conjecture the existence, for each infinite cardinal α, of pseudocompact groups of weight α with no proper, dense, pseudocompact subgroups. As our Abstract indicates, however, the available evidence runs otherwise: it is shown in [7] that every zero-dimensional pseudocompact Abelian group of uncountable weight (in particular, every pseudocompact, Abelian torsion group of uncountable weight) does have a proper, dense, pseudocompact subgroup. This has led us to consider the connected case, with the results listed in the Abstract.

The five results enunciated in the Abstract and labelled (a), (b), (c), (d) and (e) are proved below in Theorems 4.3(a), 5.5, Corollary 5.8(ii) and Theorems 6.1 and 7.1, respectively.

Every topological group of countable weight is metrizable [17, 8.3], and every pseudocompact metrizable space is compact (see [12, 8.5.13(c)] or [14, 9.7]). It follows that a pseudocompact group of weight $\leqslant\omega$ has no proper, dense, pseudocompact subgroup. For the most part, therefore, we are concerned in this paper only with topological groups of uncountable weight.

1.2. Present status of the problem

Consider now the following three statements:

(1) every zero-dimensional pseudocompact Abelian group of uncountable weight has a proper, dense, pseudocompact subgroup;

(2) every connected, pseudocompact Abelian group of uncountable weight has a proper, dense, pseudocompact subgroup; and,

(3) every pseudocompact Abelian group of uncountable weight has a proper, dense, pseudocompact subgroup.

The situation at present concerning these three statements is: (1) is true [7], (2) and (3) are as yet undetermined. While attempting unsuccessfully to establish (2) in full generality, we were led to consider the natural question whether (3) follows from (1) and (2). (This question is "natural" because it happens frequently in the theory of topological groups that a theorem is established by treating separately the zero-dimensional and the connected cases and then "pasting together" to achieve full generality. See [17] for examples of this procedure.) Two avenues of approach for deriving (3) from (1) and (2) attracted our attention, as follows, but each appears ineluctably blocked.

(i) The technique: Given G as in (3) with identity component C, attempt to apply (1) to find a proper, dense pseudocompact subgroup H of G/C and then, with ϕ the natural homomorphism from G onto G/C, use $\phi^{-1}(H)$ as the desired subgroup of G. The difficulties: First, in contrast with the situation in compact groups, a totally disconnected pseudocompact group need not be zero-dimensional (see Corollary 7.7 below for elucidation of this point). Thus G/C need not be zero-dimensional. Second, given a continuous, open surjective homomorphism $h: G\to K$ with G and K pseudocompact Abelian groups and H a dense, pseudocompact subgroup of K, the group $h^{-1}(H)$ need not be pseudocompact; in Remark 7.5 below we describe briefly a construction to this effect taken from [7].

(ii) The technique: Given G and C as in (i), and assuming further that C is divisible, let h be a (possibly discontinuous) homomorphism from G onto the divisible group C such that $h(x)=x$ for all $x\in C$, use (2) to find a proper, dense, pseudocompact subgroup H of C, and use $h^{-1}(H)=H\times\ker h$ as the desired subgroup of G. The difficulties: First, the group C need not be pseudocompact. Indeed, we show in Theorem 7.6 that many nonpseudocompact groups arise as the identity component of a pseudocompact group. Second, even if the connected component C is pseudocompact, it need not be divisible; we give an example to this effect in Remark 7.2 below.

We draw attention explicitly to a question which remains unsolved: Does every pseudocompact, Abelian group G such that $\omega < wG \leq \mathfrak{c}$ have a proper, dense pseudocompact subgroup?

1.3. Future considerations

Despite the many "positive" results now available, including those in this paper, we are ambivalent on the question of whether every pseudocompact Abelian group of uncountable weight has a proper, dense, pseudocompact subgroup. As our results make clear, the simplest or most accessible counterexample, if one exists, may be a (connected) G_δ-dense subgroup G of $T^{(\mathfrak{c}^+)}$ such that $|G| = \mathfrak{c}$; see Question 8.6 below. We hope to return to this matter in a later communication.

2. Prerequisites

In this section we fix notation, cite the results we need from the literature, and establish some elementary, preliminary lemmas. The proofs we omit are available in [3] or [17].

Since we are concerned chiefly with Abelian groups, we use additive notation throughout (even in discussing groups which are, perhaps, non-Abelian); the identity of every group is denoted 0. (As in Section 1, we make an exception in the case of the circle group T, where we use the usual multiplicative notation and denote the identity by 1.)

The torsion subgroup of an Abelian group G is denoted tG or $t(G)$. That is,

$$tG = \{x \in G\colon \text{there is } n \in \mathbb{Z}\backslash\{0\} \text{ such that } nx = 0\}.$$

A topological group G is said to be *totally bounded* (by some authors: *precompact*) if for every nonempty open $U \subseteq G$ there is a finite $F \subseteq G$ such that $G = F + U$. It is a theorem of Weil [26] that a topological group G is totally bounded if and only if G is (topologically isomorphic with) a dense subgroup of a compact group K; when these conditions are satisfied the group K is unique in the obvious sense [26] and we write $K = \bar{G}$.

2.1. Theorem [2, 20, 16, 18]. *Let K be a compact group with $wK = \alpha \geq \omega$ and let G be a dense subgroup of K (so, $K = \bar{G}$). Then*

(a) $|K| = 2^\alpha$;

(b) $dK = \log \alpha$; *and*

(c) $wK = \chi(K) = \chi G = wG$.

The following result gives a convenient characterization of those totally bounded groups which are pseudocompact.

2.2. Theorem [8]. *Let G be a totally bounded group. The following statements are equivalent*:

(a) *G is pseudocompact*;

(b) *G is G_δ-dense in $\bar{G}$*;

(c) *$\bar{G} = \beta G$.*

2.3. Corollary. *Let G be a pseudocompact group and H a dense subgroup of G. Then*

(a) *G is connected if and only if $\bar{G}$ is connected*; *and*

(b) *H is pseudocompact if and only if H is G_δ-dense in G.*

2.4. Theorem [9]. *Let G be a compact, connected Abelian group such that $wG > \omega$. Then there is a continuous homomorphism from G onto $\boldsymbol{T}^{(\omega^+)}$.*

The following easy result allows us to give a simplified proof of a theorem from [7] in Theorem 2.7(d).

For a space X a *zero-set* of X is a set of the form $f^{-1}(\{0\})$ with $f \in C(X, \boldsymbol{R})$.

2.5. Lemma. *Let X be a pseudocompact space and let Z be a zero-set of X. Then $\mathrm{cl}_{\beta X} Z$ is a zero-set of βX.*

Proof. Let $Z = f^{-1}(\{0\})$ with bounded $f \in C(X, \boldsymbol{R})$ and let $\bar{f} \in C(\beta X, \boldsymbol{R})$ satisfy $f \subseteq \bar{f}$. Then $\mathrm{cl}_{\beta X} Z \subseteq \bar{f}^{-1}(\{0\})$, and if there is

$$p \in \bar{f}^{-1}(\{0\}) \backslash \mathrm{cl}_{\beta X} Z,$$

then, choosing $g \in C(\beta X, \boldsymbol{R})$ such that $g(p) = 0$ and $g \equiv 1$ on Z and writing

$$h = |\bar{f}| + g,$$

we have $h \in C(\beta X, \boldsymbol{R})$, $h(p) = 0$, and $h > 0$ on X; this contradicts the hypothesis that X is pseudocompact. $\square$

We use the word "normal" only in its algebraic sense. In detail: A subgroup N of a group G is normal if and only if $aN = Na$ for all $a \in G$.

We adopt the following notation from [7].

