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A THEOREM ON FUNCTION SPACES

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ABSTRACT. Let X and Y be normal and first countable spaces, such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Suppose $X^{(\alpha)}$ is countably compact for some $\alpha < \omega_1$. We prove that if $\alpha = 1$ then $Y^{(\alpha)}$ is also countably compact. The first countability condition in this result is essential. We also present examples that if α is not a prime component, then $Y^{(\alpha)}$ need not to be countably compact.

0. INTRODUCTION

Let X and Y be Tychonov spaces. By $C(X)$ we denote the set of all realvalued continuous functions on X. We endow $C(X)$ with a topological vectorspace-structure by considering it to be a subspace of $\mathbb{R}^X$. With this topology we denote $C(X)$ by $C_p(X)$.

In [1] Arhangelskiĭ proved that if $C_p(X)$ is linearly homeomorphic to $C_p(Y)$, and X is compact, then Y is compact. In addition, if X is pseudocompact then Y is pseudocompact. This means in particular that if X and Y are normal then X is countably compact if and only if Y is countably compact. In this note we prove that if X and Y are both normal and first countable such that $C_p(X)$ is linearly homeomorphic to $C_p(Y)$, then $X^{(1)}$ is countably compact if and only if $Y^{(1)}$ is countably compact ($X^{(1)}$ is the set of accumulation points of X). Our technique is inspired by Arhangelskiĭ [1] and Baars, de Groot, van Mill and Pelant [3]. We give two examples showing that our result is "best possible". There exist a first countable normal space X and a normal space Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic but $X^{(1)}$ is not countably compact and $Y^{(1)}$ is countably compact. In addition, there exist two metric spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic but $X^{(2)}$ is compact while $Y^{(2)}$ is not compact ($X^{(2)}$ is the second derivative of X).

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1. Preliminaries

In this section we give some results from Baars and de Groot [2], and results and definitions from Arhangelskiĭ [1], which we use in section 2.

Let X be a toplogical space and A a subset of X. Let $Y = Y_{X,A}$ be the quotientspace obtained from X by identifying A to one point, say ∞. Let $C_{p,A}(X)$ be the subspace of $C_p(X)$ consisting of those functions which vanish on A, and let $C_{p,0}(Y)$ be the subspace of $C_p(Y)$ consisting of those functions which are zero at ∞.

If two linear spaces X and Y are linearly homeomorphic then we denote that by $X \sim Y$.

1.1 **Lemma** [2]. *Let X be a space and A a subset of X. Then $C_{p,A}(X) \sim C_{p,0}(Y)$.*

For a topological space X we define for every ordinal α the *α-th derivative* $X^{(\alpha)}$ by transfinite induction as follows: (see [5])

(a) $X^{(0)} = X$ and $X^{(1)} = \{x \in X | x$ is an accumulation point of $X\}$.

(b) If α is a successor, say $\alpha = \beta + 1$, then $X^{(\alpha)} = (X^{\beta)})^{(1)}$.

(c) If α is a limit ordinal then $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

An ordinal α is a *prime component* whenever for all ordinals β and δ with $\alpha = \beta + \delta$ we have $\delta = 0$ or $\delta = \alpha$. For every ordinal α denote by α' the largest prime component which is less than or equal to α.

By $C_{p,0}([1,\alpha])$ we mean the subspace of $C_p([1,\alpha])$ consisting of those functions which are zero at α.

The next lemma and theorem can be found in [2].

1.2 **Lemma.** *Let α be an ordinal. Then $C_{p,0}([1,\alpha]) \sim C_p([1,\alpha])$.*

1.3 **Theorem.** *Let $\omega \leqq \alpha$, $\beta < \omega_1$. Then $C_p([1,\alpha]) \sim C_p([1,\beta])$ iff $\alpha \leqq \beta < \alpha^\omega$.*

The following definitions can be found in [1]. Let X and Y be Tychonov spaces, and ϕ: $C(X) \to C(Y)$ a linear mapping. For every $y \in Y$, the *support* of y in X is defined to be the set supp(y) of all $x \in X$ satisfying the condition that for every neighborhood U of x, there is an $f \in C(X)$ such that $f(X\backslash U) = \{0\}$ and $\phi(f)(y) \neq 0$. For a subset A of Y, we denote $\bigcup_{y\in A}$ supp(y) by supp$A$. Furthermore ϕ is said to be *effective* if for every f, $g \in C(X)$ and $y \in Y$, such that f and g coincide on a neighborhood of supp(y), $\phi(f)(y) = \phi(g)(y)$.