2.6. Notation. Let G be a topological group. Then

$$\Lambda(G) = \{N \subseteq G\colon N \text{ is a closed, normal, } G_\delta\text{-subgroup of } G\}.$$

Here we collect together the information needed later about groups in the set $\Lambda(G)$ with G pseudocompact.

2.7. Theorem. *Let G be a pseudocompact group such that $wG > \omega$, and let $N \in \Lambda(G)$. Then*

(a) *G/N is compact and metrizable*;

(b) *N is a zero-set of G*;

(c) *$\bar{N}$ is a zero-set of $\bar{G}$*;

(d) [7, 6.2] *N is pseudocompact*; *and*

(e) *$wN = wG$.*

Proof. (a) is proved in [9, 3.3].

(b) The natural surjection $\phi : G \to G/N$ is continuous, so from the fact that $\{N\}$ is a zero-set of G/N it follows that $\phi^{-1}(\{N\})$, which is N, is a zero-set of G.

(c) Since $\bar{G} = \beta G$ by Theorem 2.2, statement (c) is immediate from (b) and Lemma 2.5.

(d) From the fact that $\bar{N}$ is a G_δ-set of $\bar{G}$ and G is G_δ-dense in $\bar{G}$ (Lemma 2.5) it is immediate that N is G_δ-dense in $\bar{N}$. Hence, by Lemma 2.5 (applied to N), the group N is pseudocompact.

(e) Since local weight χ and pseudocharacter ψ agree on compact spaces, Theorem 2.1(c) yields

$$wN = w\bar{N} = \psi\bar{N} \quad \text{and} \quad \psi\bar{G} = w\bar{G} = wG.$$

Now from (c) it follows that $\bar{N}$ is a G_δ-set of $\bar{G}$, so from

$$\omega < \psi\bar{G} \leq \omega + \psi(\bar{N})$$

follows $\psi\bar{G} = \psi\bar{N}$ and hence $wN = wG$, as required. □

2.8. Notation. Let $X = \langle X, \mathcal{T}\rangle$ be a topological space. The *P-space topology on X determined by $\mathcal{T}$* is the smallest topology $P\mathcal{T}$ on X such that $\mathcal{T} \subseteq P\mathcal{T}$ and every G_δ-subset of $\langle X, P\mathcal{T}\rangle$ is $P\mathcal{T}$-open.

The set X with the $P\mathcal{T}$ topology is denoted PX.

Given $X = \langle X, \mathcal{T}\rangle$, the $\mathcal{T}$-G_δ-sets are a base for PX. Thus a subset Y of X is dense in PX if and only if Y is G_δ-dense in X, and the following statements are immediate from Theorem 2.2.

2.9. Theorem [6]. *Let G be a pseudocompact group and H a dense subgroup of G. Then H is pseudocompact if and only if H is dense in PG.*

2.10. Theorem. *Let G be a pseudocompact group with $wG = \alpha \geq \omega$. Then*

(a) $w(PG) \leq \alpha^\omega$;

(b) *G has a dense, pseudocompact subgroup H such that $|H| \leq \alpha^\omega$*; *and*

(c) *if $|G| > \alpha^\omega$, then G has a proper, dense pseudocompact subgroup.*

A space X is a *Baire space* if every intersection of countably many dense, open subsets of X is dense. The following theorem derives from these two facts: (a) PX is a Baire space whenever X is compact; and (b) a G_δ-dense subspace of a Baire space is a Baire space.

2.11. Theorem (see [7]). *Let G be a pseudocompact group. Then G and PG are Baire spaces (and hence of second category).*

The core of the following theorem appears in [6, 2.5] and perhaps elsewhere; we include a proof in order that the present paper be self-contained.

2.12. Theorem. *Let G be a pseudocompact group such that $wG > \omega$. Then*
(a) $|G| \geq c$;
(b) $\mathrm{cf}(d(PG)) > \omega$; *and*
(c) *if G has no proper, dense pseudocompact subgroup, then* $\mathrm{cf}(|G|) > \omega$.

Proof. (a) There is $N \in \Lambda(G)$ such that $|G/N| \geq \omega$ (and hence $w(G/N) \geq \omega$). Since G/N is compact by Theorem 2.7(a) we have $|G/N| \geq c$ by Theorem 2.1(a) and hence

$$|G| \geq |G/N| \geq c.$$

(b) Let H be a dense, pseudocompact subgroup of G (that is, a dense subgroup of PG) such that

$$|H| = d(PG) = \gamma,$$

suppose that $\mathrm{cf}(\gamma) = \omega$ (so that $\gamma > c$ by part (a)) and let $\{H_n : n < \omega\}$ be a sequence of subgroups of H such that

$$|H_n| < \gamma \quad \text{and} \quad \bigcup_{n<\omega} H_n = H.$$

Let K_n denote the PH-closure of H_n. Since $H = \bigcup_{n<\omega} K_n$ and PH is a Baire space (Theorem 2.11) there is $n < \omega$ such that K_n has nonempty interior in PH. There is $N \in \Lambda$ such that $N \subseteq K_n$ and from Theorem 2.7(a) we have

$$|H/K_n| \leq |H/N| \leq c < \gamma.$$

It then follows that

$$d(PG) \leq d(PH) \leq dK_n |H/K_n| \leq |H_n| c < \gamma,$$

a contradiction.

(c) Clearly a group G as in (c) satisfies $|G| = d(PG)$, so (c) follows from (b). □

2.13. Lemma. *Let G be a pseudocompact group and let $G = \bigcup_{n<\omega} A_n$.*
(a) *There is $n < \omega$ such that PG-int$(PG$-cl $A_n) \neq \emptyset$;*
(b) *there are $n < \omega$, $p \in G$ and $N \in \Lambda(G)$ such that $A_n \cap (p + N)$ is G_δ-dense in $p + N$; and*
(c) *if the sets A_n are subgroups of G one may take $p = 0$ in* (b).

Proof. (a) Simply restates a portion of Theorem 2.11.

(b) Choosing n as in (a), and choosing $p \in G$ and $N \in \Lambda(G)$ such that

$$p + N \subseteq PG\text{-int}(PG\text{-cl}\, A_n),$$

we see that $p + N$ is open-and-closed in PG and hence

$$p + N = PG\text{-cl}(A_n \cap (p + N));$$

this gives (b).

(c) To derive (c) from (b) it is enough to show that if K is any topological group with subgroups A and B such that $A \cap (x + B)$ is dense in $x + B$ for some $x \in K$, then $A \cap B$ is dense in B (we here take $K = PG$, $A = A_n$, $B = N$ and $x = p$). Indeed if $q \in A \cap (x + B)$, then $q + B = x + B$ so one may choose $x = q \in A$. Clearly $(A - x) \cap B$ is dense in B, i.e., $A \cap B$ is dense in B. $\square$

2.14. Definition. Let G be an Abelian group.

(a) An indexed set $\{H_\eta : \eta < \gamma\}$ of subgroups of G is *independent* if

$$H_\eta \cap \langle \bigcup \{H_\xi : \xi \neq \eta\} \rangle = \{0\}$$

for each $\eta < \gamma$, and

(b) a subset X of G is *independent* if $\{\langle x \rangle : x \in X\}$ is an independent set of subgroups of G.

Of course, a set $\{H_\eta : \eta < \gamma\}$ of subgroups of G is independent if and only if the relation

$$\sum \{k_\eta x_\eta : \eta \in F\} = 0$$

(with finite $F \subseteq \gamma$, $x_\eta \in H_\eta$, $k_\eta \in \mathbb{Z}$) implies $k_\eta x_\eta = 0$ for each $\eta \in F$.

We continue this introductory section with an elementary lemma which will allow us to find large independent subsets of certain Abelian groups.

Lemma 2.15. *Let G be an Abelian group such that $G \neq tG$ and let κ be an uncountable cardinal number. Then the following statements are equivalent:*

(a) *There is an independent $X \subseteq G$ such that $|X| = \kappa$ and $X \cap tG = \emptyset$;*

(b) $|G/tG| \geq \kappa$.

Proof. (a)$\Rightarrow$(b). Clearly the function from X into G/tG given by

$$x \to x + tG$$

is one-to-one.

(b)$\Rightarrow$(a). Let X be a maximal independent subset of $G \backslash tG$ and suppose that $|X| < \kappa$; from $\kappa > \omega$ follows $|\langle X \rangle| < \kappa$ so there is

$$Y = \{p_\xi : \xi < \kappa\} \subseteq G \backslash \langle X \rangle$$

such that $\xi < \eta < \kappa$ implies $p_\xi - p_\eta \notin tG$.

We claim there is $\xi<\kappa$ such that $n\in\mathbb{Z}\backslash\{0\}$ implies $np_\xi\notin\langle X\rangle$. If for every $\xi<\kappa$ there is $n_\xi\in\mathbb{Z}\backslash\{0\}$ such that $n_\xi p_\xi\in\langle X\rangle$, then since the function $\xi\to\langle n_\xi, n_\xi p_\xi\rangle$ from κ into $(\mathbb{Z}\backslash\{0\})\times\langle X\rangle$ is not one-to-one there are distinct ξ, $\eta<\kappa$ and $n\in\mathbb{Z}\backslash\{0\}$ such that $np_\xi=np_\eta$. This contradicts the independence of X and establishes the claim. The set $X\cup\{p_\xi\}$ is then independent, contrary to the maximality of X. $\square$

The referee has observed that Lemma 2.15 is essentially the familiar statement that G and G/tG have equal torsion-free ranks; see in this connection [17, pp. 439-444].

The hypothesis $G\neq tG$ of Lemma 2.15 is satisfied by every nondegenerate Abelian group with a connected pseudocompact group topology. Indeed we have this simple result.

2.16. Theorem. *Let G be a connected, totally bounded Abelian group such that $|G|>1$. Then $|G/tG|\geq c$.*

Proof. Let $0\neq x\in G$ and let h be a continuous homomorphism from G to T such that $h(x)\neq 1$. From $h[G]=T$ and $h[tG]\subseteq tT$ follows

$$|G/tG|\geq|T/tT|=c. \qquad \square$$

2.17. Remark. It is worthwhile to record an elementary argument showing that if G is a pseudocompact Abelian group such that $G\neq tG$, then the coset space G/tG is uncountable. Let

$$G_n=\bigcup_{0<k\leq n}\{x\in G\colon kx=0\}$$

for $n<\omega$, note that G_n is a closed subgroup of G, and note also that G_n is nowhere dense in G. (Indeed, suppose that some G_n satisfies int $G_n\neq\emptyset$, so that G_n is open in G. Since G is totally bounded the coset space G/G_n is finite—say $|G/G_n|=m<\omega$—and then $mnx=0$ for all $x\in G$, a contradiction.) From $tG=\bigcup_{n<\omega}G_n$ and the fact that G is of second category (Theorem 2.11) it follows that $|G/tG|\geq\omega^+$, as asserted.

In fact, more can be said (in ZFC) about such G: the inequality $|G/tG|\geq c$ holds. When G has a nontrivial connected subspace, this follows easily from Theorem 2.16. When G is totally disconnected, a proof can be constructed either by appealing appropriately to the structure theory for compact Abelian groups [17] or by arguing as in [25]; the latter approach shows that in fact a suitable quotient of G contains an independent Cantor set (that is, a homeomorph of $\{0,1\}^\omega$). Since we do not need the relation $|G/tG|\geq c$ explicitly here, we omit the proof. We hope to return to this inequality and some of its consequences in a later paper.

3. The set $\Lambda(G)$ when G is connected

We have seen (Theorem 2.7(d)) that a closed, normal G_δ-subgroup N of a pseudocompact group G is itself pseudocompact. When in addition G is connected, however, the subgroup N need not be connected. For an example of this phenomenon with $wG = \alpha > \omega$ take

$$G = \{x \in T^\alpha : |\{\xi < \alpha : x_\xi \notin tT\}| \leq \omega\},$$

fix $\eta < \alpha$, and define

$$N = \{x \in G : x_\eta = \pm 1\}.$$

Since G contains densely the Σ-product defined in Section 1.1, G itself is pseudocompact and hence $\beta G = T^\alpha$ (Theorem 2.2); thus G is connected. (It is easy to see, alternatively, that each element of G is connected by an arc to the identity of G.) Here $N \in \Lambda(G)$, and N is not connected. It is clear, however, that there exists a connected $M \subseteq N$ such that $M \in \Lambda(G)$: one may choose

$$M = \{x \in N : x_\eta = 1\}.$$

We will see in Theorem 3.4 that this situation is typical: when $G = \langle G, \mathcal{T}\rangle$ is itself connected (and pseudocompact), for every $N \in \Lambda(G)$ there is a connected $M \in \Lambda(G)$ such that $M \subseteq N$. Alternatively expressed: the $\mathcal{T}$-connected, $P\mathcal{T}$-open sets are basic for PG.

3.1. Lemma. *Let G be a pseudocompact group, and let $H \in \Lambda(\bar{G})$ and $M = H \cap G$. Then*

(a) $M \in \Lambda(G)$;
(b) *M is G_δ-dense in H*;
(c) $\bar{M} = H$;
(d) *M is a pseudocompact group*; *and*
(e) *if H is connected, then M is connected.*

Proof. (a) is obvious. For (b), note simply that every nonempty G_δ-subset U of H, since it is a nonempty G_δ-subset of G, satisfies

$$U \cap M = U \cap G \neq \emptyset.$$

Now (c) is immediate, (d) follows from (b), (c) and Corollary 2.3, and (e) follows from (c), (d) and Theorem 2.2. □

We prove a lemma to be used in Lemma 3.3.

3.2. Lemma. *Let X and Y be compact spaces, let A be a compact G_δ-set in X, and let h be a continuous, open function from X onto Y. Then $h[A]$ is a compact G_δ-set in Y.*

Proof. Write

$$A = \bigcap_{n<\omega} U_n = \bigcap_{n<\omega} \mathrm{cl}_X\, U_n$$

with U_n open in X and with $\mathrm{cl}_X\, U_{n+1} \subseteq U_n$ for $n < \omega$; clearly

$$h[A] \subseteq \bigcap_{n<\omega} h[U_n] = \bigcap_{n<\omega} h[\mathrm{cl}_X\, U_n].$$

If $q \in Y$ satisfies $q \in h[\mathrm{cl}_X\, U_n]$ for each n, then from the nesting of the sets $\mathrm{cl}_X\, U_n$ follows $h^{-1}(\{q\}) \cap A \neq \emptyset$ and hence $q \in h[A]$. Thus $h[A] = \bigcap_{n<\omega} h[U_n]$, as required. □

3.3 Lemma. *Let G be a compact, connected Abelian group and $N \in \Lambda(G)$, and let H be the component of 0 in N. Then $H \in \Lambda(G)$.*

Proof. If H is not a G_δ-set of G, then G/H is not metrizable and hence $w(G/H) > \omega$. It follows from Theorem 2.4 that there is a continuous homomorphism h from G/H onto $T^{(\omega^+)}$. Since the continuous surjections $h: G/H \to T^{(\omega^+)}$ and $\phi: G \to G/H$ are both open functions, by Lemma 3.2 the set

$$K = (h \circ \phi)[N]$$

is a compact G_δ-set of $T^{(\omega^+)}$; hence there is countable $S \subseteq \omega^+$ such that

$$K = \pi_S^{-1}(\pi_S[K]).$$

Denoting by 1_S the identity of the group T^S and writing $A = \{1_S\} \times T^{(\omega^+ \setminus S)}$, we have $A \subseteq K$, A is connected, and $|A| = 2^{(\omega^+)}$.