A subset A of X is said to be *bounded* if for every $f \in C(X)$, $f(A)$ is bounded in $\mathbb{R}$.

1.4 **Proposition.** ([1] *Arhangelskiĭ*). *Let X and Y be Tychonov spaces and ϕ: $C_p(X) \to C_p(Y)$ a linear homeomorphism. Then*

(a) *ϕ is effective,*

(b) *if A is a bounded subset of Y, then* $\operatorname{supp} A$ *is bounded in* X.

For details about ordinals we refer to [5] and [6].

2. Function spaces

In this section we prove the results, announced in the Introduction.

2.1 **Lemma.** *Let X and Y be Tychonov spaces and $\phi\colon C_p(X) \to C_p(Y)$ a homeomorphism. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C_p(X)$ such that f_n converges pointwise to a discontinuous function $f \in \mathbb{R}^X$. Suppose $g\colon Y \to \mathbb{R}$ is an accumulation point of the set $\{\phi(f_n) \mid n \in \mathbb{N}\}$. Then g is not continuous.*

Proof. Since $\{f_n \mid n \in \mathbb{N}\}$ is closed and discrete in $C_p(X)$ we have $\{\phi(f_n) \mid n \in \mathbb{N}\}$ is closed and discrete in $C_p(Y)$. □

2.2 **Theorem.** *Let X and Y be topological spaces which are both normal and first countable and let $C_p(X)$ and $C_p(Y)$ be linearly homeomorphic. Then $X^{(1)}$ is countably compact if and only if $Y^{(1)}$ is countably compact.*

Proof. Suppose $X^{(1)}$ is not countably compact and $Y^{(1)}$ is countably compact. Since $X^{(1)}$ is not sequentially compact, there exists a closed discrete set $F = \{x_n \mid n \in \mathbb{N}\}$ in $X^{(1)}$. For every $n \in \mathbb{N}$ let $\{U_j^n \mid j \in \mathbb{N}\}$ be a decreasing open base at x_n and f_j^n a Urysohn function such that $f_j^n(x_n) = 1$ and $f_j^n(X \backslash U_j^n) = 0$. Then $f_j^n \to \chi_{x_n}$ pointwise, where χ_{x_n} is the characteristic function of x_n. Notice that χ_{x_n} is discontinuous. Furthermore let $\phi\colon C_p(X) \to C_p(Y)$ be a linear homeomorphism and let $g_j^n = \phi(f_j^n)$.

Claim. For every $y \in Y$ and $n \in \mathbb{N}$, the set $\{g_j^n(y) \mid j \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

Suppose not. Then there are $y \in Y$ and $n \in \mathbb{N}$, such that without loss of generality for every $k \in \mathbb{N}$ there is $j_k \in \mathbb{N}$, with $g_{j_k}^n(y) \geqq 2^k$. The function $f = \sum_{k=1}^{\infty} 2^{-k} f_{j_k}^n \in C_p(X)$, so $\phi(f) = \sum_{k=1}^{\infty} 2^{-k} g_{j_k}^n \in C_p(Y)$. But then we have a contradiction since $\phi(f)(y) = \sum_{k=1}^{\infty} 2^{-k} g_{j_k}^n(y) = \infty$.

For every $y \in Y$, let A_y be compact in $\mathbb{R}$ such that $\{g_j^n(y) \mid j \in \mathbb{N}\} \subset A_y$. Then $\prod_{y\in Y} A_y$ is a compact subset of $\mathbb{R}^Y$. Since $\{g_j^n \mid j \in \mathbb{N}\} \subset \prod_{y\in Y} A_y$, $\{g_j^n \mid j \in \mathbb{N}\}$ has an accumulation point σ_n. By Lemma 2.1, σ_n is discontinuous, say at y_n. Notice that $y_n \in Y^{(1)}$. Since $Y^{(1)}$ is sequentially compact, without loss of generality we may assume that there is $y \in Y$ such that $y_n \to y$. Let $\{V_n \mid n \in \mathbb{N}\}$ be a decreasing open base at y. Without loss of generality $y_n \in V_n$.

Since Y is first countable, for every $n \in \mathbb{N}$ there is a sequence $(y_k^n)_k$ in V_n such that $y_k^n \to y_n$ and

$$(*) \qquad \sigma_n(y_k^n) \nrightarrow \sigma_n(y_n).$$

Let $K = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{y_n, y_k^n\} \cup \{y\}$. Then K is compact. Indeed, let $\mathscr{V}$ be an open cover of K. There is $V \in \mathscr{V}$ with $y \in V$. There is $n_0 \in \mathbb{N}$ such that $y \in V_{n_0} \subset V$. Then $\bigcup_{n \geqq n_0} \bigcup_{k \in \mathbb{N}} \{y_n, y_k^n\} \cup \{y\} \subset V$. Since $\bigcup_{n < n_0} \bigcup_{k \in \mathbb{N}} \{y_n, y_k^n\}$ is compact, we are done.