The set $N/H = \phi[N]$ is a compact, zero-dimensional space, and since

$$h \mid N/H : N/H \to K$$

is an open function the space K is also zero-dimensional. This contradiction completes the proof. □

We extend the result just proved to pseudocompact groups.

3.4. Theorem. *Let G be a connected, pseudocompact Abelian group, let $N \in \Lambda(G)$ and let C be the component of 0 in N. Then $C \in \Lambda(G)$.*

Proof. Let H be the component of 0 in $\bar{N}$ and set $M = H \cap G$. Since $\bar{N} \in \Lambda(\bar{G})$ by Theorem 2.7(c) we have $H \in \Lambda(\bar{G})$ by Lemma 3.3 and hence $M \in \Lambda(G)$; thus it is enough to show $M = C$.

Since $H \subseteq \bar{N}$ and N is closed in G we have

$$M = H \cap G \subseteq N$$

and hence $M \subseteq C$ by Lemma 3.1(e); and from

$$C \subseteq N \subseteq \bar{N}$$

follows $C \subseteq H$ and hence $C \subseteq H \cap G = M$, as required. □

4. Connected groups G such that $wG \leqslant c$

We say as in [4, 5] that a pseudocompact group *fragments* into β-many dense, pseudocompact subgroups if there is an independent family $\{H_\eta: \eta < \beta\}$ of dense, pseudocompact subgroups of G. The existence of such a family (with $\beta = c$) has been shown in special cases—for G dense in T^c in [4], for example, and for $|tG| < |G| = \beta = (wG)^\omega$ in [5]. In this section we improve the results of [4, 5] and we show that every connected, pseudocompact Abelian group G such that $\omega^+ \leqslant wG \leqslant c$ has a proper, dense, pseudocompact subgroup.

4.1. Lemma. *Let G be an Abelian group and let $\{H_\eta: \eta < \gamma\}$ be an independent set of subgroups of G with each $|H_\eta| \leqslant \gamma$. Let N be a subgroup of G containing an independent set X such that*

$$|X| = \beta > \gamma, \qquad \beta > \omega \quad \text{and} \quad X \cap tG = \emptyset.$$

Then for each $p \in G$ there is $\{x_\eta: \eta < \gamma\} \subseteq p + N$ such that $\{\langle H_\eta \cup \{x_\eta\}\rangle: \eta < \gamma\}$ is an independent set of subgroups of G.

Proof. (In the terminology of [17, Appendix A], the conditions on N reduce simply to the statement that the torsion-free rank of N, denoted $r_0(N)$, satisfies $r_0(N) > \max\{\gamma, \omega\}$.)

We claim first, writing

$$J_0 = H_0 \quad \text{and} \quad K_0 = \langle \bigcup \{H_\eta: 0 < \eta < \gamma\}\rangle,$$

that there is

$$x = x_0 \in p + X \subseteq p + N,$$

such that $\langle J_0 \cup \{x\}\rangle$ and K_0 are independent. If not, then for every $x \in p + X$ there are $m \in \mathbb{Z}$, $r \in J_0$ and $s \in K_0$ such that

$$r + mx = s \neq 0;$$

since J_0 and K_0 are independent we have $m \neq 0$. The map $x \to \langle m, r, s\rangle$ from $p + X$ into

$$(\mathbb{Z}\backslash\{0\}) \times J_0 \times K_0$$

is not one-to-one so there are distinct $x, x' \in p + X$ and m, r, s such that

$$r + mx = r + mx' = s$$

and hence $m(x - x') = 0$. From

$$x - x' \in (p + X) - (p + X) \subseteq X - X \subseteq N \subseteq G\backslash tG$$

then follows $m = 0$, a contradiction establishing the claim. We define x_0 as furnished by the claim.

Suppose now that $\zeta < \gamma$ and that x_η has been defined for $\eta < \zeta$ so that

$$\{\langle H_\eta \cup \{x_\eta\}\rangle: \eta < \zeta\} \cup \{H_\eta: \zeta \leqslant \eta < \gamma\}$$

is an independent set of groups. We define

$$J_\zeta = H_\zeta \quad \text{and} \quad K_\zeta = \langle \bigcup \{H_\eta \cup \{x_\eta\}: \eta < \zeta\} \cup \bigcup \{H_\eta: \zeta < \eta < \gamma\}\rangle$$

and we note, arguing exactly as in the claim, that there is $x_\zeta \in p + X$ such that $\langle J_\zeta \cup \{x_\zeta\}\rangle$ and H_ζ are independent.

The definition of x_η for $\eta < \gamma$ is complete. It is clear that $\{\langle H_\eta \cup \{x_\eta\}\rangle: \eta < \gamma\}$ is an independent set of groups, as required. □

4.2. Theorem. *Let G be a pseudocompact, Abelian group with $wG = \alpha > \omega$ and $|G| = \beta \geq \alpha^\omega$. Suppose further that each $N \in \Lambda(G)$ contains an independent set X such that*

$$|X| = \beta \quad \textit{and} \quad X \cap tG = \emptyset$$

(that is, each $N \in \Lambda(G)$ satisfies $r_0(N) = |G|$). Then G contains an independent set $\{H_\eta: \eta < \beta\}$ of (proper) dense, pseudocompact subgroups.

Proof. Since $w(PG) \leq \alpha^\omega$ (Theorem (2.10(a)) there is a set $\mathcal{R}$ of nonempty G_δ-subsets of G (each of the form $p + N$ for some $p \in G$, $N \in \Lambda(G)$) such that $|\mathcal{R}| \leq \alpha^\omega$ and such that each nonempty G_δ-subset of G contains an element of $\mathcal{R}$. We write $\mathcal{R} = \{U(\xi): \xi < \beta\}$ with each element of $\mathcal{R}$ listed β-many times.

Now for $0 \leq \eta \leq \zeta < \beta$ we will define $x(\eta, \zeta)$ and $H(\eta, \zeta)$ so that

(i) $x(\eta, \zeta) \in U(\zeta)$,

(ii) $H(\eta, \zeta) = \langle \{x(\eta, \zeta'): \eta \leq \zeta' \leq \zeta\}\rangle$, and

(iii) $\{H(\eta, \zeta): \eta \leq \zeta\}$ is independent, for $\zeta < \beta$.

We proceed by recursion on $\zeta < \beta$.

Choose $x(0,0) \in U(0)$ and set $H(0,0) = \langle\{x(0,0)\}\rangle$; then (i), (ii) and (iii) are satisfied for $\zeta = 0$.

Now let $\xi < \beta$ and suppose that (i), (ii) and (iii) are satisfied for $\zeta < \xi$. We will define $x(\eta, \xi)$ and $H(\eta, \xi)$ for $\eta \leq \xi$. First, define

$$H'(\eta, \xi) = \bigcup \{H(\eta, \zeta): \eta \leq \zeta < \xi\} \quad \text{for } \eta < \xi, \quad \text{and} \quad H'(\xi, \xi) = \{0\}.$$

Then $\{H'(\eta, \xi): \eta \leq \xi\}$ is an independent set of $|\xi + 1|$-many subgroups of G, each of cardinality not exceeding $\omega \cdot |\xi|$. It then follows from Lemma 4.1, with $\{H'(\eta, \xi): \eta \leq \xi\}$, $\max\{|\xi + 1|, \omega \cdot |\xi|\}$ and $U(\xi)$ replacing $\{H_\eta: \eta < \gamma\}$, γ and $p + N$, respectively, that there is

$$\{x(\eta, \xi): \eta \leq \xi\} \subseteq U(\xi)$$

such that, defining

$$H(\eta, \xi) = \langle H'(\eta, \xi) \cup \{x(\eta, \xi)\}\rangle,$$

the set $\{H(\eta, \xi): \eta \leq \xi\}$ is an independent set of subgroups of G.