Since K is compact, it is bounded in Y. So by Proposition 1.4, $\overline{\operatorname{supp} K}$ is bounded in X. Since F is closed and discrete and X is normal, F is not bounded. This implies that there is $n \in \mathbb{N}$ such that $x_n \notin \overline{\operatorname{supp} K}$. Since X is regular there is $j_0 \in \mathbb{N}$ and a neighborhood V of $\overline{\operatorname{supp} K}$ such that $U_{j_0}^n \cap V = \varnothing$. So for every $z \in K$ and $j \geqq j_0$, f_j^n and the zero function on X are equal on V, which is a neighborhood of $\operatorname{supp}(z)$. Since ϕ is linear and effective, this implies that $g_j^n(z) = 0$ for every $j \geqq j_0$ and $z \in K$. But then $\sigma_n(y_k^n) = 0$ and $\sigma_n(y_n) = 0$, which gives a contradiction with $(*)$. □

By $X \oplus Y$ or $\oplus_{i=1}^{\infty} X_i$ we denote the topological sum of the topological spaces X and Y or $X_i (i \in \mathbb{N})$, respectively.

2.3 **Example.** In this example we show that the first countability condition in Theorem 2.2 is essential.

Let $X = \oplus_{i=1}^{\infty} [1, \omega]_i$. Let $A = X^{(1)}$ and $Y = Y_{X,A}$ the quotient space obtained from X by identifying A to single point, say ∞. Then X is clearly first countable and normal, and Y is normal but not first countable. By Lemma 1.1 we have $C_{p,A}(X) \sim C_{p,0}(Y)$. Furthermore we have

$$\begin{aligned} C_{p,A}(X) &\sim \prod_{i=1}^{\infty} C_{p,0}([1, \omega])_i \\ &\sim \prod_{i=1}^{\infty} C_p([1, \omega]) \quad \text{(Lemma 1.2a)} \\ &\sim C_p(X). \end{aligned}$$

Notice that for every Tychonov space Z and for every $z \in Z$, $C_p(Z) \sim C_{p,0}(Z) \times \mathbb{R}$, where $C_{p,0}(Z)$ consists of those functions in $C_p(Z)$ which vanish at z. So by Lemma 1.2, $C_p([1, \omega]) \sim C_p([1, \omega]) \times \mathbb{R}$. This implies $C_p(X) \sim C_p(X) \times \mathbb{R}$. So

$$\begin{aligned} C_p(X) &\sim C_p(X) \times \mathbb{R} \\ &\sim C_{p,A}(X) \times \mathbb{R} \\ &\sim C_{p,0}(Y) \times \mathbb{R} \\ &\sim C_p(Y). \end{aligned}$$

However $X^{(1)} = A$ is not countably compact, and $Y^{(1)} = \{\infty\}$ is countably compact.

From Theorem 2.2 and the result in [1] for normal spaces, that if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic and X is countably compact, then Y is countably compact, one could conjecture the following: Let α be an arbitary ordinal. If X and Y are both normal and first countable spaces such that

$C_p(X)$ and $C_p(Y)$ are linearly homeomorphic and $X^{(\alpha)}$ is countably compact, then $Y^{(\alpha)}$ is countably compact.

In the next example we show that if α is not a prime component, then the conjecture is false.

2.4 **Example.** Let $\alpha < \omega_1$ be an ordinal which is not a prime component. Observe that in this situation $1 \leqq \alpha' < \alpha$.

Let $X = \oplus_{i=1}^{\infty}[1, \omega^{\alpha'}]_i$ and $Y = \oplus_{i=1}^{\infty}[1, \omega^{\alpha}]_i$. By Theorem 1.3, $C_p[1, \omega^{\alpha'}] \sim C_p[1, \omega^{\alpha}]$, so that $C_p(X) \sim C_p(Y)$. But $Y^{(\alpha)} \approx \mathbb{N}$ (see [2] or [6] p. 155) which is not countably compact, and $X^{(\alpha)} = \varnothing$ which is countably compact.

Questions. (1) Is the above conjecture true for prime components?

(2) Does Theorem 2.2 still hold if normal is replaced by Tychonov?

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