The definition of $x(\eta, \zeta)$ and $H(\eta, \zeta)$ for $\eta \leq \zeta < \beta$ is complete. We define

$$H_\eta = \bigcup \{H(\eta, \zeta): \eta \leq \zeta < \beta\} \quad \text{for } \eta < \beta.$$

It is clear that $\{H_\eta: \eta < \beta\}$ is an independent set of subgroups of G. For each $\eta < \beta$ and $U \in \mathscr{R}$ there is ζ such that $\eta \leq \zeta < \beta$ and $U = U(\zeta)$; we have

$$x(\eta, \zeta) \in H_\eta \cap U.$$

It follows that H_η is G_δ-dense in G, and hence pseudocompact (Theorem 2.9). □

4.3. Theorem. *Let G be a connected, pseudocompact Abelian group such that $\omega < wG \leq \mathfrak{c}$. Then*

(a) *G has a proper, dense, pseudocompact subgroup*; *and*

(b) *there is an independent set $\{H_\eta: \eta < \mathfrak{c}\}$ of $\mathfrak{c}$-many (proper) dense, pseudocompact subgroups of G.*

Proof. It is enough to prove (b).

Since $(wG)^\omega = \mathfrak{c}$, by Theorem 2.10(b) there is a dense, pseudocompact subgroup G' of G such that $|G'| \leq \mathfrak{c}$. Then $|G'| = \mathfrak{c}$ by Theorem 2.12(a), and G' is connected by Corollary 2.3. Thus to deduce the result from Theorem 4.2 it is enough to assume $G = G'$ (hence, $|G| = \mathfrak{c}$) and to show that every $N \in \Lambda(G)$ contains an independent set X such that $|X| = \mathfrak{c}$ and $X \cap tG = \emptyset$.

Given such N, let a connected $M \in \Lambda(G)$ satisfy $M \subseteq N$ (Theorem 3.4), note that since $wM > \omega$ by Theorem 2.7(e) there is $x \in M$ such that $x \neq 0$, and let $h: G \to T$ be a continuous homomorphism such that $h(x) \neq 1$. As in Theorem 2.16 we have $h[M] = T$, and to complete the proof it is enough to choose an independent subset Y of $T \backslash tT$ such that $|Y| = \mathfrak{c}$ and then a subset X of M such that $h|X$ is one-to-one from X onto Y. □

5. Connected groups G such that $|G| \geq (wG)^\omega$

We have observed in Theorem 2.10 that every pseudocompact group G such that $wG = \alpha \geq \omega$ and $|G| > \alpha^\omega$ has a proper, dense, pseudocompact subgroup (of cardinality not exceeding α^ω). In this section, taking G connected and $wG > \omega$, we achieve the same conclusion for $|G| = \alpha^\omega$.

For $n < \omega$ we define $\phi_n: G \to G$ by

$$\phi_n(x) = nx,$$

and we write

$$t_n G = \ker \phi_n = \{x \in G: nx = 0\}.$$

5.1. Theorem. *Let G be a connected, pseudocompact Abelian group and A a subgroup of G such that $A + tG$ is G_δ-dense in G. Then there is a connected $C \in \Lambda(G)$ such that $A \cap C$ is G_δ-dense in C.*

Proof. We assume without loss of generality, the statement being obvious otherwise, that $wG > \omega$. We write

$$H = A + tG = \bigcup_{n<\omega} A_n$$

with $A_n = A + t_nG$, we note from Corollary 2.3 that H is a connected, pseudocompact group, and we use Lemma 2.13 to find $n < \omega$ and $N \in \Lambda(H)$ such that $A_n \cap N$ is G_δ-dense in N. Let C be the component of 0 in N. Then $C \in \Lambda(G)$ by Theorem 3.4, and C is pseudocompact by Theorem 2.7(d). From $C \in \Lambda(N)$ it follows that $A_n \cap C$ is G_δ-dense in C and hence in $\bar{C}$; hence

$$\phi_n[A_n \cap C] \text{ is } G_\delta\text{-dense in } \phi_n[\bar{C}]. \tag{$*$}$$

Now $\bar{C}$ is compact and connected, hence divisible [17, 24.25]; thus $\phi_n[\bar{C}] = \bar{C}$. And $\phi_n[t_nG] = \{0\}$, so

$$\begin{aligned}\phi_n[A_n \cap C] &\subseteq \phi_n[A + t_nG] \cap \phi_n[C] \subseteq \langle \phi_n[A] \cup \{0\}\rangle \cap C \\ &\subseteq A \cap C \subseteq C \subseteq \bar{C} = \phi_n[\bar{C}].\end{aligned}$$

It is then immediate from $(*)$ that $A \cap C$ is G_δ-dense in $\bar{C}$, as required. □

5.2. Corollary. *Let G be a connected, pseudocompact Abelian group and A a subgroup of G such that $A + tG$ is G_δ-dense in G. Then there is $E \subseteq G$ with $|E| \leqslant \mathfrak{c}$ such that $\langle A + E\rangle$ is G_δ-dense in G.*

Proof. By Theorem 5.1 there is (connected) $N \in \Lambda(G)$ such that $A \cap N$ is G_δ-dense in N. Since G/N is a compact metric space (Theorem 2.7(a)) there is $E \subseteq G$ with $|E| \leqslant \mathfrak{c}$ such that every $p \in G$ satisfies $E \cap (p + N) \neq \emptyset$. It is clear that E is as required. □

We indicated in Theorem 2.16 that every nondegenerate, pseudocompact, connected Abelian group G satisfies $|G/tG| \geqslant \mathfrak{c}$. Here we give a stronger conclusion.

5.3. Definition. Let G be a pseudocompact Abelian group. Then

$$\# G = \min\{|A| : \langle A \cup tG\rangle \text{ is } G_\delta\text{-dense in } G\}.$$

5.4. Lemma. *Let G be a connected, pseudocompact Abelian group such that $|G| > 1$, and let $N \in \Lambda(G)$. Then $\# N \geqslant \mathfrak{c}$.*

Proof. Since $wG < \omega$ is impossible, and the case $wG = \omega$ is handled by Theorem 2.1(a), we assume $wG > \omega$. Let C be the component of 0 in N. Then $C \in \Lambda(G)$ (Theorem 3.4), so $wC > \omega$ (Theorem 2.7(e)) and hence $|C| > 1$. Let $0 \neq x \in C$ and let h be a continuous homomorphism from N into $\boldsymbol{T}$ such that $h(x) \neq 1$; clearly

$$\boldsymbol{T} \subseteq h[C] \subseteq h[N] \subseteq \boldsymbol{T}.$$

Now let $\langle A \cup tN\rangle$ be G_δ-dense in N. Then $h[\langle A \cup tN\rangle] = \boldsymbol{T}$ and from $h[tN] \subseteq t\boldsymbol{T}$ and $|t\boldsymbol{T}| = \omega$ it follows that $|h[\langle A\rangle]| = \mathfrak{c}$ and hence $|A| \geqslant \mathfrak{c}$, as required. □

The following theorem is our principal result.

5.5. Theorem. *Let G be a connected, pseudocompact Abelian group with $wG = \alpha > \omega$ and $|G| = \beta \geq \alpha^{\omega}$. Then either*

(i) *G contains a (proper) dense, pseudocompact subgroup H such that $|H| < \beta$, or*

(ii) *G contains an independent set $\{H_\eta : \eta < \beta\}$ of (proper) dense, pseudocompact subgroups.*

Proof. We consider two cases.

Case 1. There is a connected $N \in \Lambda(G)$ such that $\# N < \beta$. Then $\beta > c$ by Lemma 5.4. From the definition of $\# N$ and Corollary 5.2 (applied to N in place of G) there are $A \subseteq N$ with $|A| = \# N$ and $E \subseteq N$ with $|E| \leq c$ such that $\langle A \cup E \rangle$ is G_δ-dense in N. From Theorem 2.7(a), there is $F \subseteq G$ with $|F| \leq c$ such that

$$G/N = \{x + N : x \in F\}.$$

It is then clear, writing $H = \langle A \cup E \cup F \rangle$, that H is G_δ-dense in G and that

$$|H| \leq \# N + c + c < \beta;$$

thus (i) holds.

Case 2. Every connected $N \in \Lambda(G)$ satisfies $\# N \geq \beta$. Then every such N satisfies $|N/tN| \geq \beta$, so according to Theorem 3.4 every (not necessarily connected) $N \in \Lambda(G)$ satisfies $|N/tN| \geq \beta$. It follows from Lemma 2.15 that every $N \in \Lambda(G)$ contains an independent set X such that $|X| = \beta$ and $X \cap tG = \emptyset$, and (ii) is immediate from Theorem 4.2. □

Theorem 5.5 yields another route to Theorem 4.3. Here are the details.

5.6. Corollary. *Let G be a connected, pseudocompact Abelian group such that $\omega < wG \leq c$. Then there is an independent set $\{H_\eta : \eta < c\}$ of c-many (proper) dense, pseudocompact subgroups of G.*

Proof. As in the proof of Theorem 4.3, we may assume $|G| = c$. According to Theorem 2.12(a) no dense, pseudocompact subgroup H of G can satisfy (i) of Theorem 5.5, so (ii) holds with $\beta = c$. □

The singular cardinals hypothesis (SCH) is the statement that $\kappa^\lambda \leq \kappa^+ 2^\lambda$ for all infinite cardinals κ and λ. That it can be useful in connection with the study of dense subspaces has been demonstrated by Cater, Erdős and Galvin [1]. Concerning the strength of this axiom, they write in part (here we paraphrase slightly): "Clearly, SCH follows from the generalized continuum hypothesis, but is much weaker. In fact, models of set theory violating SCH are not easy to come by; Prikry and Silver (see Jech [19, Section 37] and Magidor [22, 23]) have constructed such models assuming the existence of very large (e.g., supercompact) cardinals, and Devlin and Jensen [11] have shown that some large cardinal assumption is necessary."

In the following theorem we use SCH to replace the hypothesis $\beta \geqslant \alpha^{\omega}$ of Theorem 5.5 with the more natural assumption $\beta \geqslant \alpha$.

5.7. Theorem. *Assume SCH. Let G be a connected, pseudocompact Abelian group such that $|G| \geqslant wG > \omega$. Then G has a proper, dense, pseudocompact subgroup.*

Proof. We set $wG = \alpha$ and $|G| = \beta$. According to Theorems 4.3 and 2.12(c) the conclusion holds in case $\alpha \leqslant \mathfrak{c}$ or $\mathrm{cf}(\beta) = \omega$, so we assume $\alpha > \mathfrak{c}$ and $\mathrm{cf}(\beta) > \omega$; according to Corollary 5.6 it is enough to show $\beta \geqslant \alpha^{\omega}$. If $\beta < \alpha^{\omega}$, then from SCH follows

$$\alpha \leqslant \beta < \alpha^{\omega} \leqslant \alpha^{+} 2^{\omega} = \alpha^{+}$$

and hence $\alpha = \beta$. From $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) > \omega$ it follows that for every countable subset A of α there is $\xi < \alpha$ such that $A \subseteq \xi$, so we have

$$\alpha^{\omega} \leqslant \sum_{\xi < \alpha} |\xi|^{\omega} \leqslant \sum_{\xi < \alpha} |\xi|^{+} 2^{\omega} \leqslant \alpha \cdot \mathfrak{c} = \alpha = \beta,$$

as required. $\square$

A cardinal number α is said to be a *strong limit cardinal* if $2^{\gamma} < \alpha$ whenever $\gamma < \alpha$. We consider next some consequences of Theorem 5.7 for groups G whose weight is a strong limit cardinal.

5.8. Corollary. *Let G be a connected, pseudocompact Abelian group such that $wG = \alpha > \omega$ and α is a strong limit cardinal. If any one of the following conditions holds, then G has a proper, dense, pseudocompact subgroup:*

(i) *SCH*;
(ii) $\mathrm{cf}(\alpha) > \omega$;
(iii) $\alpha^{\omega} \leqslant \alpha^{+}$;
(iv) $\alpha^{+} = 2^{\alpha}$.

Proof. Set $\beta = |G|$. Since α is a strong limit cardinal and G is dense in $\bar{G}$ and $|\bar{G}| = 2^{\alpha}$ (Theorem 2.1(a)), we have $\beta \geqslant \alpha$.

If (i) holds, use Theorem 5.7.

If (ii) holds, then as in the proof of Theorem 5.7 we have

$$\alpha^{\omega} \leqslant \sum_{\xi < \alpha} |\xi|^{\omega} \leqslant \sum_{\xi < \alpha} 2^{|\xi|} \leqslant \alpha \cdot \alpha = \alpha$$

and hence $\beta \geqslant \alpha^{\omega}$; thus Theorem 5.5 applies.

If (iii) holds but $\beta < \alpha^{\omega}$, then from

$$\alpha \leqslant \beta < \alpha^{\omega} \leqslant \alpha^{+}$$

we have $\beta = \alpha$. The desired conclusion now follows from Theorem 2.12(c) in case $\mathrm{cf}(\beta) = \omega$, and from (ii) in case $\mathrm{cf}(\beta) > \omega$.

If (iv) holds, then (iii) holds.

The proof is complete. $\square$

6. Groups G with $|tG| > c$

Our success heretofore has been based in part on the fact that the groups G we have been considering have a torsion subgroup tG which in an appropriate sense is small—so small, for example, that $G/tG \geq c$ in Theorem 4.3, or $|G/tG| \geq |G|$ in Theorem 5.5(ii). In this section, in contrast, we achieve the same conclusion (that is, the existence of a proper, dense, pseudocompact subgroup) when tG is suitably large. The theorem we prove here appears to stand in isolation; in particular, its proof does not depend on the constructions introduced in Section 5.

6.1. Theorem. *Let G be a connected, pseudocompact Abelian group such that $|tG| > c$. Then G has a proper, dense, pseudocompact subgroup.*

Proof. Let A be a subgroup of G which is maximal with respect to the property $A \cap tG = \{0\}$. For $0 \neq n \in \mathbb{Z}$ define $\phi_n : \bar{G} \to \bar{G}$ as before:

$$\phi_n(p) = np.$$

We note that since $\bar{G}$ is connected and hence divisible [17, 24.25], the function ϕ_n is a surjection onto $\bar{G}$. We claim that for every $x \in G$ there is $n \in \mathbb{Z}\backslash\{0\}$ such that $\phi_n(x) \in A$. For $x \in A$ this is clear. If $x \in G\backslash A$ there are $a \in A$, $k \in \mathbb{Z}$, and $p \in tG$ such that

$$a + kx = p \neq 0;$$

from $a \neq p$ follows $k \neq 0$ and if $m \in \mathbb{Z}\backslash\{0\}$ satisfies $mp = 0$, then with $n = km$ we have $n \neq 0$ and

$$\phi_n(x) = nx = mp - ma = -ma \in A,$$

as required. The claim is proved.

Now define

$$A_n = \{x \in G: \phi_n(x) \in A\}.$$

From Lemma 2.13(c) there are $n < \omega$ and $N \in \Lambda(G)$ such that $A_n \cap N$ is G_δ-dense in N, so from Theorem 2.7(a) there is $E \subseteq G$ such that $|E| \leq c$ and $\langle A_n \cup E\rangle$ is G_δ-dense in G. Since $\langle A_n \cup E\rangle$ is G_δ-dense in $\bar{G}$, the group $\phi_n[\langle A_n \cup E\rangle]$ is then G_δ-dense in $\phi_n[\bar{G}] = \bar{G}$ and hence in the intermediate space G. It follows, writing $F = \phi_n[E]$ and noting

$$\phi_n[\langle A_n \cup E\rangle] \subseteq \langle \phi_n[A_n] + \phi_n[E]\rangle \subseteq A + \langle F\rangle,$$

that $|\langle F\rangle| \leq c$ and $A + \langle F\rangle$ is a dense, pseudocompact subgroup of G. Since each coset of A contains at most one element of tG the inclusion $tG \subseteq A + \langle F\rangle$ is false and hence the containment $A + \langle F\rangle \subseteq G$ is proper. The proof is complete. □

6.2. Remark. It is tempting to hope or believe that every sufficiently large (connected, pseudocompact, Abelian) group G will satisfy the hypothesis $|tG| > \mathfrak{c}$ of Theorem 6.1, but this is naive. Indeed, it is clear from [17, 25.8] that for every cardinal number $\alpha \geqslant \omega$ there is a connected, compact Abelian group $G(\alpha)$ such that $wG(\alpha) = \alpha$ and $t(G(\alpha)) = \{0\}$: the Abelian "$\boldsymbol{a}$-adic solenoid" $\Sigma_{\boldsymbol{a}}$ of [17, 10.12] is connected, compact and torsion-free, so $G(\alpha) = (\Sigma_{\boldsymbol{a}})^{\alpha}$ is such a group.

7. Miscellaneous theorems and examples

The results we give here serve to illuminate the foregoing theorems and to place them in perspective. Perhaps the principal result of this section is Corollary 7.7, which shows that there exist totally disconnected pseudocompact groups of positive dimension. This (somewhat unexpected) phenomenon explains in part our difficulty in piecing together a solution to the general "proper dense subgroup" problem for pseudocompact groups from the restricted results available for the connected case and the zero-dimensional case.

Let us first establish Result (e) of the Abstract.

7.1. Theorem. *Let G be a pseudocompact, connected Abelian group. If G is not divisible, then G has a proper, dense, pseudocompact subgroup.*

Proof. Since $\bar{G}$ is connected and hence divisible, the continuous homomorphism

$$\phi_n : \bar{G} \to \bar{G}$$

given by $\phi_n(x) = nx$ is surjective; further, $\phi_n[G]$ is G_δ-dense in $\phi_n[\bar{G}] = \bar{G}$ and hence in the intermediate group G. It follows that if G is not divisible, then for some $n < \omega$ the group $\phi_n[G]$ is a proper, dense, pseudocompact subgroup of G. $\square$

7.2. Remark. Although divisibility and connectedness are equivalent for compact, Abelian groups [17, 24.25], it is easy to find connected, pseudocompact Abelian groups G which are not divisible. For an example of weight $\alpha \geqslant \omega$, choose a nondivisible subgroup D of $\boldsymbol{T}$ and set

$$G = \{x \in \boldsymbol{T}^{\alpha} : |\{\xi < \alpha : x_\xi \notin D\}| \leqslant \omega\}.$$

(That G is pseudocompact and connected follows just as in the case of the group considered in the first paragraph of Section 3.)

The relation between divisibility and connectivity for pseudocompact groups is easily clarified (Theorem 7.4).

7.3. Lemma. *Let G be a divisible, totally bounded Abelian group. Then $\bar{G}$ is divisible.*

Proof. Given $p \in \bar{G}$ and $0 \neq n \in \mathbb{Z}$, let x_λ be a net in G such that $x_\lambda \to p$ and find $y_\lambda \in G$ such that $ny_\lambda = x_\lambda$. Because $\bar{G}$ is compact the net y_λ has a subnet converging to some point $q \in \bar{G}$. It is then clear that $nq = p$, as required. $\square$

7.4. Theorem. *Let G be a divisible, pseudocompact Abelian group. Then G is connected.*

Proof. $\bar{G}$ is divisible (Lemma 7.3) and hence connected [17, 24.25], so G is connected (Corollary 2.3). □

7.5. Remark. In Section 1.2(i) above we claimed the existence of a 4-tuple $\langle G, K, h, H\rangle$ with G and K pseudocompact groups, h a continuous, open, surjective homomorphism from G onto K, and H a dense, pseudocompact subgroup of K such that $h^{-1}(H)$ is not pseudocompact. The interested reader may consult [7, 4.10(b)] for a complete, detailed proof of the validity of the following construction.

For $i = 1, 2$ let K_i be a compact, Abelian group with a proper, dense subgroup H_i such that H_1 is pseudocompact, K_2 is metrizable, and the groups K_i/H_i are algebraically isomorphic. Let

$$q_i : K_i \to K_i/H_i = E$$

be the natural homomorphisms, define

$$G = \{\langle k_1, k_2\rangle : k_i \in K_i,\ q_1(k_1) = q_2(k_2) \in E\},$$

and define $h : G \to K$ by $h(k_1, k_2) = k_1$. The 4-tuple

$$\langle G, K_1, h, H_1\rangle = \langle G, K, h, H\rangle$$

is as required.

We show next, as promised in Section 1.2(ii), that many nonpseudocompact groups arise as the identity component of a pseudocompact group.

7.6. Theorem. *Let K be a compact, connected, torsion-free Abelian group and let H be a connected subgroup of K. Then there is a pseudocompact group G such that H and the identity component of G are topologically isomorphic.*

Proof. Let $\alpha = |K|^{\omega}$, to avoid the trivial case assume $\alpha > 1$, let M be a compact, torsion-free, zero-dimensional metrizable group [17, 25.8], and set $J = K \times M^{\alpha}$. The relation $wJ = \alpha$ has two consequences: first, each $N \in \Lambda(J)$ satisfies $|N| = 2^{\alpha}$ (Theorems 2.1(a) and 2.7(e)); and second, there is, just as in Theorem 4.2, a set $\{U(\xi) : \xi < \alpha\}$ of nonempty G_δ-subsets of J such that each nonempty G_δ-subset of J contains one of the sets $U(\xi)$. We assume without loss of generality that $0 \in U(0)$.

For $\xi < \alpha$ we will define x_ξ and A_ξ so that

(i) $x_\xi \in U(\xi)$,

(ii) $A_\xi = \langle\{x_\eta : \eta \leq \xi\}\rangle$, and

(iii) $A_\xi \cap (K \times \{0\}) = \{0\}$.

We proceed by recursion. We set $x_0 = 0$ and $A_0 = \{0\}$, and we note that (i), (ii) and (iii) are satisfied for $\xi = 0$. Now let $\zeta < \alpha$ and suppose that (i), (ii), (iii) are satisfied for $\xi < \zeta$. We then claim, writing

$$A'_\zeta = \langle\{x_\xi : \xi < \zeta\}\rangle$$

and noting

$$A'_\zeta \cap (K\times\{0\})=\{0\}, \tag{**}$$

that there is $x\in U(\zeta)$ such that

$$\langle A'_\zeta\cup\{x\}\rangle\cap(K\times\{0\})=\{0\}.$$

If the claim fails, then for every $x\in U(\zeta)$ there are $a\in A'_\zeta$, $n\in\mathbb{Z}$ and $k\in K$ such that

$$a+nx=\langle k,0\rangle\neq 0;$$

from (**) we have $n\neq 0$. Since $|U(\zeta)|=2^\alpha$ the map $x\to\langle a,n,k\rangle$ from $U(\zeta)$ into

$$A'_\zeta\times(\mathbb{Z}\backslash\{0\})\times K$$

is not one-to-one so there are distinct x, $x'\in U(\zeta)$ and a, n, k such that

$$a+nx=a+nx'=\langle k,0\rangle\neq 0.$$

It follows that ($n\neq 0$ and) $n(x-x')=0$, contrary to the hypothesis that J is torsion-free. The claim is established. We define x_ξ as furnished by the claim and we note that, with A_ζ defined by (ii) for $\zeta=\xi$, conditions (i), (ii) and (iii) are satisfied for $\zeta=\xi$.

Now define

$$A=\bigcup_{\xi<\alpha}A_\xi=\langle\{x_\xi:\xi<\alpha\}\rangle\quad\text{and}\quad G=\langle A\cup(H\times\{0\})\rangle,$$

and let C denote the identity component of G. Since A (and hence G) is G_δ-dense in the compact group J, the group G is pseudocompact (Corollary 2.3(b)). Since M^α is zero-dimensional we have

$$H\times\{0\}\subseteq C\subseteq K\times\{0\},$$

and from $A\cap(K\times\{0\})=\{0\}$ follows

$$C=H\times\{0\},$$

as required. □

By the *dimension* of a space X we mean dim X, the Čech–Lebesgue or covering dimension of X. The relation $\dim X=\dim\beta X$ holds for every (Tychonoff) space X [12, 7.2.17], and the spaces X for which $\dim X=0$ are exactly those spaces X for which βX is zero-dimensional (in the sense that βX has a basis of open-and-closed sets). It is clear that if X is a (Tychonoff) space and a point $x\in X$ has an open-and-closed local basis in X, then x has an open-and-closed local basis in βX. It is then immediate from Theorem 2.2 above that a pseudocompact group G is zero-dimensional if and only if $\dim G=0$.

A space X is *totally disconnected* if every connected subset Y of X satisfies $|Y|\leq 1$. As is well known (see [17, 3.5 or 14, 16.17] for a proof), a compact (Hausdorff) space X satisfies $\dim X=0$ if and only if X is totally disconnected.

The fact that a totally disconnected group need not be zero-dimensional was first noted by Erdős [13], who showed that the group G of "rational points" in separable Hilbert space l^2 satisfies $\dim G = 1$. (Relevant details of this example are examined in [17, 7.18 and 14, Exercise 16L].) Van Mill [24] shows that for every integer n there is a totally disconnected group G such that $\dim G = n$. As with the group of Erdős [13], van Mill's groups have the strong property that every two points are contained in disjoint open-and-closed sets. In our next result, a consequence of Theorem 7.6, we show that totally disconnected groups of positive dimension may in addition be chosen to be pseudocompact.

7.7. Corollary. *For $n < \omega$ there is a totally disconnected, pseudocompact Abelian group $G(n)$ such that $\dim G(n) = n$. The groups $G(n)$ may be chosen so that every two points are contained in disjoint open-and-closed sets.*

Proof. In the construction of Theorem 7.6 choose K so that $\dim K = n$ and take $H = \{0\}$. Retaining the notation of Theorem 7.6 we have $|C| = 1$, so that G is totally disconnected, while from

$$\beta A = \beta G = J$$

follows

$$\dim G = \dim J = \dim K + \dim M^\alpha = \dim K + 0 = n + 0 = n,$$

as required.

From

$$G \cap (K \times \{0\}) = \{0\}$$

it follows that the projection

$$\pi : K \times M^\alpha \to M^\alpha$$

is one-to-one on G. Since every two points of M^α are contained in disjoint open-and-closed sets, the same is true of G. $\square$

7.8. Remark. The authors did not determine whether every totally bounded, totally disconnected group has the stronger property that every two points are contained in disjoint open-and-closed subsets.

7.9. Remark. It is noted in [7] that for many pseudocompact Abelian groups G there are topological group topologies $\mathcal{T}$ and $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T} \neq \mathcal{T}'$. Indeed, for many pseudocompact groups $G = \langle G, \mathcal{T} \rangle$ the existence of such a topology $\mathcal{T}'$ is equivalent to the condition that G admits a proper, dense, pseudocompact subgroup [7, 5.8]. Here we record a simple criterion sufficient to ensure the availability of a strictly coarser pseudocompact group topology.

7.10. Theorem. *Let $G = \langle G, \mathcal{T}\rangle$ be a pseudocompact group such that $|G| < wG$. Then there is a pseudocompact topological group topology $\mathcal{T}'$ for G such that $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{T}' \neq \mathcal{T}$.*

Proof. For $0 \neq x \in G$ there are a compact metric group K_x and a continuous homomorphism $h_x : G \to K_x$ such that $h_x(x) \neq 0$ in K_x. (This is essentially a statement of the Gel'fand–Raĭkov theorem [17, 22.12]. When G is Abelian one may choose $K_x = \boldsymbol{T}$ for each x.) The evaluation map

$$e : G \to K = \prod_x K_x$$

given by $(ep)_x = h_x(p)$ is an algebraic isomorphism, so the topology $\mathcal{T}'$ induced by K on G is a (Hausdorff) topological group topology. Since $\mathcal{T}' \subseteq \mathcal{T}$, the identity function from $\langle G, \mathcal{T}\rangle$ onto $\langle G, \mathcal{T}'\rangle$ is continuous and hence $\langle G, \mathcal{T}'\rangle$ is pseudocompact. Since $w(G, \mathcal{T}) > |G|$ while

$$w(G, \mathcal{T}') \leq wK \leq \omega \cdot |G| = |G|,$$

we have $\mathcal{T}' \neq \mathcal{T}$, as required. $\square$

8. Questions

For the convenience of the research-oriented reader we collect here, approximately in descending order of generality, those problems related to the present paper which we have been unable to solve. In the interest of simplicity, let us say that a pseudocompact group is *extremal* if it admits no proper, dense, pseudocompact subgroup. As indicated earlier, every pseudocompact group of countable weight is extremal, and we do not know if there are other extremal pseudocompact groups; the principal result of [17] is that no zero-dimensional pseudocompact Abelian group of uncountable weight is extremal.

8.1. Question. Is every pseudocompact group of uncountable weight nonextremal?

8.2. Question. Is every pseudocompact Abelian group of uncountable weight nonextremal?

8.3. Question. Is every pseudocompact Abelian connected group of uncountable weight nonextremal?

8.4. Question. Is every pseudocompact Abelian group of weight $\leq \boldsymbol{c}$ nonextremal?

8.5. Question. Is every connected pseudocompact Abelian group of cardinality $\leq \boldsymbol{c}$ nonextremal?

Since there is no extremal, connected Abelian group whose weight α satisfies $\omega < \alpha \leq \boldsymbol{c}$, and since $(\boldsymbol{c}^+)^\omega = \boldsymbol{c}^+$, the most accessible context not settled by Theorems

4.3 and 5.5 is the case of a connected group G such that $|G| = c$ and $wG = c^+$. Specifically, we ask:

8.6. Question. Is there an extremal, pseudocompact group G such that $|G| = c$ and G is dense in $T^{(c^+)}$?

Acknowledgement

We are pleased to thank the referee for several helpful comments, both mathematical and stylistic.

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